Problem Set 7 Solutions

Problem 1

First, we calculate the contribution to the steady-state error due only to disturbance: Rearranging the block diagram to show D(s) as the input,

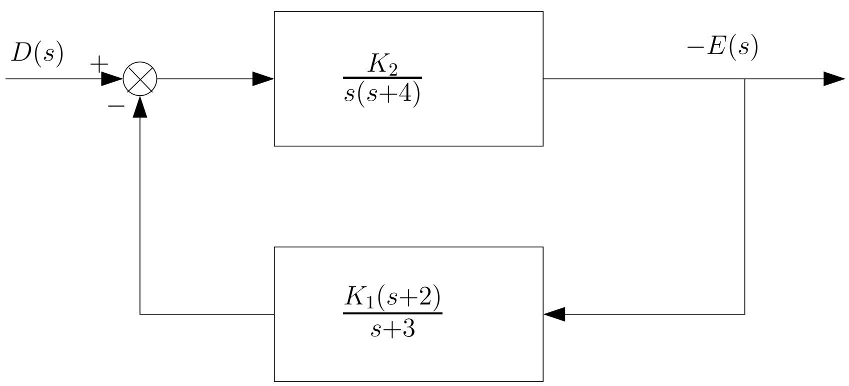


Figure 1: Rearranged block diagram.

Therefore,

$$-E(s) = D(s)\frac{\frac{K_2}{s(s+4)}}{1+\frac{K_1K_2(s+2)}{s(s+3)(s+4)}} = D(s)\frac{K_2(s+3)}{s(s+3)(s+4)+K_1K_2(s+2)}$$

Next, we calculate the contribution to the steady-state error due to the reference input $R(s)$:

$$E_R(s) = R(s)\frac{1}{1+\frac{K_1K_2(s+2)}{s(s+3)(s+4)}}$$

For $D(s) = \frac{1}{s}$, $e_D(\infty) = \lim_{s\to 0} sE_D(s) = -\frac{3}{2K_1}$.
This identity holds *provided* $E_D(s)$ has poles in the LHP or at most one pole in the origin.
For $R(s) = 1/s^2$, $e_R(\infty) = \lim_{s\to 0} sE_R(s) = \frac{1}{\frac{K_1K_2}{6}} = \frac{6}{K_1K_2}$.

This identity holds *provided* $E_R(s)$ has poles in the LHP or at most one pole in the origin.
Design: $e_D(\infty) = -0.000012 = -\frac{3}{2K_1}$, or $K_1 = 125000$. Similarly, $e_R(\infty) = 0.003 = \frac{6}{K_1K_2}$, or $K_2 = 0.016$
Note that, for these values of K_1 and K_2 to be acceptable, both $E_R(s)$ and $E_D(s)$ must satisfy the property above (have poles in the LHP and at most one pole in the origin). It is easy to check that this is the case.

Problem 2

a. $E(s) = R(s) - C(s)$. But, $C(s) = [R(s) - C(s)H(s)]G_1(s)G_2(s) + D(s)$.
Solving for $C(s)$,

$$C(s) = \frac{R(s)G_1(s)G_2(s)}{1+G_1(s)G_2(s)H(s)} + \frac{D(s)}{1+G_1(s)G_2(s)H(s)}$$

Substituting into $E(s)$,

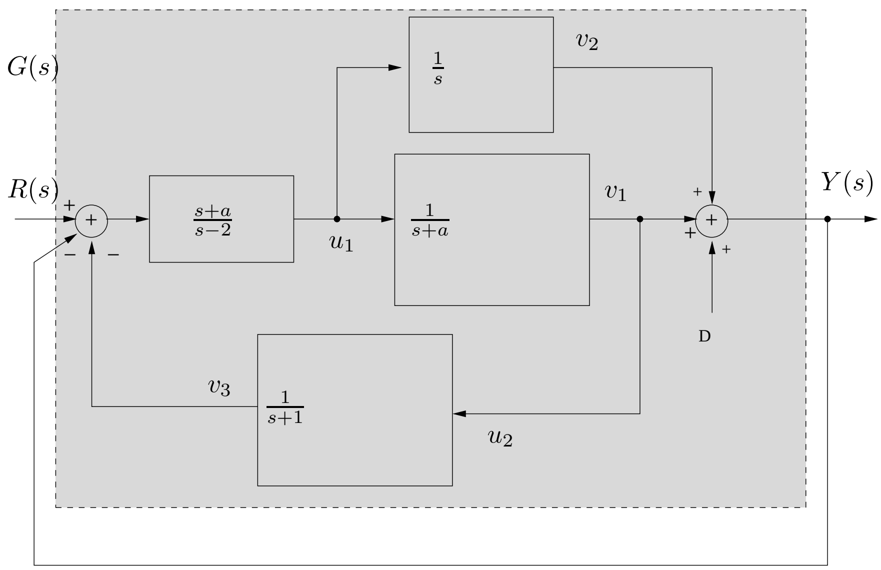
$$E(s) = \left[1 - \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)}\right]R(s) - \frac{1}{1 + G_1(s)G_2(s)H(s)}D(s)$$

b. For $R(s) = D(s) = \frac{1}{s}$,

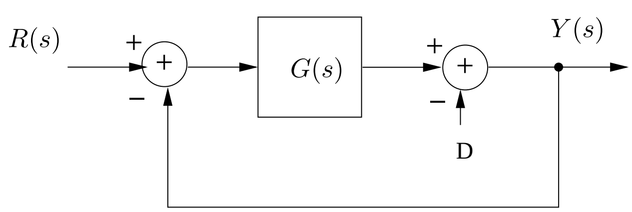
$$e(\infty) = \lim_{s \to 0} sE(s) = 1 - \frac{\lim_{s\to 0} G_1(s)G_2(s)}{1 + \lim_{s\to 0} G_1(s)G_2(s)H(s)} - \frac{1}{1 + \lim_{s\to 0} G_1(s)G_2(s)H(s)}$$

Problem 3

Trivial state space to transfer function conversions yield the following block diagram:



The block diagram of the system after block simplifications is:



with G defined above. Applying superposition and the Final Value Theorem one easily arrives at the following:

$$e(\infty) = 0.$$

Problem 4

In order to calculate the steady-state tracking error, we calculate the error signal $E(s) = R(s) - Y(s)$,

$$\begin{aligned}
E(s) =& [1 - T(s)]R(s) \\
=& \frac{1}{1+G(s)} R(s) \\
=& \frac{s(\tau s+1)}{s(\tau s+1)+K} \cdot \frac{R_0}{s^2} \\
=& \frac{\tau s+1}{s(\tau s+1)+K} \cdot \frac{R_0}{s}.
\end{aligned}$$

Next, in order to apply the Final Value Theorem, we have to check that the poles of $E(s)$ are in the left half plane (LHP) and there is at most one pole in the origin. In this case $E(s)$ has one pole in the origin and two poles given by the roots of the polynomial

$$s(\tau s+1) + K = 0 \iff \tau s^2 + s + K = 0.$$

Since the polynomial is second-order, without calculating its roots we know that they are contained in the LHP if and only if the coefficients of the polynomial are positive. Since this is the case, we conclude that the hypothesis of the Final Value Theorem is satisfied and, hence,

$$e(\infty) = \lim_{s\to 0} sE(s) = \frac{R_0}{K}.$$

Problem 5

(a) The poles of the system are the roots of

$$s^4 + Ks^3 + (Kb+K)s^2 + K(a+b)s + Kab.$$

Form the Routh array:

$$\begin{array}{c|ccc}
s^4 & 1 & Kb+K & Kab \\
s^3 & K & K(a+b) & 0 \\
s^2 & Kb+K-(a+b) & Kab & 0 \\
s & \dfrac{[Kb+K-(a+b)]K(a+b)-K^2ab}{Kb+K-(a+b)} & 0 & 0 \\
s^0 & Kab & 0 & 0
\end{array}$$

Conditions for BIBO stability are obtained by imposing that all entries in the first column of the Routh array be positive, i.e., (recall that we already have that $K > 0, a > 0, b > 0$),

$$\begin{aligned}
&Kb + K - (a+b) > 0 \\
&[Kb+K-(a+b)]K(a+b) - K^2ab > 0.
\end{aligned}$$

(b) Just apply the Final Value Theorem to obtain the following:

$$y_{ss} = \lim_{s\to 0} sY(s) = 1.$$

Problem 6

We calculate $E(s)$,

$$\begin{aligned}
E(s) &= \frac{1}{1+T(s)} R(s) = \frac{1}{1+\frac{K(s+2)}{s^2(s+5)}} \, \frac{1}{s^3} = \frac{s^2(s+5)}{s^2(s+5)+K(s+2)} \, \frac{1}{s^3} \\
&= \frac{s+5}{s(s^3+5s^2+Ks+2K)}.
\end{aligned}$$

Before using the Final Value Theorem one has to check that the poles of $E(s)$ are in the LHP or there is at most one pole in the origin. In this case $E(s)$ has one pole in the origin and three poles given by the roots of

$$s^3 + 5s^2 + Ks + 2K = 0.$$

Using the Routh criterion one can find conditions on K such that the roots of this polynomial are in the LHP with at most one root at $s = 0$, and then apply the Final Value Theorem. The Routh table is:

$$\begin{array}{c|cc} s^3 & 1 & K \\ s^2 & 5 & 2K \\ s & \frac{3}{5}K & 0 \\ s^0 & 2K & 0 \end{array}$$

from which it is immediately deduced that the FVT can be applied for any $K > 0$. By applying the Final Value Theorem we get

$$e(\infty) = \frac{1}{K_a} = \frac{5}{2K}.$$

Hence, in order for $e(\infty) = 0.01$, we choose $K = 250$.

Problem 7

(a)

$$Y(s) = \frac{1}{s-2} \cdot \frac{1}{s}$$

Since one pole of the TF is in RHP, it follows that the output $y(t) \to \infty$ as $t \to \infty \Rightarrow e(\infty) = \lim_{t\to\infty} u(t) - y(t) = 1 - \infty = -\infty$.

Note: The Final value Theorem **can not be** applied here. Applying it would not only give a wrong result, but would also represent an important conceptual mistake.

(b) We notice that

$$\begin{aligned} E(s) &= U(s) - Y(s) \\ &= U(s) - \frac{\frac{s(s+1)}{s^2(s+2)}}{1 + \frac{s^2(s+1)}{s^2(s+2)}} U(s) \\ &= \left[1 - \frac{s+1}{s(s+2) + s(s+1)}\right] U(s) \\ &= \frac{s^2 + 2s + s^2 + s - s - 1}{s(2s+3)} \cdot \frac{1}{s} \end{aligned}$$

$E(s)$ has two poles at the orgin $\Rightarrow e(t) \to \infty$ as $t \to \infty$.

Note: The Final value Theorem **can not be** applied here. Applying it would represent an important conceptual mistake.

(c)

$$\begin{aligned} Y(s) &= Y_R(s) + Y_D(s) \\ Y_R(s) &= \frac{1}{s(s+1)+1} U(s) \\ Y_D(s) &= \frac{1}{1 + \frac{1}{s(s+1)}} D(s) = \frac{s(s+1)}{s(s+1)+1} D(s) \\ E(s) &= U(s) - Y(s) = \frac{1}{s}\left[1 - \frac{1}{s(s+1)+1} - \frac{s(s+1)}{s(s+1)+1}\right] \\ &= \frac{1}{s}\left[\frac{s^2 + s + 1 - 1 - s^2 - s}{s(s+1)+1}\right] = 0 \Rightarrow e(\infty) = 0 \end{aligned}$$

Problem 8

(i) The poles of the closed-loop system are the roots of the polynomial

$$s^4 + 8s^3 + 17s^2 + (K+10)s + Ka = 0.$$

We form the Routh array

$$\begin{array}{c|ccc} s^4 & 1 & 17 & Ka \\ s^3 & 8 & K+10 & 0 \\ s^2 & \frac{126-K}{8} & Ka & 0 \\ s & \star & 0 & 0 \\ s^0 & Ka & 0 & 0 \end{array}$$

where

$$\star = \frac{(K+10)(126-K) - 64Ka}{126-K}.$$

We impose that there be no sign variations in the first column of the array

$$\begin{aligned} &K < 126 \\ &(K+10)(126-K) - 64Ka > 0 \\ &Ka > 0. \end{aligned}$$

(ii) Under the assumption that $\lim_{t\to\infty} e(t)$ exists, such limit is

$$\lim_{t\to\infty} e(t) = \frac{1}{K_v} R,$$

where

$$K_v = \lim_{s\to 0} \frac{K(s+a)}{(s+1)(s+2)(s+5)} = \frac{aK}{10}.$$

We thus need

$$aK \geq 40.$$

(iii) $20 \leq K < 30$.

Problem 9

(i) The closed-loop system has two poles. They are the roots of the polynomial

$$s^2 + Ks + 1 + K = 0.$$

Using the formula

$$\%OS = \exp(-\pi\zeta/\sqrt{1-\zeta^2}),$$

we want the damping ration of the poles of the closed-loop system to satisfy

$$1 \geq \zeta > 0.83.$$

It is thus sufficient to choose K such that $\zeta = 1$. From the polynomial above we find ζ and ω_n by identifying coefficients as follows

$$2\zeta\omega_n = K \quad w_n^2 = 1 + K.$$

We get

$$\zeta = \frac{K}{2\sqrt{1+K}}.$$

By imposing $\zeta = 1$ we get $K^2 - 4K - 4 = 0$. The positive root of this polynomial is $K = 2 + 2\sqrt{2}$.

(ii) We have $R(s) = \frac{R}{s^2+1}$ and

$$E(s) = R\frac{s^2 + p^2}{(s^2 + 1)(s^2 + Ks + p^2 + Kz)}.$$

We notice that if $p \neq 1$ then $E(s)$ has two poles at $\pm j$ and hence $\lim_{t\to\infty} e(t)$ does not exist. Recall, in fact, that in order for such limit to exist $E(s)$ must have poles in the open LHP and at most one pole at the origin.

On the other hand, if $p = 1$, $E(s)$ becomes

$$E(s) = \frac{R}{s^2 + Ks + 1 + Kz}.$$

By imposing that all poles be in the open LHP, that is,

$$K > 0, \; 1 + Kz > 0$$

we have the desired property $\lim_{t\to\infty} e(t) = 0$.

In conclusion the most general conditions are:

$$\begin{aligned} &p = 1 \\ &K > 0 \\ &Kz + 1 > 0. \end{aligned}$$