Problem Set 8 Solutions

Problem 1

(a) We have:

$$P(s)C(s) = \frac{50}{s(s+3)(s+6)}.$$

The Nyquist contour is shown in Figure 1.

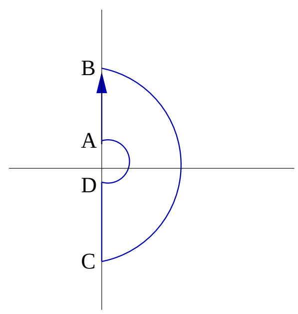

Figure 1: Nyquist contour for 1(a)

- $\overline{AB}$: Set $s = j\omega$, where $\omega : 0 \to \infty$. We have

$$\begin{aligned} PC(j\omega) &= \frac{50}{j\omega(j\omega+3)(j\omega+6)} \\ &= \frac{-450\omega}{\omega(9+\omega^2)(36+\omega^2)} + j\frac{50\omega^2 - 900}{\omega(9+\omega^2)(36+\omega^2)}. \end{aligned}$$

  Thus, we have

$$\begin{aligned} \Re(PC(j\omega)) &= \frac{-450}{(9+\omega^2)(36+\omega^2)} \\ \Im(PC(j\omega)) &= \frac{50\omega^2 - 900}{\omega(9+\omega^2)(36+\omega^2)}. \end{aligned}$$

  The plots of these functions for positive values of ω are shown in Figures 2-3.

- $\overline{BC}$: Set $s = Re^{j\theta}$ where $R \to \infty$ and $\theta : \frac{\pi}{2} \to -\frac{\pi}{2}$. Since PC is strictly proper, the semi-circle at ∞ will collapse to zero.
- $\overline{CD}$: Use rules about complex conjugate and the plot for $\overline{AB}$.
- $\overline{DA}$: Set $s = \epsilon e^{j\theta}$ where $\epsilon \to 0$ and $\theta : -\frac{\pi}{2} \to \frac{\pi}{2}$. This gives

$$PC(s)|_{s=\epsilon e^{j\theta}} \sim \frac{50}{18\epsilon}e^{-j\theta}$$

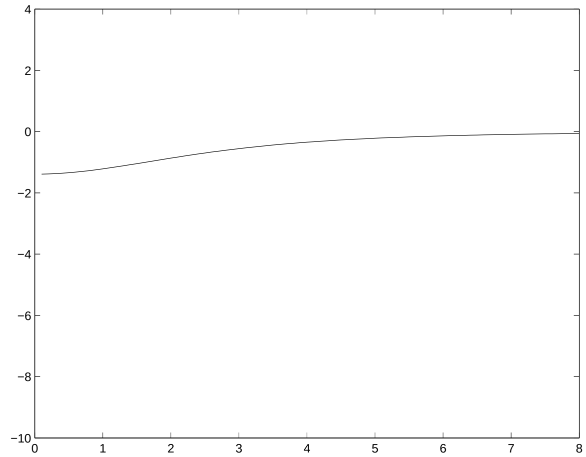

Figure 2: $\Re(PC)$ v.s. ω.

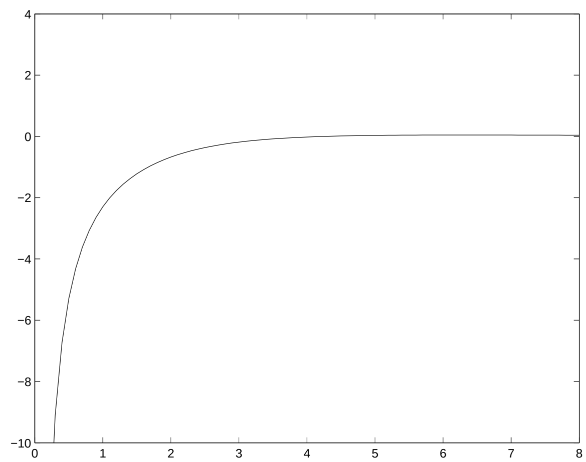

Figure 3: $\Im(PC)$ v.s. ω.

Now we can try a test point $s = \epsilon$ and we get

$$PC(\epsilon) \sim \frac{50}{18\epsilon}.$$

This is a positive real number which tells us which way the infinite radius semicircle turns after departing from point D.

Putting all this together, the Nyquist plot is shown in Figure 4. We extract the following data from the plot:

$$n = 0, \qquad e = 0, \qquad m = n - e = 0.$$

Therefore the closed loop system is stable.

(b) We have:

$$P(s)C(s) = \frac{(s+4)}{s(s+1)}.$$

The Nyquist contour is shown in Figure 5.

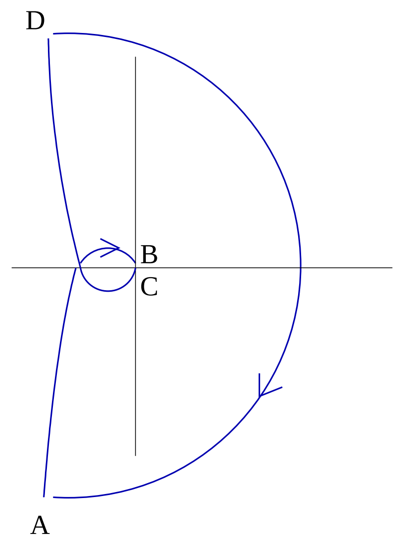

Figure 4: Nyquist plot for 1(a)

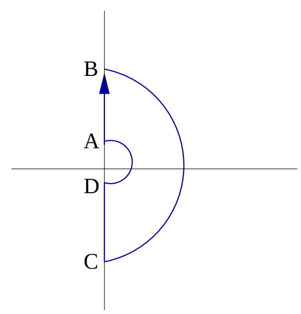

Figure 5: Nyquist contour for 1(b)

- $\overline{AB}$: Set $s = j\omega$, where $\omega : 0 \to \infty$. We have

$$\begin{aligned} PC(j\omega) &= \frac{j\omega + 4}{j\omega(j\omega+1)} \\ &= \frac{-3\omega}{\omega(1+\omega^2)} - j\frac{\omega^2+4}{\omega(1+\omega^2)}. \end{aligned}$$

  Thus, we have

$$\begin{aligned} \Re(PC(j\omega)) &= \frac{-3}{(1+\omega^2)} \\ \Im(PC(j\omega)) &= \frac{-4-\omega^2}{\omega(1+\omega^2)}. \end{aligned}$$

  The plots of these functions for positive values of ω are shown in Figures 6-7.
- $\overline{BC}$: Set $s = Re^{j\theta}$ where $R \to \infty$ and $\theta : \frac{\pi}{2} \to -\frac{\pi}{2}$. Since PC is strictly proper, the semi-circle at ∞ will collapse to zero.
- $\overline{CD}$: Use rules about complex conjugate and the plot for $\overline{AB}$.
- $\overline{DA}$: Set $s = \epsilon e^{j\theta}$ where $\epsilon \to 0$ and $\theta : -\frac{\pi}{2} \to \frac{\pi}{2}$. This gives

$$PC(s)|_{s=\epsilon e^{j\theta}} \sim \frac{4}{\epsilon} e^{-j\theta}$$

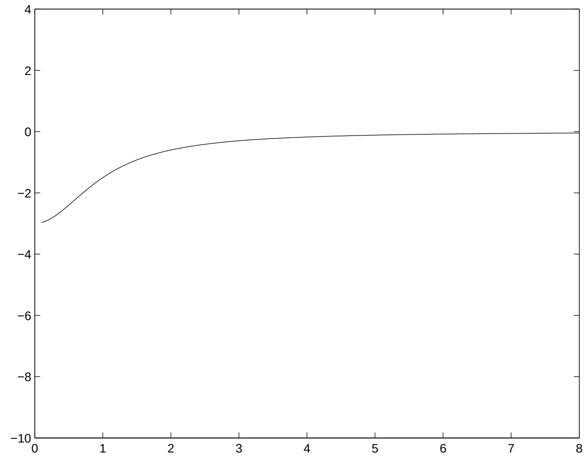

Figure 6: $\Re(PC)$ v.s. ω.

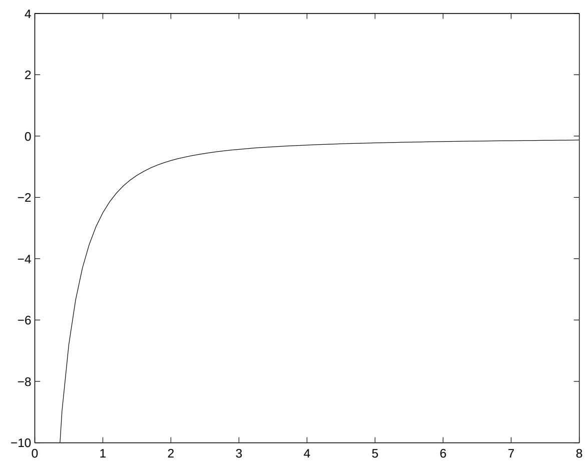

Figure 7: $\Im(PC)$ v.s. ω.

Now we can try a test point $s = \epsilon$ and we get

$$PC(\epsilon) \sim \frac{4}{\epsilon}\,.$$

This is a positive real number which tells us which way the infinite radius semicircle turns after departing from point D.

Putting all this together, the Nyquist plot is shown in Figure 8.

We extract the following data from the plot:

$$n = 0\,, \qquad e = 0\,, \qquad m = n - e = 0\,.$$

Therefore the closed loop system is stable.

(c) We have:

$$P(s)C(s) \quad = \quad \frac{20(s+3)}{s(s+1)(s+4)}$$

The Nyquist contour is shown in Figure 9.

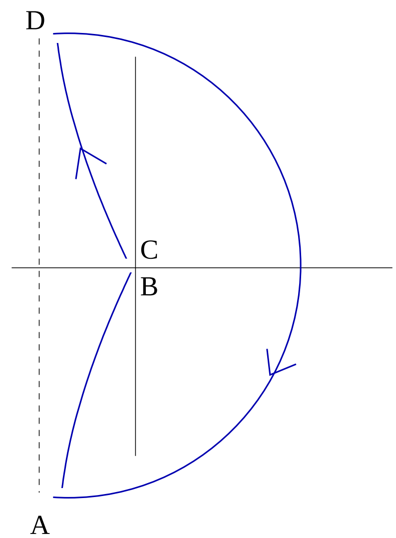

Figure 8: Nyquist plot for 1(b)

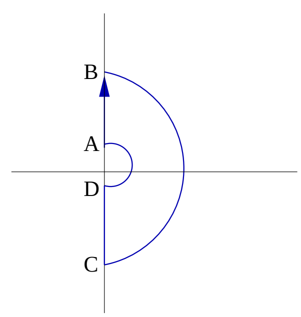

Figure 9: Nyquist contour for 1(c)

- $\overline{AB}$: Set $s = j\omega$, where $\omega : 0 \to \infty$. We have

$$\begin{aligned} PC(j\omega) &= \frac{20(j\omega+3)}{j\omega(j\omega+1)(j\omega+4)} \\ &= \frac{-20\omega^2-220}{(1+\omega^2)(16+\omega^2)} + j\frac{-40\omega^2-240}{\omega(1+\omega^2)(16+\omega^2)}. \end{aligned}$$

  Thus, we have

$$\begin{aligned} \Re(PC(j\omega)) &= \frac{-20\omega^2-220}{(1+\omega^2)(16+\omega^2)} \\ \Im(PC(j\omega)) &= \frac{-40\omega^2-240}{\omega(1+\omega^2)(16+\omega^2)}. \end{aligned}$$

  The plots of these functions for positive values of ω are shown in Figures 10-11.
- $\overline{BC}$: Set $s = Re^{j\theta}$ where $R \to \infty$ and $\theta : \frac{\pi}{2} \to -\frac{\pi}{2}$. Since PC is strictly proper, the semi-circle at ∞ will collapse to zero.
- $\overline{CD}$: Use rules about complex conjugate and the plot for $\overline{AB}$.
- $\overline{DA}$: Set $s = \epsilon e^{j\theta}$ where $\epsilon \to 0$ and $\theta : -\frac{\pi}{2} \to \frac{\pi}{2}$. This gives

$$PC(s)|_{s=\epsilon e^{j\theta}} \sim \frac{15}{\epsilon} e^{-j\theta}$$

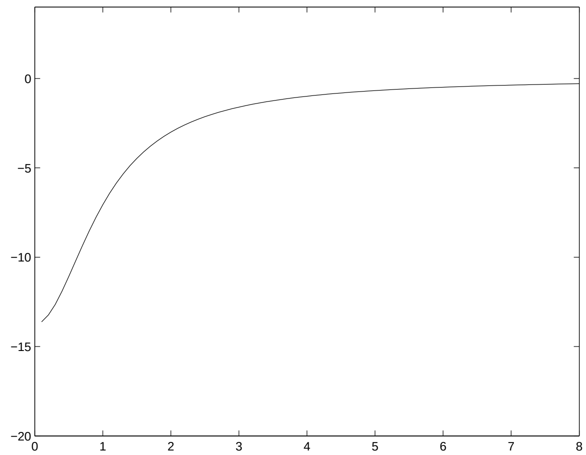

Figure 10: $\Re(PC)$ v.s. ω.

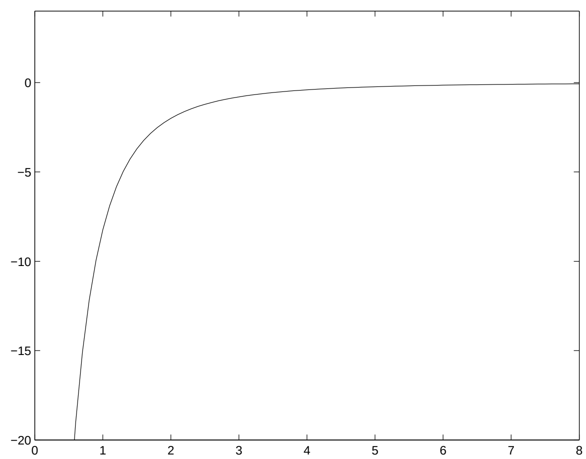

Figure 11: $\Im(PC)$ v.s. ω.

Now we can try a test point $s = \epsilon$ and we get

$$PC(\epsilon) \sim \frac{15}{\epsilon}\,.$$

This is a positive real number which tells us which way the infinite radius semicircle turns after departing from point D.

Putting all this together, the Nyquist plot is shown in Figure 12.

We extract the following data from the plot:

$$n = 0\,, \qquad e = 0\,, \qquad m = n - e = 0\,.$$

Therefore the closed loop system is stable.

(d) We have:

$$P(s)C(s) \quad = \quad \frac{100(s+5)}{s(s+3)(s^2+4)}$$

The Nyquist contour is shown in Figure 13.

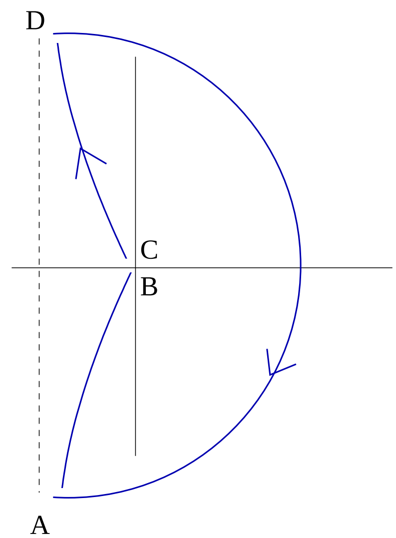

Figure 12: Nyquist plot for 1(c)

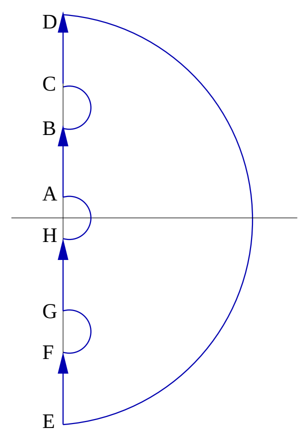

Figure 13: Nyquist contour for 1(d)

- $\overline{AB}$, $\overline{CD}$: Set $s = j\omega$, where $\omega : 0 \to \infty$. We have

$$\begin{aligned} PC(j\omega) &= \frac{100(j\omega+5)}{j\omega(j\omega+3)(-\omega^2+4)} \\ &= \frac{-200\omega}{\omega(9+\omega^2)(4-\omega^2)} + j\frac{-100(\omega^2+15)}{\omega(9+\omega^2)(4-\omega^2)}. \end{aligned}$$

  Thus, we have

$$\begin{aligned} \Re(PC(j\omega)) &= \frac{-200}{(9+\omega^2)(4-\omega^2)} \\ \Im(PC(j\omega)) &= \frac{-100(\omega^2+15)}{\omega(9+\omega^2)(4-\omega^2)}. \end{aligned}$$

  The plots of these functions for positive values of ω are shown in Figures 14-15.
- $\overline{DE}$: Set $s = Re^{j\theta}$ where $R \to \infty$ and $\theta : \frac{\pi}{2} \to -\frac{\pi}{2}$. Since PC is strictly proper, the semi-circle at ∞ will collapse to zero.

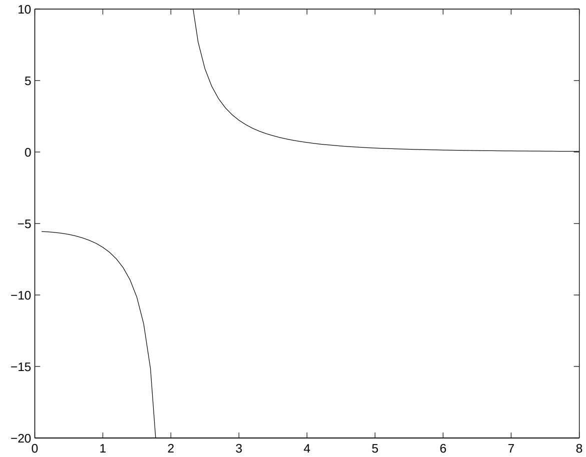

Figure 14: $\Re(PC)$ v.s. ω.

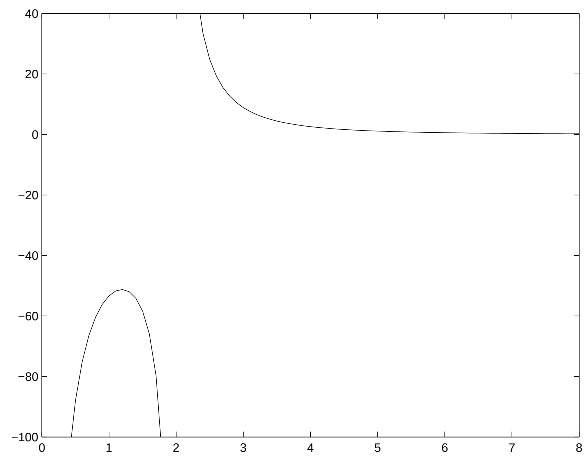

Figure 15: $\Im(PC)$ v.s. ω.

- $\overline{EF}$, $\overline{GH}$: Use rules about complex conjugate and the plot for $\overline{AB}$ and $\overline{CD}$.
- $\overline{HA}$: Set $s = \epsilon e^{j\theta}$ where $\epsilon \to 0$ and $\theta : -\frac{\pi}{2} \to \frac{\pi}{2}$. This gives

  $$PC(s)|_{s=\epsilon e^{j\theta}} \sim \frac{500}{12\epsilon} e^{-j\theta}$$

  Now we can try a test point $s = \epsilon$ and we get

  $$PC(\epsilon) \sim \frac{125}{3\epsilon}\,.$$

  This is a positive real number which tells us which way the infinite radius semicircle turns after departing from point H.

Putting all this together, the Nyquist plot is shown in Figure 16.

We extract the following data from the plot:

$$n = 0\,, \qquad e = -2\,, \qquad m = n - e = 2\,.$$

Therefore the closed loop system is not stable.

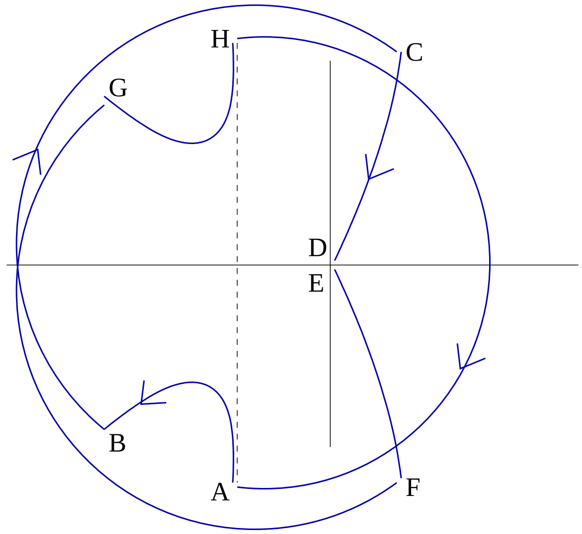

Figure 16: Nyquist plot for 1(d)

Problem 2

For this question, we must note that the Nyquist plots for $P(s)C(s)$ and $KP(s)C(s)$, where K is a scalar, are the same up to the scaling factor K. This means that the contours are the same except each point is scaled by a factor K, i.e. a point $a+bj$ on the Nyquist plot of $P(s)C(s)$ will be scaled to the point $Ka+Kbj$ for the Nyquist plot of $KP(s)C(s)$.

(a) We have:

$$P(s)C(s) \quad = \quad \frac{K}{(s+2)(s+4)(s+6)}.$$

First set $K=1$. The Nyquist contour is shown in Figure 17.

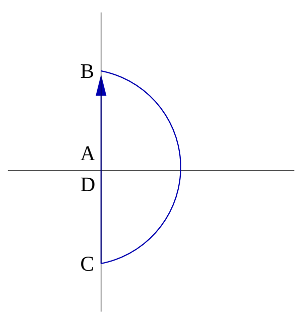

Figure 17: Nyquist contour for 2(a)

- $\overline{AB}$: Set $s=j\omega$, where $\omega : 0 \to \infty$. We have

$$\begin{aligned} PC(j\omega) &= \frac{1}{(j\omega+2)(j\omega+4)(j\omega+6)} \\ &= \frac{48-12\omega^2}{(4+\omega^2)(16+\omega^2)(36+\omega^2)} + j\frac{\omega(\omega^2-44)}{(4+\omega^2)(16+\omega^2)(36+\omega^2)}. \end{aligned}$$

Thus, we have

$$\begin{aligned}\Re(PC(j\omega)) &= \frac{48 - 12\omega^2}{(4+\omega^2)(16+\omega^2)(36+\omega^2)} \\ \Im(PC(j\omega)) &= \frac{\omega(\omega^2 - 44)}{(4+\omega^2)(16+\omega^2)(36+\omega^2)}.\end{aligned}$$

The plots of these functions for positive values of ω are shown in Figures 18-19.

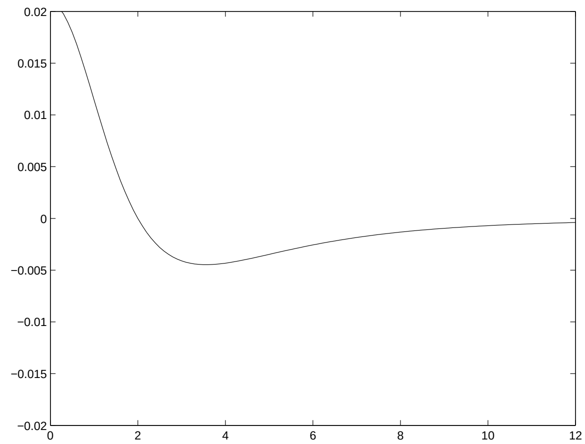

Figure 18: $\Re(PC)$ v.s. ω.

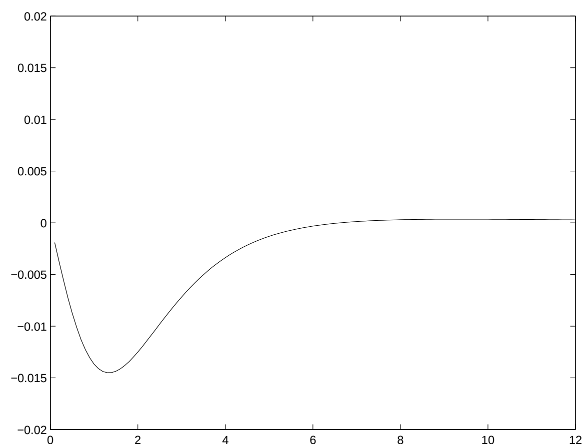

Figure 19: $\Im(PC)$ v.s. ω.

- $\overline{BC}$: Set $s = Re^{j\theta}$ where $R \to \infty$ and $\theta : \frac{\pi}{2} \to -\frac{\pi}{2}$. Since PC is strictly proper, the semi-circle at ∞ will collapse to zero.
- $\overline{CD}$: Use rules about complex conjugate and the plot for $\overline{AB}$.

Putting all this together, the Nyquist plot is shown in Figure 20. The crossing point of the plot with the positive real axis is at $\frac{1}{48}$ and with the negative real axis is at $\frac{48-12*44}{(4+44)(16+44)(36+44)} = -\frac{1}{480}$.

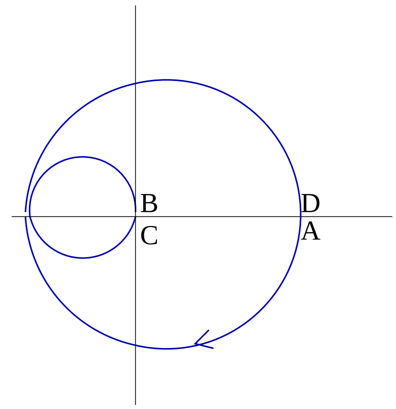

Figure 20: Nyquist plot for 2(a)

We extract the following data from the plot:

$$
\begin{array}{rlll}
-\dfrac{1}{K} < -\dfrac{1}{480}: & n = 0\,, & e = 0\,, & m = n - e = 0 \\
-\dfrac{1}{480} < -\dfrac{1}{K} < 0: & n = 0\,, & e = -2\,, & m = n - e = 2 \\
0 < -\dfrac{1}{K} < \dfrac{1}{48}: & n = 0\,, & e = -1\,, & m = n - e = 1 \\
-\dfrac{1}{K} > \dfrac{1}{48}: & n = 0\,, & e = 0\,, & m = n - e = 0\,.
\end{array}
$$

Overall the closed loop system is stable when $-48 < K < 480$.

Now let's try using the Routh criterion. The numerator of $1 + C(s)P(s)$ is $s^3 - 12s^2 + 44s + 48 + K$. We know the zeros of this polynomial are the poles of the closed loop system. The Routh array for this polynomial is:

$$
\begin{array}{ll}
1 & 44 \\
12 & 48 + K \\
\frac{48+K-528}{12} & 0 \\
48 + K & 0
\end{array}
$$

By the Routh-Hurwitz stability criterion, all the zeros of the above polynomial are in the OLHP if all the entries in the first column of the Routh array have the same sign. Looking at the array, this implies that $-48 < K < 480$, which is the same result as the Nyquist stability criterion.

(b) We have:

$$
P(s)C(s) \quad = \quad \frac{K(s^2 - 4s + 13)}{s(s+2)(s+4)}\,.
$$

First set $K = 1$. The Nyquist contour is shown in Figure 21.

- $\overline{AB}$: Set $s = j\omega$, where $\omega : 0 \to \infty$. We have

$$
PC(j\omega) \quad = \quad \frac{10(\omega^2 - 11)}{(4+\omega^2)(16+\omega^2)} + j\frac{-\omega^4 + 45\omega^2 - 104}{\omega(4+\omega^2)(16+\omega^2)}\,.
$$

Thus, we have

$$
\begin{aligned}
\Re(PC(j\omega)) &= \frac{10(\omega^2 - 11)}{(4+\omega^2)(16+\omega^2)} \\
\Im(PC(j\omega)) &= \frac{-\omega^4 + 45\omega^2 - 104}{\omega(4+\omega^2)(16+\omega^2)}\,.
\end{aligned}
$$

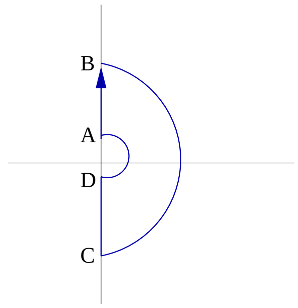

Figure 21: Nyquist contour for 2(b)

The plots of these functions for positive values of ω are shown in Figures 22-23. These graphs have the feature that $\Re(PC(j\omega)) = 0$ at $\omega = 3.3$ and $\Im(PC(j\omega)) = 0$ at $\omega = 1.56$ and $\omega = 6.5$. Also, at $\omega = 1.56$, $\Re(PC(j\omega)) = -.7223$ and at $\omega = 6.5$, $\Re(PC(j\omega)) = 0.116$. Finally, at $\omega = 0$, $\Re(PC(j\omega)) = -\frac{110}{64}$.

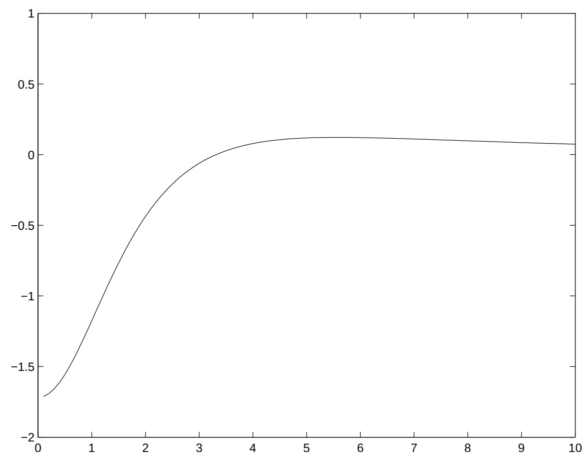

Figure 22: $\Re(PC)$ v.s. ω.

- $\overline{BC}$: Set $s = Re^{j\theta}$ where $R \to \infty$ and $\theta : \frac{\pi}{2} \to -\frac{\pi}{2}$. Since PC is strictly proper, the semi-circle at ∞ will collapse to zero.
- $\overline{CD}$: Use rules about complex conjugate and the plot for $\overline{AB}$.
- $\overline{DA}$: Set $s = \epsilon e^{j\theta}$ where $\epsilon \to 0$ and $\theta : -\frac{\pi}{2} \to \frac{\pi}{2}$. This gives

  $$PC(s)|_{s=\epsilon e^{j\theta}} \sim \frac{13}{8\epsilon} e^{-j\theta}$$

  Now we can try a test point $s = \epsilon$ and we get

  $$PC(\epsilon) \sim \frac{13}{8\epsilon}.$$

  This is a positive real number which tells us which way the infinite radius semicircle turns after departing from point D.

Putting all this together, the Nyquist plot is shown in Figure 24. The crossing point of the plot with the positive real axis is at 0.116 and with the negative real axis is at $-.7223$.

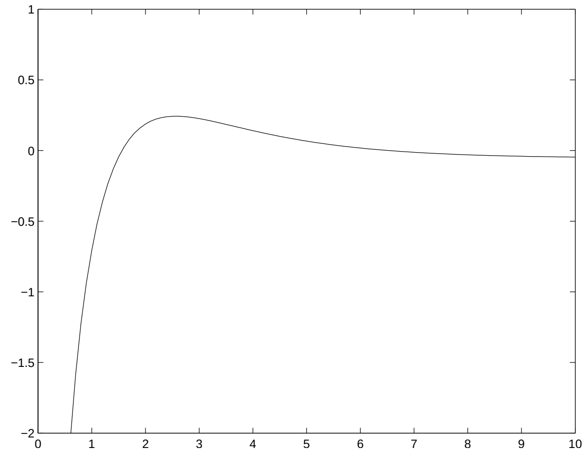

Figure 23: $\Im(PC)$ v.s. ω.

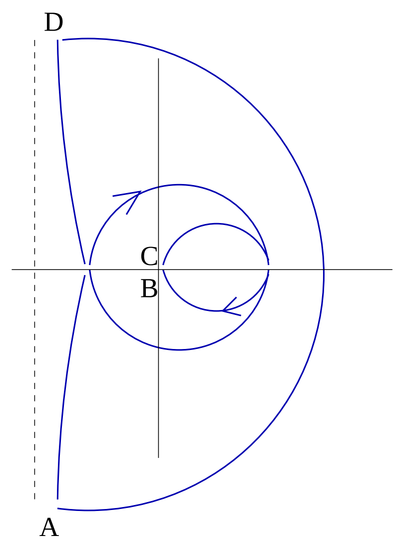

Figure 24: Nyquist plot for 2(b)

We extract the following data from the plot:

$$\begin{aligned}
-\frac{1}{K} < -.7223: &\qquad n=0\,, \quad e=0\,, \quad m=n-e=0\\
-.7223 < -\frac{1}{K} < 0: &\qquad n=0\,, \quad e=-2\,, \quad m=n-e=2\\
0 < -\frac{1}{K} < 0.116: &\qquad n=0\,, \quad e=-3\,, \quad m=n-e=3\\
-\frac{1}{K} > 0.116: &\qquad n=0\,, \quad e=-1\,, \quad m=n-e=1\,.
\end{aligned}$$

Overall the closed loop system is stable when $0 < K < 1.384$.

Now let's try using the Routh criterion. The numerator of $1+C(s)P(s)$ is $s^3+(6+K)s^2+(8-4K)s13K$.

The Routh array for this polynomial is:

$$\begin{array}{ll} 1 & 8-4k \\ 6+K & 13K \\ \frac{(6+K)(8-4K)-13K}{6+K} & 0 \\ 13K & 0 \end{array}$$

We need all of the entries in the first column to be positive. This implies that $K > 0$ by the fouth entry. The third entry being positive is equivalent to $-4K^2 - 29K + 48 > 0$. The roots of this polynomial are -8.64 and 1.39. We find that $-4K^2 - 29K + 48 > 0$ when $-8.64 < K < 1.39$. So, the closed loop system is stable when $0 < K < 1.39$, which is the same result as the Nyquist stability criterion.

(c) We have:

$$P(s)C(s) \quad = \quad \frac{K(s-1)(s-2)}{(s+1)(s+2)}.$$

First set $K = 1$. The Nyquist contour is shown in Figure 25.

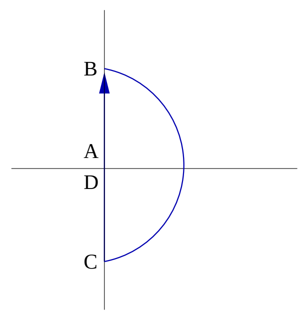

Figure 25: Nyquist contour for 2(c)

- $\overline{AB}$: Set $s = j\omega$, where $\omega : 0 \to \infty$. We have

$$PC(j\omega) \quad = \quad \frac{(\omega^4 - 13\omega^2 + 4)}{(1+\omega^2)(4+\omega^2)} + j\frac{6\omega(\omega^2-2)}{(1+\omega^2)(4+\omega^2)}.$$

  Thus, we have

$$\begin{aligned} \Re(PC(j\omega)) &= \frac{(\omega^4 - 13\omega^2 + 4)}{(1+\omega^2)(4+\omega^2)} \\ \Im(PC(j\omega)) &= \frac{6\omega(\omega^2-2)}{(1+\omega^2)(4+\omega^2)}. \end{aligned}$$

  The plots of these functions for positive values of ω are shown in Figures 26-27. We note that at $\omega = \sqrt{2}$, $\Re(PC(j\omega)) = -1$.
- $\overline{BC}$: Set $s = Re^{j\theta}$ where $R \to \infty$ and $\theta : \frac{\pi}{2} \to -\frac{\pi}{2}$. The semi-circle at ∞ collapses to the point $(1, 0)$.
- $\overline{CD}$: Use rules about complex conjugate and the plot for $\overline{AB}$.

Putting all this together, the Nyquist plot is shown in Figure 28. The crossing point of the plot with the positive real axis is at 1 and with the negative real axis is at -1.

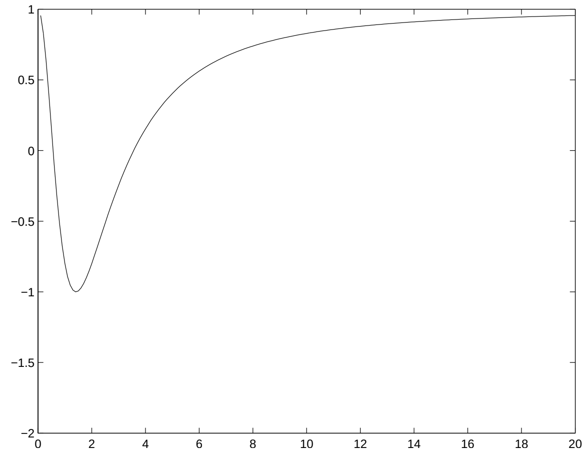

Figure 26: $\Re(PC)$ v.s. ω.

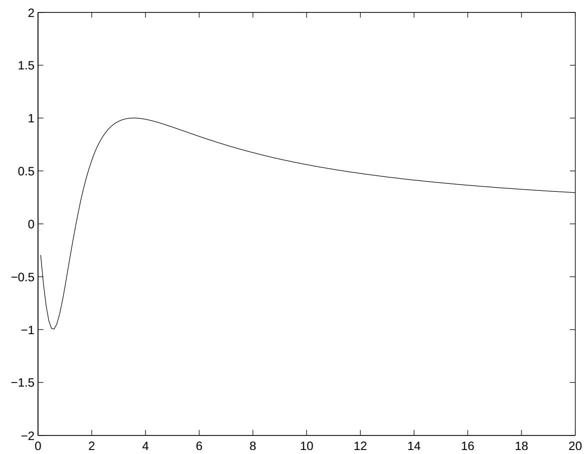

Figure 27: $\Im(PC)$ v.s. ω.

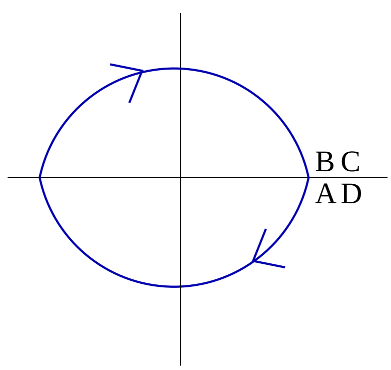

Figure 28: Nyquist plot for 2(c)

We extract the following data from the plot:

$$\begin{aligned}
-\frac{1}{K} < -1: &\qquad n=0\,, \quad e=0\,, \quad m=n-e=0 \\
-1 < -\frac{1}{K} < 1: &\qquad n=0\,, \quad e=-2\,, \quad m=n-e=2 \\
-\frac{1}{K} > 1: &\qquad n=0\,, \quad e=0\,, \quad m=n-e=0\,.
\end{aligned}$$

Overall the closed loop system is stable when $-1 < K < 1$.

Now let's try using the Routh criterion. The numerator of $1+C(s)P(s)$ is $(1+K)s^2+(3-3K)s+(2+2K)$. The Routh array for this polynomial is:

$$\begin{array}{ll} 1+K & 2+2K \\ 3-3K & 0 \\ 2+2K & 0 \end{array}$$

We need all of the entries in the first column to have the same sign. To get all the entries positive, we need $-1 < K < 1$. To get all the entries negative, we need $K < -1$ and $K > 1$, which has no solution. So, the closed loop system is stable when $-1 < K < 1$, which is the same result as the Nyquist stability criterion.