Problem Set 2 Solutions

Problem 1

The equilibrium point is

$$\bar{\mathbf{x}} = \begin{bmatrix} 1 \\ -\frac{1}{e} \\ -\frac{2}{e} \end{bmatrix}.$$

Set $\delta\mathbf{x} = \begin{bmatrix} x_1 - 1 \\ x_2 + \frac{1}{e} \\ x_3 + \frac{2}{e} \end{bmatrix}$, $\delta u = u - 1$, $\delta y = y - 1 + \frac{1}{e}$. Then, the linearized system is

$$\begin{aligned} \dot{\delta\mathbf{x}} &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 1 \\ -1 & e & 0 \end{bmatrix} \delta\mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \delta u \\ y &= [1 \;\; 1 \;\; 0]\delta\mathbf{x}. \end{aligned}$$

Problem 2

1. $f(t) = 3t^2 e^{-t} \cdot \mathbf{1}(t)$. Use P7 twice:
$$F(s) = \frac{6}{(s+1)^3}.$$

2. $f(t) = \sin t \cos t \cdot \mathbf{1}(t)$. Note that $\sin t \cos t = 1/2 \; \sin(2t)$. Then,
$$F(s) = \frac{1}{s^2+4}.$$

3. $f(t) = \sin(t-3) \cdot \mathbf{1}(t)$. Note: you cannot use P5 here! Rather, write $f(t) = \sin t \cos 3 \cdot \mathbf{1}(t) - \cos t \sin 3 \cdot \mathbf{1}(t)$ so that:
$$F(s) = \frac{\cos 3}{s^2+1} - \frac{\sin 3 \cdot s}{s^2+1}.$$

4. $f(t) = \sin(t-3) \cdot \mathbf{1}(t-3)$. Here you should use P5:
$$F(s) = \frac{1}{s^2+1} e^{-3s}.$$

5. $f(t) = t\sin(t-3)\mathbf{1}(t)$. Use the identity $\sin(t-3) = \sin t \cos 3 - \cos t \sin 3$ and P7:
$$F(s) = \frac{\cos 3 \cdot 2s}{(s^2+1)^2} + \frac{\sin 3(s^2+1) - \sin 3 \cdot 2s^2}{(s^2+1)^2}.$$

6. $f(t) = (t-3)e^{t-3}\mathbf{1}(t)$. You can't use P5 here. Rather, write $f(t) = te^t e^{-3}\mathbf{1}(t) - 3e^t e^{-3}\mathbf{1}(t)$ and use P7:
$$F(s) = \frac{e^{-3}}{(s-1)^2} - 3\,\frac{e^{-3}}{s-1}.$$

7. $f(t) = (t-3)e^{t-3}\mathbf{1}(t-3)$. Here you should use P7 first and then P5:

$$F(s) = \frac{e^{-3\,s}}{(s-1)^2}.$$

8. $f(t) = te^{-at} + 2t\cos t$. Since

$$\mathcal{L}\{te^{-at}\} = \frac{1}{(s+a)^2}$$
$$\mathcal{L}\{t\cos at\} = \frac{s^2-a^2}{(s^2+a^2)^2}$$

hence

$$\begin{aligned}\mathcal{L}\{te^{-at} + 2t\cos t\} &= \mathcal{L}\{te^{-at}\} + \mathcal{L}\{2t\cos t\}\\ &= \frac{1}{(s+a)^2} + 2\frac{s^2-1}{(s^2+1)^2}\end{aligned}$$

9. $f(t) = t^2 + e^{-at}\sin bt$.

$$\begin{aligned}\mathcal{L}\{t^2 + e^{-at}\sin bt\} &= \mathcal{L}\{t^2\} + \mathcal{L}\{e^{-at}\sin bt\}\\ &= \frac{2}{s^3} + \frac{b}{(s+a)^2+b^2}\end{aligned}$$

Problem 3

1. $F(s) = \frac{s^2+s+1}{(s+1)(s+2)(s+3)}$. Write

$$F(s) = \frac{a_1}{s+1} + \frac{a_2}{s+2} + \frac{a_3}{s+3}.$$

Compute:

$$a_1 = \left.\frac{s^2+s+1}{(s+2)(s+3)}\right|_{s=-1} = \frac{1}{2},$$
$$a_2 = \left.\frac{s^2+s+1}{(s+1)(s+3)}\right|_{s=-2} = -3,$$
$$a_3 = \left.\frac{s^2+s+1}{(s+1)(s+2)}\right|_{s=-3} = \frac{7}{2},$$

so that

$$f(t) = \left(\frac{1}{2}e^{-t} - 3e^{-2t} + \frac{7}{2}e^{-3t}\right)\mathbf{1}(t).$$

2. $F(s) = \left(\frac{s^2+s+1}{(s+1)(s+2)(s+3)}\right)e^{-s}$. Using the solution to the previous problem and P5, we immediately get

$$f(t) = \left(\frac{1}{2}e^{-(t-1)} - 3e^{-2(t-1)} + \frac{7}{2}e^{-3(t-1)}\right)\mathbf{1}(t-1).$$

3. $F(s) = \frac{e^{-2s}}{s-3}$. Use P5:

$$f(t) = e^{3(t-2)}\mathbf{1}(t-2).$$

4. $F(s) = \frac{e^{1-s}}{s}$. Write $F(s) = \frac{e^{-s}\cdot e^{1}}{s}$ and use P5:

$$f(t) = e\cdot\mathbf{1}(t-1).$$

5. $F(s) = \frac{1}{s^6}$.

$$\mathcal{L}^{-1}\{\frac{1}{s^6}\} = \frac{t^5}{5!} = \frac{t^5}{120}$$

6. $F(s) = \frac{10}{s(s+1)(s+10)}$.

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{10}{s(s+1)(s+10)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{\frac{10}{9}}{s+1} + \frac{\frac{1}{9}}{s+10}\right\} \\ &= \left\{1 + \frac{10}{9}e^{-t} + \frac{1}{9}e^{-10t}\right\}1(t)\end{aligned}$$

7. $F(s) = \frac{1}{s(s+2)^2}$

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s(s+2)^2}\right\} &= \mathcal{L}^{-1}\left\{\frac{\frac{1}{4}}{s} - \frac{\frac{1}{4}}{s+2} - \frac{\frac{1}{2}}{(s+2)^2}\right\} \\ &= \frac{1}{4}\left\{1 - e^{-2t} - 2te^{-2t}\right\}1(t)\end{aligned}$$

8. $F(s) = \frac{2(s+2)}{(s+1)(s^2+4)}$ Write

$$\frac{2(s+2)}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}$$

Therefore,

$$A = \left.\frac{2(s+2)}{s^2+4}\right|_{s=-1} = \frac{2}{5},$$

and

$$\begin{aligned}A(s^2+4) + (Bs+C)(s+1) &= 2(s+2) \\ \Longrightarrow (A+B)s^2 + (B+C)s + (C+4A) &= 0s^2 + 2s + 4.\end{aligned}$$

Now, equate the coefficients to obtain

$$B = -A = -\frac{2}{5}, \qquad C = 2 - B = 2 + \frac{2}{5} = \frac{12}{5}.$$

(Check: $C + 4A = \frac{12}{5} + \frac{8}{5} = 4$.)

Since

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}, \qquad \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2},$$

we have

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{A}{s+1} + \frac{Bs+C}{s^2+4}\right\} &= Ae^{-t} + B\cos 2t + \frac{C}{2}\sin 2t, \\ \therefore \qquad \mathcal{L}^{-1}\left\{\frac{2(s+2)}{(s+1)(s^2+4)}\right\} &= \frac{2}{5}e^{-t} - \frac{2}{5}\cos 2t + \frac{6}{5}\sin 2t.\end{aligned}$$

Problem 4

Given $F(s) = \frac{2-s}{(s-1)(s+2)}$, find $\lim_{t\to+\infty} f(t)$.

Let us try to employ the Final Value Theorem. This says that, IF the limit $\lim_{t\to+\infty} f(t)$ exists and is finite, then it is given by

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{2s - s^2}{(s-1)(s+2)} = 0.$$

We now need to find out whether the limit exists and is finite. It turns out that the limit isn't finite and so the Final Value Theorem predicts the wrong limit value! Note, indeed, that if we write the partial fraction expansion of $F(s)$ we get

$$F(s) = \frac{A}{s-1} + \frac{B}{s+2}.$$

We don't need to find A and B for the purpose of solving this problem. Taking the inverse Laplace transform of $F(s)$ above we get

$$f(t) = Ae^t \mathbf{1}(t) + Be^{-2t}\mathbf{1}(t),$$

where $\mathbf{1}(t)$ denotes the unit step function. Now, it is clear that $\lim_{t\to+\infty} f(t) = \infty$.

Problem 5

1.

$$\begin{aligned} \ddot{y} + 2\dot{y} + 5y &= e^{-t}\mathbf{1}(t) \\ y(0) = \dot{y}(0) &= 0 \end{aligned}$$

$$\begin{aligned} Y(s) &= \left(\frac{1}{s+1}\right)\left(\frac{1}{s^2+2s+5}\right) \\ &= \left[\frac{1}{s+1}\right]\left[\frac{1}{(s+1+2j)(s+1-2j)}\right] \\ &= \frac{c_1}{s+1} + \frac{c_2}{s+1+2j} + \frac{c_3}{s+1-2j} \end{aligned}$$

$$c_1 = \tfrac{1}{2j}\cdot\tfrac{1}{-2j} = \tfrac{1}{4}, \quad c_2 = \tfrac{1}{-2j}\cdot\tfrac{1}{-4j} = -\tfrac{1}{8}, \quad c_3 = c_2^* = -\tfrac{1}{8}$$

$$\begin{aligned} y(t) &= \left[\frac{1}{4}e^{-t} - \frac{1}{8}e^{(-1-2j)t} - \frac{1}{8}e^{(-1+2j)t}\right]\mathbf{1}(t) \\ &= \frac{1}{4}\left[e^{-t} - \frac{1}{2}e^{-t}\left(e^{-2jt} + e^{2jt}\right)\right]\mathbf{1}(t) \\ &= \frac{1}{4}e^{-t}\left[1 - \cos 2t\right]\mathbf{1}(t) \end{aligned}$$

2.

$$\begin{aligned}
\ddot{y} - y &= \mathbf{1}(t) \\
y(0) = \dot{y}(0) &= 1 \\
&\Downarrow \\
s^2 Y(s) - s - 1 - Y(s) &= \frac{1}{s} \\
\left(s^2 - 1\right) Y(s) &= s + \frac{1}{s} + 1 \\
&\Updownarrow \\
Y(s) &= \frac{s^2 + s + 1}{s(s^2 - 1)} \\
&= \frac{s^2 + s + 1}{s(s+1)(s-1)} \\
&= \frac{c_1}{s} + \frac{c_2}{s+1} + \frac{c_3}{s-1}
\end{aligned}$$

$$c_1 = -1 \quad c_2 = \tfrac{1}{-1\cdot(-2)} = \tfrac{1}{2} \quad c_3 = \tfrac{3}{2}$$

$$\begin{aligned}
&\Downarrow \\
y(t) \quad &= \quad \left[-1 + \frac{1}{2} e^{-t} + \frac{3}{2} e^{t}\right] \mathbf{1}(t)
\end{aligned}$$

3.

$$\begin{aligned}
&y^{(4)} + 4y^{(3)} + 8y^{(2)} + 8y^{(1)} + 4y = 3\dot{u} + 4u, \ u = \mathbf{1}(t) \\
&y(0) = \ldots = y^{(3)}(0) = 0
\end{aligned}$$

$$\begin{aligned}
Y(s) &= \frac{3s+4}{s^4 + 4s^3 + 8s^2 + 8s + 4} \frac{1}{s} \\
&= \frac{1}{s} \frac{3s+4}{\left[(s+1)^2 + 1\right]^2} \\
&= \frac{k}{s} + \frac{c_1}{s+1+j} + \frac{c_2}{(s+1+j)^2} + \frac{d_1}{s+1-j} + \frac{d_2}{(s+1-j)^2}
\end{aligned}$$

$$k = 1 \quad c_1 = -\frac{1}{2} - \frac{1}{4} j \quad c_2 = -\frac{1}{4} - \frac{1}{2} j \quad d_1 = c_1^* = -\frac{1}{2} + \frac{1}{4} j \quad d_2 = c_2^* = -\frac{1}{4} + \frac{1}{2} j$$

$$\begin{aligned}
y(t) &= \left[1 + \left(-\frac{1}{2} - \frac{1}{4} j\right) e^{(-1-j)t} + \left(-\frac{1}{4} - \frac{1}{2} j\right) t e^{(-1-j)t} + \left(-\frac{1}{2} + \frac{1}{4} j\right) e^{(-1+j)t} + \left(-\frac{1}{4} + \frac{1}{2} j\right) t e^{(-1+j)t}\right] \mathbf{1}(t) \\
&= \left[1 - e^{-t}\left(\cos t + \frac{1}{2}\sin t + t\cos t + t\sin t\right)\right] \mathbf{1}(t)
\end{aligned}$$

4. $\ddot{y} + \dot{y} + 3y(t) = 0; y(0) = \alpha, \dot{y}(0) = \beta.$

$$\begin{aligned}
&\mathcal{L}\{\ddot{y}(t) + \dot{y}(t) + 3y(t) = 0\} \\
&\Rightarrow s^2 Y(s) - s\alpha - \beta + sY(s) - \alpha + 3Y(s) = 0
\end{aligned}$$

$$
\begin{aligned}
Y(s) &= \frac{\alpha(s+1)+\beta}{s^2+s+3} \\
&= \frac{A}{s+0.5+j1.6583} + \frac{A^*}{s+0.5-j1.6583}
\end{aligned}
$$

$$
\begin{aligned}
A &= \left.\frac{\alpha(s+1)+\beta}{s+0.5-j1.6583}\right|_{s=-0.5-j1.6583} \\
&= \alpha(0.5+j0.1508) + j0.3015\beta
\end{aligned}
$$

$$
A^* = \alpha(0.5-j0.1508) - j0.3015\beta
$$

$$
\begin{aligned}
y(t) &= [\alpha(0.5+j0.1508) + j0.3015\beta]e^{-(0.5+j1.6583)t} \\
&\quad + [\alpha(0.5-j0.1508) - j0.3015\beta]e^{-(0.5-j1.6583)t} \\
&= e^{-\frac{t}{2}}\{\alpha\cos(1.6583t) + (0.3015\alpha + 0.6030\beta)\sin(1.6583t)\}1(t)
\end{aligned}
$$