ECE356S Linear Systems & Control

Bruce Francis

Course notes, Version 2.0, January 2008

Preface

This is the first Engineering Science course on control. It may be your one and only course on control, and therefore the aim is to give both some breadth and some depth. Mathematical models of mostly mechanical systems is an important component. There's also an introduction to state models and state-space theory. Finally, there's an introduction to design in the frequency domain.

The sequels to this course are ECE557 Systems Control, which develops the state-space method in continuous time, and ECE411 Real-Time Computer Control, which treats digital control via both state-space and frequency-domain methods.

There are several computer applications for solving numerical problems in this course. The most widely used is MATLAB, but it's expensive. I like Scilab, which is free. Others are Mathematica (expensive) and Octave (free).

Contents

1 Introduction **1**
1.1 Problems 1

2 Mathematical Models of Systems **3**
2.1 Block Diagrams 3
2.2 State Models 5
2.3 Linearization 15
2.4 Simulation 18
2.5 The Laplace Transform 19
2.6 Transfer Functions 27
2.7 Interconnections 30
2.8 Problems 32

3 Linear System Theory **35**
3.1 Initial-State-Response 35
3.2 Input-Response 38
3.3 Total Response 40
3.4 Logic Notation 41
3.5 Lyapunov Stability 44
3.6 BIBO Stability 48
3.7 Frequency Response 52
3.8 Problems 54

4 Feedback Control Theory **57**
4.1 Closing the Loop 57
4.2 The Internal Model Principle 65
4.3 Principle of the Argument 67
4.4 Nyquist Stability Criterion (1932) 69
4.5 Examples 70
4.6 Stability and Bode Plots 71
4.7 Problems 89

5 Introduction to Control Design **95**
5.1 Loopshaping 95
5.2 Lag Compensation 97
5.3 Lead Compensation 102

5.4 Loopshaping Theory 106
5.4.1 Bode's phase formula 106
5.4.2 The waterbed effect 108
5.5 Problems 109

Chapter 1

Introduction

A familiar example of a control system is the one we all have that regulates our internal body temperature at 37° C. As we move from a warm room to the cold outdoors, our body temperature is maintained. Other familiar examples of control systems:

- autofocus mechanism in cameras
- cruise control system in cars
- anti-lock brake system (ABS) and other traction control systems in cars
- thermostat temperature control systems.

More widely, control systems are in every manufacturing process. One very interesting application domain is vehicle motion control, and maybe the best example of this is helicopter flight control.

This course is about the modeling of systems and the analysis and design of control systems.

1.1 Problems

1. List three examples of control systems not mentioned in class. Make your list as diverse as possible.

2. Go to

   http://www.modelaviation.co.uk/heli/models/hoverfly/hoverfly.htm

   Imagine yourself flying the model helicopter shown. Draw the block diagram with you in the loop.

3. Imagine a wooden box of cube shape that can balance itself on one of its edges; while it's balanced on the edge, if you tapped it lightly it would right itself. Think of a mechatronic system to put inside the box to do this stabilization. Draw a schematic diagram of your mechatronic system.

4. Historically, control systems go back at least to ancient Greek times. More recently, in 1769 a feedback control system was invented by James Watt: the flyball governor for speed control of a steam engine. Sketch the flyball governor and explain in a few sentences how it works.

[References: Many books on control; also, many interesting websites, for example,

http://www.uh.edu/engines/powersir.htm

Stability of this control system was studied by James Clerk Maxwell, whose equations you know and love.]

5. The topic of vehicle formation is of current research interest. See for example

   http://www.path.berkeley.edu

   This is a very simple instance of such a problem.

   Consider two cars that can move in a horizontal straight line. Let car #1 be the leader and car #2 the follower. The goal is for car #1 to follow a reference speed and for car #2 to maintain a specified distance behind. Discuss how this might be done. (We'll study this example more later.)

Chapter 2

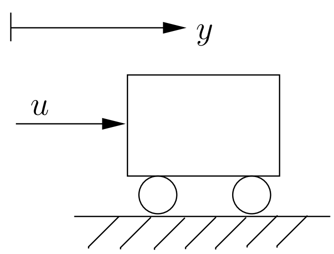
Mathematical Models of Systems

The best (only?) way to learn this subject is bottom up—from specific examples to general theory. So we begin with mathematical models of physical systems. We mostly use mechanical examples since their behaviour is easier to understand than, say, electromagnetic systems, because of our experience in the natural world. We have an intuitive understanding of Newtonian mechanics from having played volleyball, skateboarded, etc.

2.1 Block Diagrams

Block diagrams are fundamental in control analysis and design. We can see how signals affect each other.

Example 2.1.1 The simplest vehicle to control is a cart on wheels:

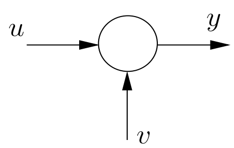


This is a **schematic diagram**, not a block diagram. Assume the cart can move only in a straight line on a flat surface. (There may be air resistance to the motion and other friction effects.) Assume a force u is applied to the cart and let y denote the position of the cart measured from a stationary reference position. Then u and y are functions of time t and we could indicate this by $u(t)$ and $y(t)$. But we are careful to distinguish between u, a function, and the value $u(t)$ of this function at time t. We regard the functions u and y as **signals**.

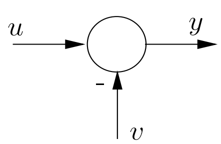
Newton's second law tells us that there's a mathematical relationship between u and y, namely, $u = M\ddot{y}$. We take the viewpoint that the force can be applied independently of anything else, that is, it's an input. Then y is an output. We represent this graphically by a **block diagram**:

So a block diagram has arrows representing signals and boxes representing system components; the boxes represent functions that map inputs to outputs. Suppose the cart starts at rest at the origin at time 0, i.e., $y(0) = \dot{y}(0) = 0$. Then the position depends only on the force applied. However y at time t depends on u not just at time t, but on past times as well. So we can write $y = F(u)$, i.e., y is a function of u, but we can't write $y(t) = F(u(t))$. $\square$

Block diagrams also may have **summing junctions**:

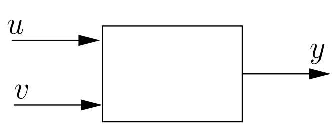


stands for $y = u + v$



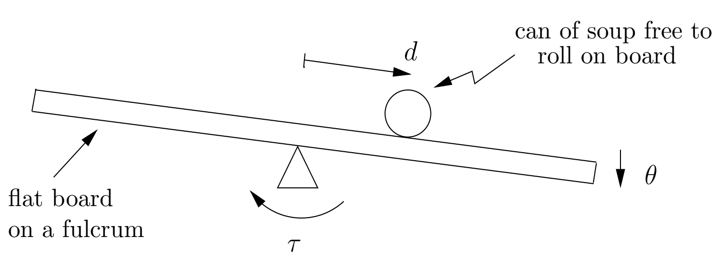
stands for $y = u - v$

Also, we may need to allow a block to have more than one input:

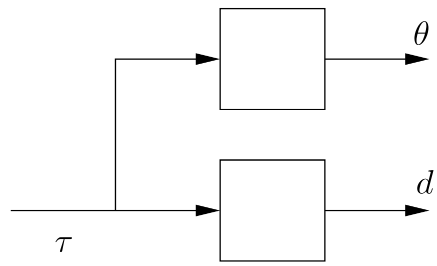


This means that y is a function of u and v, $y = F(u, v)$.

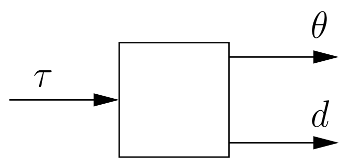
Example 2.1.2



Suppose a torque τ is applied to the board. Let θ denote the angle of tilt and d the distance of roll. Then both θ and d are functions of τ. The block diagram could be



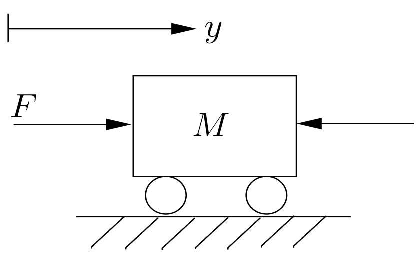
or



2.2 State Models

In control engineering, the system to be controlled is termed the **plant**. For example, in helicopter flight control, the plant is the helicopter itself plus its sensors and actuators. The control system is implemented in an onboard computer. The design of a flight control system for a helicopter requires first the development of a mathematical model of the helicopter dynamics. This is a very advanced subject, well beyond the scope of this course. We must content ourselves with much simpler plants.

Example 2.2.1 Consider a cart on wheels, driven by a force F and subject to air resistance:



Typically air resistance creates a force depending on the velocity $\dot{y}$; let's say this force is a possibly nonlinear function $D(\dot{y})$. Assuming M is constant, Newton's second law gives

$$M\ddot{y} = F - D(\dot{y}).$$

We are going to put this in a standard form. Define what are called state variables:

$$x_1 := y, \quad x_2 := \dot{y}.$$

Then

$$\begin{aligned}
\dot{x}_1 &= x_2\\
\dot{x}_2 &= \frac{1}{M}F - \frac{1}{M}D(x_2)\\
y &= x_1.
\end{aligned}$$

These equations have the form

$$\dot{x} = f(x,u), \quad y = h(x) \tag{2.1}$$

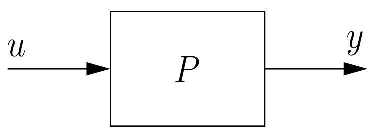
where

$$x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ u := F$$

$$f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2, \ f(x_1, x_2, u) = \begin{bmatrix} x_2 \\ \\ \frac{1}{M}u - \frac{1}{M}D(x_2) \end{bmatrix}$$

$$h : \mathbb{R}^2 \to \mathbb{R}, \ h(x_1, x_2) = x_1.$$

The function f is nonlinear if D is; h is linear. Equation (2.1) constitutes a state model of the system, and x is called the **state** or state vector. The block diagram is



Here P is a possibly nonlinear system, u (applied force) is the input, y (cart position) is the output, and

$$x = \begin{bmatrix} \text{cart pos'n} \\ \text{cart velocity} \end{bmatrix}$$

is the state of P. (We'll define state later.)

As a special case, suppose the air resistance is a linear function of velocity:

$$D(x_2) = D_0 x_2, \ D_0 \text{ a constant.}$$

Then f is linear:

$$f(x,u) = Ax + Bu, \quad A := \begin{bmatrix} 0 & 1 \\ 0 & -D_0/M \end{bmatrix}, \ B := \begin{bmatrix} 0 \\ 1/M \end{bmatrix}.$$

Defining $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$, we get the state model

$$\dot{x} = Ax + Bu, \quad y = Cx. \tag{2.2}$$

This model is of a linear, time-invariant (LTI) system. □

It is convenient to write vectors sometimes as column vectors and sometimes as n-tuples, i.e., ordered lists. For example

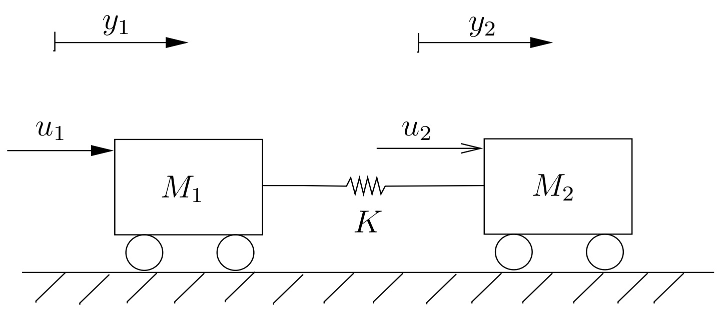
$$x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad x = (x_1, x_2).$$

We shall use both.

Generalizing the example, we can say that an important class of models is

$$\begin{aligned} \dot{x} &= f(x,u),\ f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \\ y &= h(x,u),\ h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p. \end{aligned}$$

This model is nonlinear, time-invariant. The input u has dimension m, the output y dimension p, and the state x dimension n. An example where $m = 2, p = 2, n = 4$ is



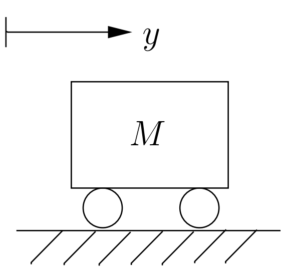
$$u = (u_1, u_2),\ y = (y_1, y_2),\ x = (y_1, \dot{y}_1, y_2, \dot{y}_2).$$

The LTI special case is

$$\begin{aligned} \dot{x} &= Ax + Bu,\ A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m} \\ y &= Cx + Du,\ C \in \mathbb{R}^{p\times n},\ D \in \mathbb{R}^{p\times m}. \end{aligned}$$

Now we turn to the concept of the **state of a system**. Roughly speaking, $x(t_0)$ encapsulates all the system dynamics up to time t_0, that is, no additional prior information is required. More precisely, the concept is this: For any t_0 and t_1, with $t_0 < t_1$, knowing $x(t_0)$ and knowing $\{u(t) : t_0 \leq t \leq t_1\}$, we can compute $x(t_1)$, and hence $y(t_1)$.

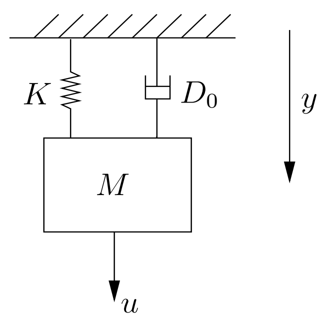
Example 2.2.2



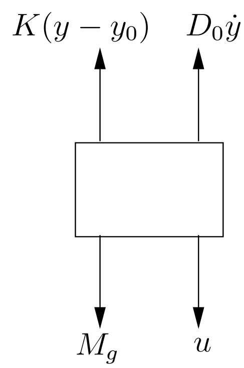
no force; no air resistance

If we were to try simply $x = y$, then knowing $x(t_0)$ without $\dot{y}(t_0)$, we could not solve the initial value problem for the future cart position. Similarly $x = \dot{y}$ won't work. Since the equation of motion, $M\ddot{y} = 0$, is second order, we need two initial conditions at $t = t_0$, implying we need a 2-dimensional state vector. In general for mechanical systems it is customary to take x to consist of positions and velocities of all masses. □

Example 2.2.3 mass-spring-damper



free-body :



$$M\ddot{y} = u + Mg - K(y - y_0) - D_0\dot{y}$$

$$\text{state} \quad x = (x_1, x_2), \quad x_1 = y, \; x_2 = \dot{y}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{M}u + g - \frac{K}{M}x_1 + \frac{K}{M}y_0 - \frac{D_0}{M}x_2$$

$$y = x_1$$

This has the form

$$\dot{x} = Ax + Bu + c$$

$$y = Cx$$

where

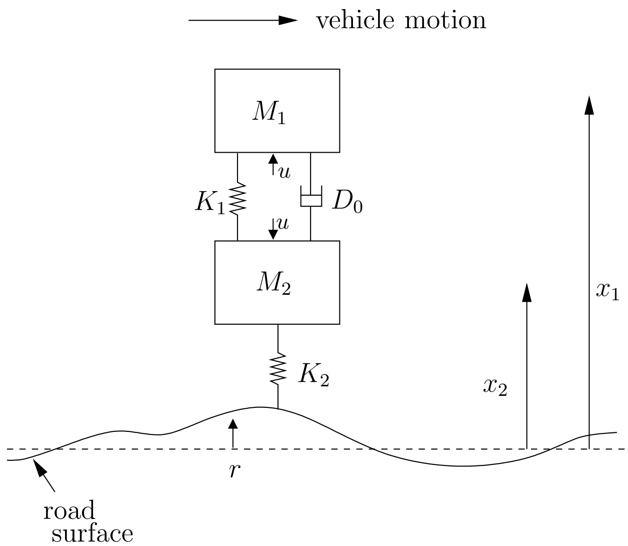
$$A = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{D_0}{M} \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}, \; c = \begin{bmatrix} 0 \\ g + \frac{K}{M}y_0 \end{bmatrix} = \text{const. vector}$$

$$C = [1 \;\; 0]\,.$$

The constant vector c is known, and hence is taken as part of the system rather than as a signal. □

Example 2.2.4 Active suspension

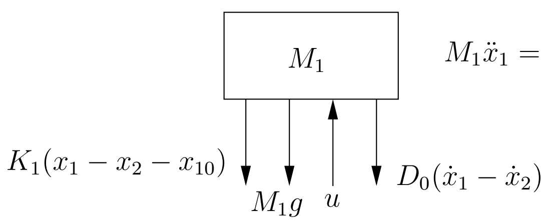
This example concerns active suspension of a vehicle for passenger comfort.



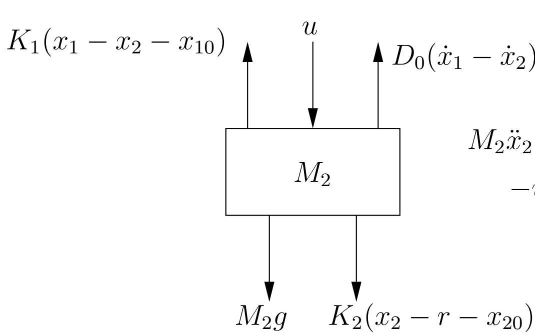
M_1 = mass of chassis & passengers

M_2 = mass of wheel carriage

To derive the equations of motion, bring in free body diagrams:



$$M_1 \ddot{x}_1 = u - K_1(x_1 - x_2 - x_{10}) - M_1 g - D_0(\dot{x}_1 - \dot{x}_2)$$



$$\begin{aligned} M_2 \ddot{x}_2 &= K_1(x_1 - x_2 - x_{10}) + D_0(\dot{x}_1 - \dot{x}_2) \\ &\quad -u - M_2 g - K_2(x_2 - r - x_{20}) \end{aligned}$$

Define $x_3 = \dot{x}_1, x_4 = \dot{x}_2$. Then the equations can be assembled as

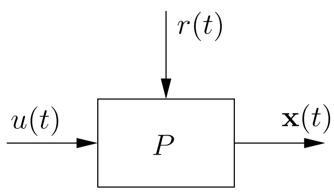
$$\dot{x} = Ax + B_1 u + B_2 r + c_1 \qquad (2.3)$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, c_1 = \begin{bmatrix} 0 \\ 0 \\ \frac{K_1}{M_1}x_{10} - g \\ -\frac{K_1}{M_2}x_{20} - g + \frac{K_2}{M_2}x_{20} \end{bmatrix} = \text{constant vector}$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{K_1}{M_1} & \frac{K_1}{M_1} & -\frac{D_0}{M_1} & \frac{D_0}{M_1} \\ \frac{K_1}{M_2} & -\frac{K_1+K_2}{M_2} & \frac{D_0}{M_2} & -\frac{D_0}{M_2} \end{bmatrix}, \ B_1 = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M_1} \\ -\frac{1}{M_2} \end{bmatrix}, \ B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{K_2}{M_2} \end{bmatrix}.$$

We can regard (2.3) as corresponding to the block diagram



Since c_1 is a known constant vector, it's not taken to be a signal. Here $u(t)$ is the controlled input and $r(t)$ the uncontrolled input or disturbance.

The output to be controlled might be acceleration or jerk of the chassis. Taking $y = \ddot{x}_1 = \dot{x}_3$ gives

$$y = Cx + Du + c_2 \tag{2.4}$$

where

$$C = \begin{bmatrix} -\frac{K_1}{M_1} & \frac{K_1}{M_1} & -\frac{D_0}{M_1} & \frac{D_0}{M_1} \end{bmatrix}, \ D = \frac{1}{M_1}, \ c_2 = \frac{K_1}{M_1}x_{10} - g.$$

Equations (2.3) and (2.4) have the form

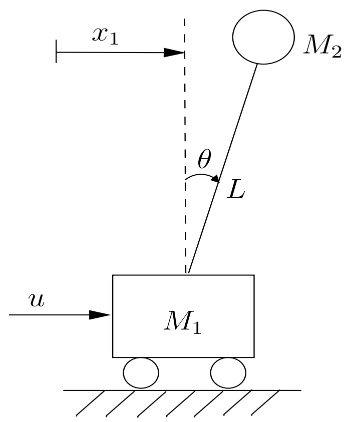
$$\dot{x} = f(x, u, r)$$

$$y = h(x, u).$$

Notice that f and h are **not** linear, because of the constants c_1, c_2. □

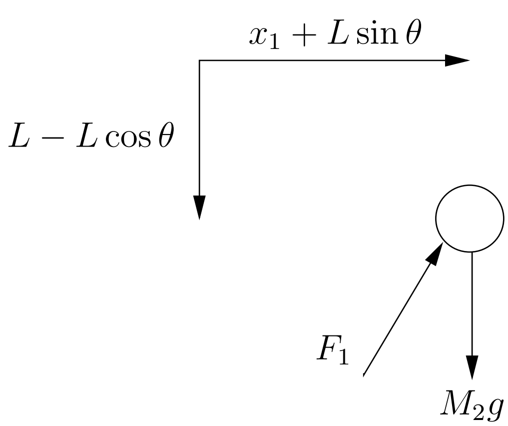
Example 2.2.5 Cart-pendulum

A favourite toy control problem is to get a cart to automatically balance a pendulum.



$$x = (x_1, x_2, x_3, x_4) = (x_1, \theta, \dot{x}_1, \dot{\theta})$$

Again, we bring in free body diagrams:



$$M_2 \frac{d^2}{dt^2}(x_1 + L\sin\theta) = F_1 \sin\theta$$

$$M_2 \frac{d^2}{dt^2}(L - L\cos\theta) = M_2 g - F_1 \cos\theta$$

$$M_1 \ddot{x}_1 = u - F_1 \sin\theta.$$

These are three equations in the four signals x_1, θ, u, F_1. Use

$$\frac{d^2}{dt^2}\sin\theta = \ddot{\theta}\cos\theta - \dot{\theta}^2 \sin\theta, \quad \frac{d^2}{dt^2}\cos\theta = -\ddot{\theta}\sin\theta - \dot{\theta}^2\cos\theta$$

to get

$$M_2 \ddot{x}_1 + M_2 L \ddot{\theta}\cos\theta - M_2 L \dot{\theta}^2 \sin\theta = F_1 \sin\theta$$

$$M_2L\ddot{\theta}\sin\theta + M_2L\dot{\theta}^2\cos\theta = M_2g - F_1\cos\theta$$

$$M_1\ddot{x}_1 = u - F_1\sin\theta.$$

We can eliminate F_1: Add the first and the third to get

$$(M_1 + M_2)\ddot{x}_1 + M_2L\ddot{\theta}\cos\theta - M_2L\dot{\theta}^2\sin\theta = u;$$

multiply the first by $\cos\theta$, the second by $\sin\theta$, add, and cancel M_2 to get

$$\ddot{x}_1\cos\theta + L\ddot{\theta} - g\sin\theta = 0.$$

Solve the latter two equations for $\ddot{x}_1$ and $\ddot{\theta}$:

$$\begin{bmatrix} M_1 + M_2 & M_2L\cos\theta \\ \cos\theta & L \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} u + M_2L\dot{\theta}^2\sin\theta \\ g\sin\theta \end{bmatrix}.$$

Thus

$$\ddot{x}_1 = \frac{u + M_2L\dot{\theta}^2\sin\theta - M_2g\sin\theta\cos\theta}{M_1 + M_2\sin^2\theta}$$

$$\ddot{\theta} = \frac{-u\cos\theta - M_2L\dot{\theta}^2\sin\theta\cos\theta + (M_1 + M_2)g\sin\theta}{L(M_1 + M_2\sin^2\theta)}.$$

In terms of state variables we have

$$\begin{aligned}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= \frac{u + M_2Lx_4^2\sin x_2 - M_2g\sin x_2\cos x_2}{M_1 + M_2\sin^2 x_2} \\
\dot{x}_4 &= \frac{-u\cos x_2 - M_2Lx_4^2\sin x_2\cos x_2 + (M_1 + M_2)g\sin x_2}{L(M_1 + M_2\sin^2 x_2)}.
\end{aligned}$$

Again, these have the form

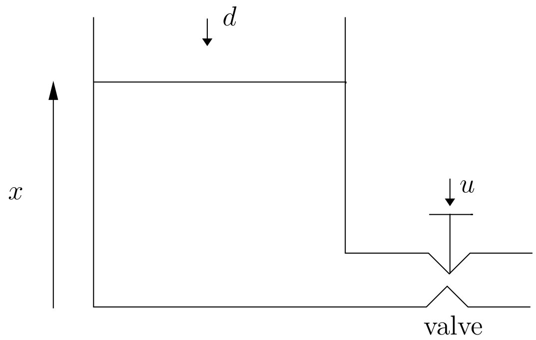
$$\dot{x} = f(x, u).$$

We might take the output to be

$$y = \begin{bmatrix} x_1 \\ \theta \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = h(x).$$

The system is highly nonlinear, though, as you would expect, it can be approximated by a linear system for $|\theta|$ small enough, say $< 5°$. □

Example 2.2.6 Level control



Let A = cross-sectional area of tank, assumed constant. Then conservation of mass:

$$A\dot{x} = d - (\text{flow rate out}).$$

Also

$$(\text{flow rate out}) = (\text{const}) \times \sqrt{\Delta \text{p}} \times (\text{area of valve opening})$$

where

$$\begin{aligned} \Delta p &= \text{pressure drop across valve} \\ &= (\text{const}) \times x. \end{aligned}$$

Thus

$$(\text{flow rate out}) = c\sqrt{x}\ u$$

and hence

$$A\dot{x} = d - c\sqrt{x}\ u.$$

The state equation is therefore

$$\dot{x} = f(x, u, d) = \frac{1}{A}d - \frac{c}{A}\sqrt{x}\ u.$$

$\square$

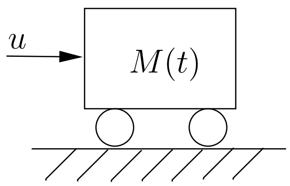
It is worth noting that not all systems have state models of the form

$$\dot{x} = f(x, u), \quad y = h(x, u).$$

Examples:

1. Differentiator $y = \dot{u}$
2. Time delay $y(t) = u(t - 1)$

3. Time-varying system



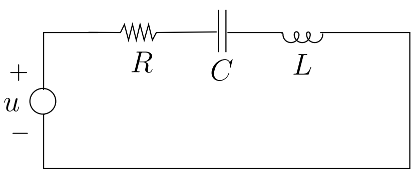
M a function of time (e.g. burning fuel)

4. PDE model, e.g. vibrating violin string with input the bow force.

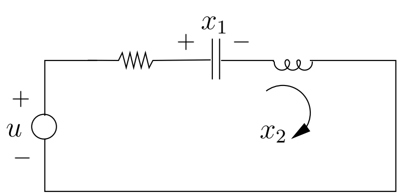
Finally, let us see how to get a state model for an electric circuit.

Example 2.2.7

An RLC circuit.



There are two energy storage elements (L, C). It is natural to take the state variables to be voltage drop across C and current through L:



Then KVL gives

$$-u + Rx_2 + x_1 + L\dot{x}_2 = 0$$

and the capacitor equation is

$$x_2 = C\dot{x}_1.$$

Thus

$$\dot{x} = Ax + Bu$$
$$A = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}.$$

□

2.3 Linearization

So far we have seen that many systems can be modeled by nonlinear state equations of the form

$$\dot{x} = f(x, u), \quad y = h(x, u).$$

(There might be disturbance inputs present, but for now we suppose they are lumped into u.) There are techniques for controlling nonlinear systems, but that's an advanced subject. However, many systems can be linearized about an equilibrium point. In this section we see how to do this. The idea is to use Taylor series.

Example 2.3.1

Let's linearize the function $y = f(x) = x^3$ about the point $x_0 = 1$. The Taylor series expansion is

$$\begin{aligned} f(x) &= \sum_{0}^{\infty} c_n (x - x_0)^n, \; c_n = \frac{f^{(n)}(x_0)}{n!} \\ &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \ldots \, . \end{aligned}$$

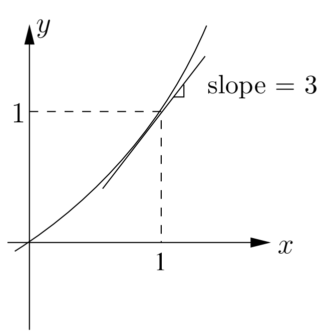
Taking only terms $n = 0, 1$ gives

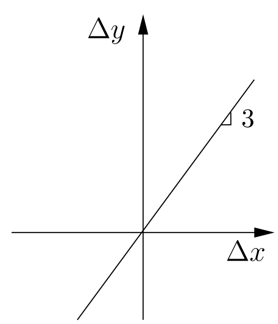
$$f(x) \approx f(x_0) + f'(x_0)(x - x_0),$$

that is

$$y - y_0 \approx f'(x_0)(x - x_0).$$

Defining $\Delta y = y - y_0, \; \Delta x = x - x_0$, we have the linearized function $\Delta y = f'(x_0)\Delta x$, or $\Delta y = 3\Delta x$ in this case.





Obviously, this approximation gets better and better as $|\Delta x|$ gets smaller and smaller. $\square$

Taylor series extend to functions $f : \mathbb{R}^n \to \mathbb{R}^m$.

Example 2.3.2

$$f : \mathbb{R}^3 \to \mathbb{R}^2, \ f(x_1, x_2, x_3) = (x_1 x_2 - 1, x_3^2 - 2x_1 x_3)$$

Suppose we want to linearize f at the point $x_0 = (1, -1, 2)$. Terms $n = 0, 1$ in the expansion are

$$f(x) \approx f(x_0) + \frac{\partial f}{\partial x}(x_0)(x - x_0),$$

where

$$\begin{aligned}
\frac{\partial f}{\partial x}(x_0) &= \text{Jacobian of } f \text{ at } x_0 \\
&= \left(\frac{\partial f_i}{\partial x_j}(x_0) \right) \\
&= \left[\begin{array}{ccc} x_2 & x_1 & 0 \\ -2x_3 & 0 & 2x_3 - 2x_1 \end{array} \right] \Bigg|_{x_0} \\
&= \left[\begin{array}{ccc} -1 & 1 & 0 \\ -4 & 0 & 2 \end{array} \right].
\end{aligned}$$

Thus the linearization of $y = f(x)$ at x_0 is $\Delta y = A \Delta x$, where

$$\begin{aligned}
A &= \frac{\partial f}{\partial x}(x_0) = \left[\begin{array}{ccc} -1 & 1 & 0 \\ -4 & 0 & 2 \end{array} \right] \\
\Delta y &= y - y_0 = f(x) - f(x_0) \\
\Delta x &= x - x_0.
\end{aligned}$$

$\square$

By direct extension, if $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, then

$$f(x, u) \approx f(x_0, u_0) + \frac{\partial f}{\partial x}(x_0, u_0) \Delta x + \frac{\partial f}{\partial u}(x_0, u_0) \Delta u.$$

Now we turn to linearizing the differential equation

$$\dot{x} = f(x, u).$$

First, **assume** there is an equilibrium point, that is, a constant solution $x(t) \equiv x_0, u(t) \equiv u_0$. This is equivalent to saying that $0 = f(x_0, u_0)$. Now consider a nearby solution:

$$x(t) = x_0 + \Delta x(t), u(t) = u_0 + \Delta u(t), \quad \Delta x(t), \Delta u(t) \text{ small}.$$

We have

$$\begin{aligned}
\dot{x}(t) &= f[x(t), u(t)] \\
&= f(x_0, u_0) + A \Delta x(t) + B \Delta u(t) + \text{higher order terms}
\end{aligned}$$

where

$$A := \frac{\partial f}{\partial x}(x_0, u_0), \ B := \frac{\partial f}{\partial u}(x_0, u_0).$$

Since $\dot{x} = \Delta\dot{x}$ and $f(x_0, u_0) = 0$, we have the linearized equation to be

$$\dot{\Delta x} = A\Delta x + B\Delta u.$$

Similarly, the output equation $y = h(x, u)$ linearizes to

$$\Delta y = C\Delta x + D\Delta u,$$

where

$$C = \frac{\partial h}{\partial x}(x_0, u_0), \;\; D = \frac{\partial h}{\partial u}(x_0, u_0).$$

Summary

Linearizing $\dot{x} = f(x, u), \; y = h(x, u)$: Select, if one exists, an equilibrium point. Compute the four Jacobians, A, B, C, D, of f and h at the equilibrium point. Then the linearized system is

$$\dot{\Delta x} = A\Delta x + B\Delta u, \;\; \Delta y = C\Delta x + D\Delta u.$$

Under mild conditions (sufficient smoothness of f and h), this linearized system is a valid approximation of the nonlinear one in a sufficiently small neighbourhood of the equilibrium point.

Example 2.3.3

$$\begin{aligned} \dot{x} &= f(x, u) = x + u + 1 \\ y &= h(x, u) = x \end{aligned}$$

An equilibrium point is composed of constants x_0, u_0 such that

$$x_0 + u_0 + 1 = 0.$$

So either x_0 or u_0 must be specified, that is, the analyst must select where the linearization is to be done. Let's say $x_0 = 0$. Then $u_0 = -1$ and

$$A = 1, B = 1, C = 1, D = 0.$$

Actually, here A, B, C, D are independent of x_0, u_0, that is, we get the same linear system at every equilibrium point. □

Example 2.3.4 Cart-pendulum

See $f(x, u)$ in Example 2.2.5. An equilibrium point

$$x_0 = (x_{10}, x_{20}, x_{30}, x_{40}), \quad u_0$$

satisfies $f(x_0, u_0) = 0$, i.e.,

$$x_{30} = 0$$

$$x_{40} = 0$$

$$u_0 + M_2 L x_{40}^2 \sin x_{20} - M_2 g \sin x_{20} \cos x_{20} = 0$$

$$-u_0 \cos x_{20} - M_2 L x_{40}^2 \sin x_{20} \cos x_{20} + (M_1 + M_2) g \sin x_{20} = 0.$$

Multiply the third equation by $\cos x_{20}$ and add to the fourth: We get in sequence

$$-M_2 g \sin x_{20} \cos^2 x_{20} + (M_1 + M_2) g \sin x_{20} = 0$$

$$(\sin x_{20})(M_1 + M_2 \sin^2 x_{20}) = 0$$

$$\sin x_{20} = 0$$

$$x_{20} = 0 \text{ or } \pi.$$

Thus the equilibrium points are described by

$$x_0 = (\text{arbitrary}, 0 \text{ or } \pi, 0, 0), \quad u_0 = 0.$$

We have to choose $x_{20} = 0$ (pendulum up) or $x_{20} = \pi$ (pendulum down). Let's take $x_{20} = 0$. Then the Jacobians compute to

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{M_2}{M_1} g & 0 & 0 \\ 0 & \frac{M_1+M_2}{M_1} \frac{g}{L} & 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M_1} \\ -\frac{1}{LM_1} \end{bmatrix}.$$

The above provides a general method of linearizing. In this particular example, there's a faster way, which is to approximate $\sin \theta = \theta, \cos \theta = 1$ in the original equations:

$$M_2 \frac{d^2}{dt^2}(x_1 + L\theta) = F_1 \theta$$

$$0 = M_2 g - F_1$$

$$M_1 \ddot{x}_1 = u - F_1 \theta.$$

These equations are already linear and lead to the above A and B. □

2.4 Simulation

Concerning the model

$$\dot{x} = f(x, u), y = h(x, u),$$

simulation involves numerically computing $x(t)$ and $y(t)$ given an initial state $x(0)$ and an input $u(t)$. If the model is nonlinear, simulation requires an ODE solver, based on, for example, the Runge-Kutta method. Scilab and MATLAB have ODE solvers and also very nice simulation GUIs, Scicos and SIMULINK, respectively.

2.5 The Laplace Transform

You already had a treatment of Laplace transforms, for example, in a differential equations or circuit theory course. Nevertheless, we give a brief review here.

Let $f(t)$ be a real-valued function defined for $t \geq 0$. Its *Laplace transform* (LT) is

$$F(s) = \int_0^\infty f(t)\mathrm{e}^{-st}dt.$$

Here s is a complex variable. Normally, the integral converges for some values of s and not for others. That is, there is a *region of convergence* (ROC). It turns out that the ROC is always an open right half-plane, of the form $\{s : \mathrm{Re}\ \ s > a\}$. Then $F(s)$ is a complex-valued function of s.

Example 2.5.1 the unit step or unit constant

$$f(t) = \begin{cases} 1 & , \quad t \geq 0 \\ 0 & , \quad t < 0 \end{cases}$$

$$F(s) = \int_0^\infty \mathrm{e}^{-st}dt = -\left.\frac{\mathrm{e}^{-st}}{s}\right|_0^\infty = \frac{1}{s}$$

$$\mathrm{ROC} : \mathrm{Re}\ \ s > 0$$

The same $F(s)$ is obtained if $f(t) = 1$ for all t, even $t < 0$. This is because the LT is oblivious to negative time. Notice that $F(s)$ has a pole at $s = 0$ on the western boundary of the ROC. □

Example 2.5.2

$$f(t) = \mathrm{e}^{at}, \quad F(s) = \frac{1}{s-a}, \quad \mathrm{ROC} : \mathrm{Re}\ \ s > a$$

□

Example 2.5.3 sinusoid

$$f(t) = \cos wt = \frac{1}{2}\left(\mathrm{e}^{jwt} + \mathrm{e}^{-jwt}\right)$$

$$F(s) = \frac{s}{s^2 + w^2}, \ \mathrm{ROC} : \mathrm{Re}\ \ s > 0$$

□

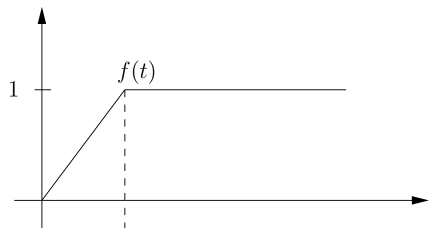
It is a theorem that $f(t)$ has a LT provided

1. it is piecewise continuous on $t \geq 0$

2. it is of exponential order, meaning there exist constants M, c such that $|f(t)| \leq M\mathrm{e}^{ct}$ for all $t \geq 0$.

The LT thus maps a class of time-domain functions $f(t)$ into a class of complex-valued functions $F(s)$. The mapping $f(t) \mapsto F(s)$ is linear.

Example 2.5.4



Thus $f = f_1 + f_2$, where f_1 is the unit ramp starting at time 0 and f_2 the ramp of slope -1 starting at time 1. By linearity, $F(s) = F_1(s) + F_s(s)$. We compute that

$$F_1(s) = \frac{1}{s^2}, \text{ Re } \ s > 0$$

$$F_2(s) = -\frac{e^{-s}}{s^2}, \text{ Re } \ s > 0.$$

Thus

$$F(s) = \frac{1 - e^{-s}}{s^2}, \text{ Re } \ s > 0.$$

□

There are tables of LTs. So in practice, if you have $F(s)$, you can get $f(t)$ using a table.

Example 2.5.5

Given $F(s) = \dfrac{3s+17}{s^2-4}$, find $f(t)$.
Sol'n

$$F(s) = \frac{c_1}{s-2} + \frac{c_2}{s+2}, \ c_1 = \frac{23}{4}, \ c_2 = -\frac{11}{4}$$

$$\Rightarrow \quad f(t) = c_1 e^{2t} + c_2 e^{-2t}$$

□

An important use of the LT is in solving initial value problems involving linear, constant-coefficient differential equations. For this it is useful to note that if

$$f(t) \leftrightarrow F(s)$$

and f is continuously differentiable at $t = 0$, then

$$\dot{f}(t) \leftrightarrow sF(s) - f(0).$$

Proof The LT of $\dot{f}(t)$ is

$$\begin{aligned}\int_0^\infty e^{-st}\dot{f}(t)dt &= e^{-st}f(t)\Big|_0^\infty + s\int_0^\infty e^{-st}f(t)dt \\ &= -f(0) + sF(s).\end{aligned}$$

□

Likewise

$$\ddot{f}(t) \longleftrightarrow s^2F(s) - sf(0) - \dot{f}(0).$$

Example 2.5.6

Solve

$$\ddot{y} + 4\dot{y} + 3y = \mathrm{e}^t,\ y(0) = 0,\ \dot{y}(0) = 2.$$

Sol'n We assume $y(t)$ is sufficiently smooth. Then

$$s^2Y(s) - 2 + 4sY(s) + 3Y(s) = \frac{1}{s-1}.$$

So

$$\begin{aligned} Y(s) &= \frac{2s-1}{(s-1)(s+1)(s+3)} \\ &= \frac{1}{8}\frac{1}{s-1} + \frac{3}{4}\frac{1}{s+1} - \frac{7}{8}\frac{1}{s+3} \\ y(t) &= \frac{1}{8}\mathrm{e}^t + \frac{3}{4}\mathrm{e}^{-t} - \frac{7}{8}\mathrm{e}^{-3t} \end{aligned}$$

□

The LT of the product $f(t)g(t)$ of two functions is *not* equal to $F(s)G(s)$, the product of the two transforms. Then what operation in the time domain does correspond to multiplication of the transforms? The answer is *convolution*. Let $f(t), g(t)$ be defined on $t \geq 0$. Define a new function

$$h(t) = \int_0^t f(t-\tau)g(\tau)d\tau,\ t \geq 0.$$

We say h is the convolution of f and g. Note that another equivalent way of writing h is

$$h(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

We also frequently use the star notation $h = f * g$. To include t in this notation, strictly speaking we should write $h(t) = (f * g)(t)$. However, it is useful sometimes (and common) to write $h(t) = f(t) * g(t)$.

Theorem 2.5.1 *The LT of $f * g$ is $F(s)G(s)$.*

Proof Let $h := f * g$. Then

$$\begin{aligned}
H(s) &= \int_0^\infty h(t)\mathrm{e}^{-st}dt \\
&= \int_0^\infty \int_0^t f(t-\tau)g(\tau)\mathrm{e}^{-st}d\tau dt \\
&= \int_0^\infty \int_\tau^\infty f(t-\tau)g(\tau)\mathrm{e}^{-st}dt d\tau \\
&= \int_0^\infty \underbrace{\left[\int_\tau^\infty f(t-\tau)\mathrm{e}^{-st}dt\right]}_{(r=t-\tau)} g(\tau)d\tau \\
&\quad \int_0^\infty f(r)\mathrm{e}^{-sr}dr\mathrm{e}^{-s\tau} \\
&= F(s)G(s).
\end{aligned}$$

□

Example 2.5.7

Consider

$$M\ddot{y} + Ky = u$$

and suppose $y(0) = \dot{y}(0) = 0$. Then

$$Ms^2Y(s) + KY(s) = U(s),$$

So

$$Y(s) = G(s)U(s),$$

where

$$G(s) = \frac{1}{Ms^2 + K}.$$

This function, $G(s)$, is called the **transfer function** of the system with input u and output y. The time-domain relationship is $y = g * u$, where $g(t)$ is the inverse LT (ILT) of $G(s)$. Specifically,

$$G(s) = \frac{1}{Ms^2 + K} \longleftrightarrow g(t) = \frac{1}{\sqrt{MK}} \sin\sqrt{\frac{K}{M}}t \quad (t \geq 0).$$

□

Now we pause to discuss the problematical object, the impulse $\delta(t)$. Let us first admit that it is not a function $\mathbb{R} \longrightarrow \mathbb{R}$, because its "value" at $t = 0$ is not a real number. Yet the impulse is so useful in applications that we have to make it legitimate. Actually, mathematicians have worked out a very nice, consistent way of dealing with the impulse. We shall borrow the main idea: $\delta(t)$ is not a function, but rather it is a way of defining the mapping $f \mapsto f(0)$ that maps a signal to its value at $t = 0$. This mapping is usually written like this:

$$\int f(t)\delta(t)dt = f(0).$$

That is, we pretend δ is a function that has this so-called sifting property. In particular, if we let $f(t) = \mathrm{e}^{-st}$, we get that the LT of δ is 1. Needless to say, we have to be careful with δ; for example, there's no way to make sense of δ^2. As long as δ is used in the sifting formula, we're on safe ground.

The formula $y = g * u$ is the time-domain relationship between u and y that is valid for any u. In particular, if $u(t) = \delta(t)$, the unit impulse, then $y(t) = g(t)$, so $Y(s) = G(s)$. Thus we see the true meaning of $g(t)$: It's the output when the input is the unit impulse and all the initial conditions are zero. We call $g(t)$ the *impulse-response function*, or the impulse response.

Example 2.5.8

Consider an RC lowpass filter with transfer function

$$G(s) = \frac{1}{\mathrm{RC}s + 1}.$$

The impulse response function is

$$g(t) = \frac{1}{\mathrm{RC}}\,\mathrm{e}^{-\frac{t}{\mathrm{RC}}} \quad (t \geq 0).$$

Now the highpass filter:

$$G(s) = \frac{\mathrm{RC}s}{\mathrm{RC}s + 1} = 1 - \frac{1}{\mathrm{RC}s + 1}$$

$$g(t) = \delta(t) - \frac{1}{\mathrm{RC}}\,\mathrm{e}^{-t/\mathrm{RC}} \quad (t \geq 0).$$

□

Inversion

As was mentioned before, in practice to go from $F(s)$ to $f(t)$ one uses a LT table. However, for a deeper understanding of the theory, one should know that there is a mathematical form of the ILT. Let

$$f(t) \longleftrightarrow F(s)$$

be a LT pair and let $\mathrm{Re}\ s > a$ be the ROC. Let σ be any real number $> a$. Then the vertical line

$$\{s : \mathrm{Re}\ \ s = \sigma\}$$

is in the ROC. The ILT formula is this:

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} \mathrm{e}^{st} F(s) ds.$$

Note that the integral is a line integral up the vertical line just mentioned.

This suggests a lovely application of Cauchy's residue theorem. Suppose $F(s)$ has the form

$$F(s) = \frac{\text{polynomial of degree} < n}{\text{polynomial of degree} = n}.$$

For example

$$F(s) = \frac{1}{Ms^2 + K}, \ n = 2$$

$$F(s) = \frac{1}{\mathrm{RC}s + 1}, \ n = 1$$

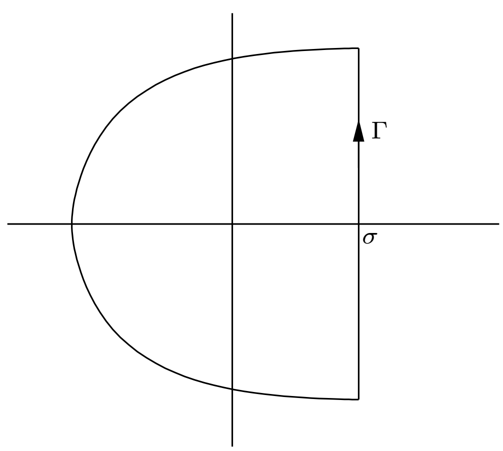
but not

$$F(s) = \frac{\mathrm{RC}s}{\mathrm{RC}s + 1}.$$

Then it can be proved that the integral up the vertical line equals the limit of the contour integral

$$\frac{1}{2\pi j} \oint_{\Gamma} \mathrm{e}^{st} F(s) ds,$$

where Γ is the semicircle



and where the limit is taken as the radius of the semicircle tends to ∞. In the limit, Γ encircles all the poles of $\mathrm{e}^{st}F(s)$. Hence by the residue theorem

$$f(t) = \Sigma \text{ residues of } \mathrm{e}^{st} F(s) \text{ at all poles.}$$

Let us review residues.

Example

$$F(s) = \frac{1}{s+1}$$

This function has a pole at $s = -1$. At all other points it's perfectly well defined. For example, near $s = 0$ it has a Taylor series expansion:

$$F(s) = F(0) + F'(0)s + \frac{1}{2}F''(0)s^2 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} F^{(k)}(0)s^k.$$

Near $s = 1$ it has a different Taylor series expansion:

$$F(s) = F(1) + F'(1)(s-1) + \frac{1}{2}F''(1)(s-1)^2 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} F^{(k)}(1)(s-1)^k.$$

And so on. Only at $s = -1$ does it not have a Taylor series. Instead, it has a Laurent series expansion, where we have to take negative indices:

$$F(s) = \sum_{k=-\infty}^{\infty} c_k (s+1)^k.$$

In fact, equating

$$\frac{1}{s+1} = \sum_{k=-\infty}^{\infty} c_k (s+1)^k$$

and matching coefficients, we see that $c_k = 0$ for all k except $c_{-1} = 1$. The coefficient c_{-1} is called the **residue** of $F(s)$ at the pole $s = -1$. □

Example

$$F(s) = \frac{1}{s(s+1)}$$

This has a pole at $s = 0$ and another at $s = -1$. At all points except these two, $F(s)$ has a Taylor series. The Laurent series at $s = 0$ has the form

$$F(s) = \sum_{k=-\infty}^{\infty} c_k s^k.$$

To determine these coefficients, first do a partial-fraction expansion:

$$F(s) = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}.$$

Then do a Taylor series expansion at $s = 0$ of the second term:

$$F(s) = \frac{1}{s} - 1 + s + \cdots .$$

Thus the residue of $F(s)$ at $s = 0$ is $c_{-1} := 1$. Similarly, to get the residue at the pole $s = -1$, start with

$$F(s) = \frac{1}{s} - \frac{1}{s+1}$$

but now do a Taylor series expansion at $s = -1$ of the first term:

$$F(s) = -\frac{1}{s+1} - 1 - (s+1) - (s+1)^2 - \cdots .$$

Thus the residue of $F(s)$ at $s = -1$ is $c_{-1} := -1$. $\square$

More generally, if p is a simple pole of $F(s)$, then the residue equals

$$\lim_{s \to p} (s-p)F(s).$$

Example

$$F(s) = \frac{1}{s^2(s+1)}$$

This has a pole at $s = 0$ of multiplicity 2 and a simple pole at $s = -1$. Partial-fraction expansion looks like

$$F(s) = \frac{1}{s^2(s+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1}.$$

We can get A and C by the usual coverup method, e.g.,

$$A = s^2 F(s)\big|_{s=0} = 1.$$

The formula for B is

$$B = \frac{d}{ds}\left(s^2 F(s)\right)\bigg|_{s=0} = -1.$$

Thus for this function, the residue at the pole $s = 0$ is $B = -1$. $\square$

Back to the ILT via residues:

$$f(t) = \sum \text{ residues of } F(s)\mathrm{e}^{st} \text{ at all its poles}, \quad t \geq 0.$$

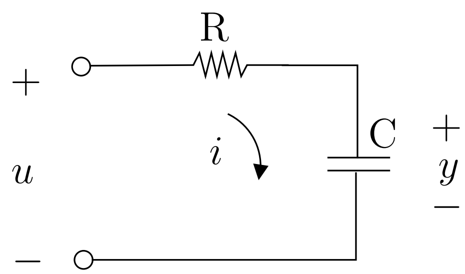
Example: $F(s) = \frac{1}{s(s-1)}$ has two poles and e^{st} has none; thus for $t \geq 0$

$$f(t) = \mathrm{Res}_{s=0} \frac{1}{s(s-1)}\mathrm{e}^{st} + \mathrm{Res}_{s=1} \frac{1}{s(s-1)}\mathrm{e}^{st} = -1 + \mathrm{e}^{t}.$$

2.6 Transfer Functions

Linear time-invariant (LTI) systems, and *only* LTI systems, have transfer functions.

Example 2.6.1 RC filter



Circuit equations:

$$-u + Ri + y = 0,\ i = C\frac{dy}{dt}$$

$$\Rightarrow\ \mathrm{RC}\dot{y} + y = u$$

Apply Laplace transforms with zero initial conditions:

$$\mathrm{RC}Y(s) + Y(s) = U(s)$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{1}{\mathrm{RC}s + 1} =: \text{transfer function.}$$

Or, by voltage-divider rule using impedances:

$$\frac{Y(s)}{U(s)} = \frac{\frac{1}{Cs}}{R + \frac{1}{Cs}} = \frac{1}{\mathrm{RC}s + 1}.$$

This transfer function is *rational*, a ratio of polynomials. □

Example 2.6.2 mass-spring-damper

$$M\ddot{y} = u - Ky - D\dot{y}$$

$$\Rightarrow\ \frac{Y(s)}{U(s)} = \frac{1}{Ms^2 + Ds + K}$$

This transfer function also is rational. □

Let's look at some other transfer functions:

$G(s) = 1$, a pure gain

$G(s) = \dfrac{1}{s}$, integrator

$G(s) = \frac{1}{s^2}$, double integrator

$G(s) = s$, differentiator

$G(s) = \mathrm{e}^{-\tau s} (\tau > 0)$, time delay; *not rational*

$G(s) = \dfrac{w_n^2}{s^2 + 2\zeta w_n s + w_n^2} (w_n > 0, \zeta \geq 0)$, standard 2^{nd} - order system

$G(s) = K_1 + \dfrac{K_2}{s} + K_3 s$, proportional-integral-derivative (PID) controller

We say a transfer function $G(s)$ is **proper** if the degree of the denominator $\geq$ that of the numerator. The transfer functions $G(s) = 1, \dfrac{1}{s+1}$ are proper, $G(s) = s$ is not. We say $G(s)$ is **strictly proper** if the degree of the denominator $>$ that of the numerator. Note that if $G(s)$ is proper then $\lim_{|s|\to\infty} G(s)$ exists; if strictly proper then $\lim_{|s|\to\infty} G(s) = 0$. These concepts extend to multi-input, multi-output systems, where the transfer function is a matrix.

Let's see what the transfer function is of an LTI state model:

$$\dot{x} = Ax + Bu, \;\; y = Cx + Du$$

$$sX(s) = AX(s) + BU(s), Y(s) = CX(s) + DU(s)$$
$$\Rightarrow X(s) = (sI - A)^{-1} BU(s)$$
$$Y(s) = [C(sI - A)^{-1} B + D] U(s).$$

We conclude that the transfer function from u to x is $(sI - A)^{-1} B$ and from u to y is

$$C(sI - A)^{-1} B + D.$$

Example 2.6.3

Two carts, one spring:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \; D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C(sI - A)^{-1} B + D = \begin{bmatrix} \dfrac{s^2+1}{s^2(s^2+2)} & \dfrac{1}{s^2(s^2+2)} \\ \dfrac{1}{s^2(s^2+2)} & \dfrac{s^2+1}{s^2(s^2+2)} \end{bmatrix}.$$

□

Let us *recap* our procedure for getting the transfer function of a system:

1. Apply the laws of physics etc. to get differential equations governing the behaviour of the system. Put these equations in state form. In general these are nonlinear.
2. Linearize about an equilibrium point.
3. Take Laplace transforms with zero initial state.

The transfer function from input to output satisfies

$$Y(s) = G(s)U(s).$$

In general $G(s)$ is a matrix: If $\dim u = m$ and $\dim y = p$ (m inputs, p outputs), then $G(s)$ is $p \times m$. In the SISO case, $G(s)$ is a scalar-valued transfer function.

There is a converse problem: Given a transfer function, find a corresponding state model. That is, given $G(s)$, find A, B, C, D such that

$$G(s) = C(sI - A)^{-1}B + D.$$

The state matrices are never unique: Each $G(s)$ has an infinite number of state models. But it is a fact that every proper, rational $G(s)$ has a state realization. Let's see how to do this in the SISO case, where $G(s)$ is 1×1.

Example 2.6.4 $G(s) = \dfrac{1}{2s^2 - s + 3}$

The corresponding differential equation model is

$$2\ddot{y} - \dot{y} + 3y = u.$$

Taking $x_1 = y, x_2 = \dot{y}$, we get

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{2}x_2 - \frac{3}{2}x_1 + \frac{1}{2}u \\ y &= x_1 \end{aligned}$$

and thus

$$A = \begin{bmatrix} 0 & 1 \\ -3/2 & 1/2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0.$$

This technique extends to

$$G(s) = \frac{1}{\text{poly of degree } n}.$$

$\square$

Example 2.6.5

$$G(s) = \frac{s-2}{2s^2 - s + 3}$$

Introduce an auxiliary signal $V(s)$:

$$Y(s) = (s-2)\underbrace{\frac{1}{2s^2 - s + 3}U(s)}_{=:V(s)}$$

Then

$$2\ddot{v} - \dot{v} + 3v = u$$
$$y = \dot{v} - 2v.$$

Defining

$$x_1 = v, x_2 = \dot{v},$$

we get

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{2}x_2 - \frac{3}{2}x_1 + \frac{1}{2}u$$
$$y = x_2 - 2x_1$$

and so

$$A = \begin{bmatrix} 0 & 1 \\ -3/2 & 1/2 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} -2 & 1 \end{bmatrix}, \ D = 0.$$

This extends to any strictly proper rational function. □

Finally, if $G(s)$ is proper but not strictly proper (deg num = deg denom), then we can write

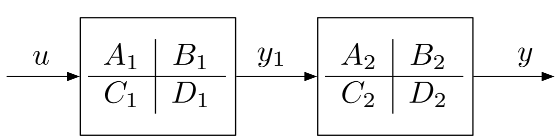
$$G(s) = c + G_1(s),$$

c = constant, $G_1(s)$ strictly proper. In this case we get A, B, C to realize $G_1(s)$, and $D = c$.

2.7 Interconnections

Frequently, a system is made up of components connected together in some topology. This raises the question, if we have state models for components, how can we assemble them into a state model for the overall system?

Example 2.7.1 series connection



This diagram stands for the equations

$$\begin{aligned}
\dot{x}_1 &= A_1x_1 + B_1u \\
y_1 &= C_1x_1 + D_1u \\
\dot{x}_2 &= A_2x_2 + B_2y_1 \\
y &= C_2x_2 + D_2y_1.
\end{aligned}$$

Let us take the overall state to be

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Then

$$\dot{x} = Ax + Bu, \; y = Cx + Du,$$

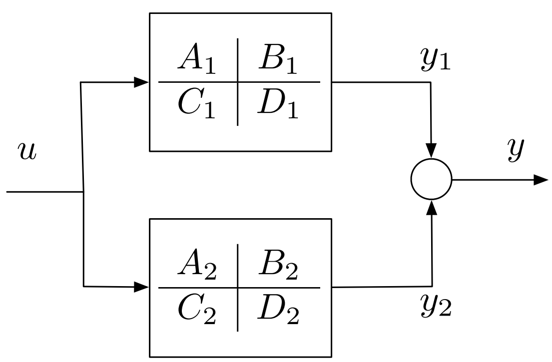
where

$$A = \begin{bmatrix} A_1 & 0 \\ B_2C_1 & A_2 \end{bmatrix}, \; B = \begin{bmatrix} B_1 \\ B_2D_1 \end{bmatrix}$$

$$C = \begin{bmatrix} D_2C_1 & C_2 \end{bmatrix}, \; D = D_2D_1.$$

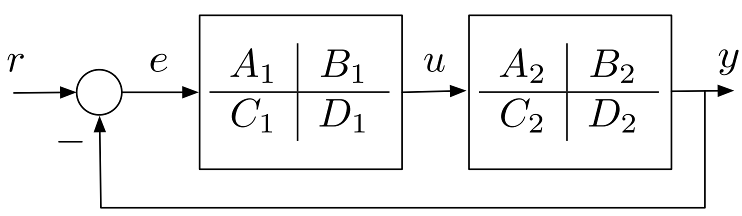
□

Parallel connection



is very similar and is left for you.

Example 2.7.2 feedback connection



$$\begin{aligned}
&\dot{x}_1 = A_1x_1 + B_1e = A_1x_1 + B_1(r - C_2x_2) \\
&\dot{x}_2 = A_2x_2 + B_2u = A_2x_2 + B_2(C_1x_1 + D_1(r - C_2x_2)) \\
&y = C_2x_2
\end{aligned}$$

Taking

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

we get

$$\dot{x} = Ax + Br, \ y = Cx$$

where

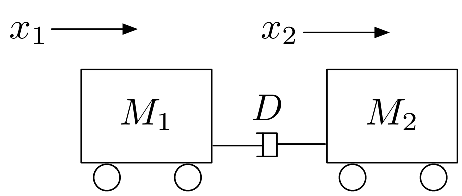
$$A = \begin{bmatrix} A_1 & -B_1C_2 \\ B_2C_1 & A_2 - B_2D_1C_2 \end{bmatrix}, \ B = \begin{bmatrix} B_1 \\ B_2D_1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & C_2 \end{bmatrix}.$$

□

2.8 Problems

1. Consider the following system with two carts and a dashpot:



   (Recall that a dashpot is like a spring except the force is proportional to the *derivative* of the change in length; D is the proportionality constant.) The input is the force u and the positions of the carts are x_1, x_2. The other state variables are $x_3 = \dot{x}_1$, $x_4 = \dot{x}_2$. Take $M_1 = 1$, $M_2 = 1/2$, $D = 1$. Derive the matrices A, B in the state model $\dot{x} = Ax + Bu$,

2. This problem concerns a beam balanced on a fulcrum. The angle of tilt of the beam is denoted $\alpha(t)$; a torque, denoted $\tau(t)$, is applied to the beam; finally, a ball rolls on the beam at distance $d(t)$ from the fulcrum.

   Introduce the parameters

J	moment of inertia of the beam
J_b	moment of inertia of the ball
R	radius of the ball
M	mass of the ball.

   The equations of motion are given to you as

$$\left(\frac{J_b}{R^2} + M \right) \ddot{d} + Mg \sin\alpha - Md\dot{\alpha}^2 = 0$$

$$(Md^2 + J + J_b)\ddot{\alpha} + 2Md\dot{d}\dot{\alpha} + Mgd\cos\alpha = \tau.$$

   Put this into the form of a nonlinear state model with input τ.

3. Continue with the same ball-and-beam problem. Find all equilibrium points. Linearize the state equation about the equilibrium point where $\alpha = d = 0$.

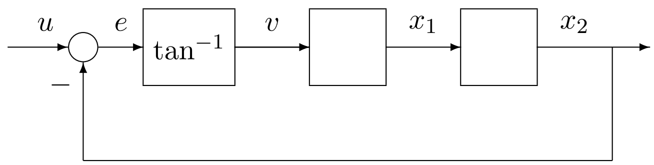
4. Let A be an $n \times n$ real matrix and $b \in \mathbb{R}^n$. Define the function

$$f(x) = x^T A x + b^T x \quad f : \mathbb{R}^n \longrightarrow \mathbb{R},$$

where ^{T} denotes transpose. Linearize the equation $y = f(x)$ at the point x_0.

5. Linearize the water-tank example.

6. Consider the following feedback control system:



The nonlinearity is the saturating function $v = \tan^{-1}(e)$, and the two blank blocks are integrators modeled by

$$\dot{x}_1 = v, \quad \dot{x}_2 = x_1.$$

Taking these state variables, derive the nonlinear state equation

$$\dot{x} = f(x, u).$$

Linearize the system about the equilibrium point where $u = 1$ and find the matrices A and B in the linear equation

$$\dot{\Delta x} = A\Delta x + B\Delta u.$$

[Hint: $\dfrac{d}{dy} \tan^{-1} y = \cos^2(\tan^{-1} y)$.]

7. Sketch the function

$$f(t) = \begin{cases} t+1, & 0 \le t \le 10 \\ -2e^t, & t > 10 \end{cases}$$

and find its Laplace transform, including the region of convergence.

8. (a) Find the inverse Laplace transform of $G(s) = \dfrac{1}{2s^2+1}$ using the residue formula.

   (b) Repeat for $G(s) = \dfrac{1}{s^2}$.

   (c) Repeat for $G(s) = \dfrac{s^2}{2s^2+1}$. [Hint: Write $G(s) = c + G_1(s)$ with $G_1(s)$ strictly proper.]

9. Explain why we don't use Laplace transforms to solve the initial value problem

$$\ddot{y}(t) + 2t\dot{y}(t) - y(t) = 1, \quad y(0) = 0, \ \dot{y}(0) = 1.$$

10. Consider a mass-spring system where $M(t)$ is a known function of time. The equation of motion in terms of force input u and position output y is

$$\frac{d}{dt}M\dot{y} = u - Ky$$

(i.e., rate of change of momentum equals sum of forces), or equivalently

$$M\ddot{y} + \dot{M}\dot{y} + Ky = u.$$

This equation has time-dependent coefficients. So there's no transfer function $G(s)$, hence no impulse-response function $g(t)$, hence no convolution equation $y = g \star u$.

    (a) Find a linear state model.

    (b) Guess what the correct form of the time-domain integral equation is. [Hint: If M is constant, the output y at time t when the input is an impulse applied at time t_0 depends only on the difference $t - t_0$. But if M is not constant, it depends on both t and t_0 separately.]

11. Consider Problem 1. Find the transfer function from u to x_1. Do it both by hand (from the state model) and by Scilab or MATLAB.

12. Find a state model (A, B, C, D) for the system with transfer function

$$G(s) = \frac{-2s^2 + s + 1}{s^2 - s - 4}.$$

13. Consider the parallel connection of G_1 and G_2, the LTI systems with transfer functions

$$G_1(s) = \frac{10}{s^2 + s + 1}, \quad G_2(s) = \frac{1}{0.1s + 1}.$$

    (a) Find state models for G_1 and G_2.

    (b) Find a state model for the overall system.

Chapter 3

Linear System Theory

In the preceding chapter we saw nonlinear state models and how to linearize them about an equilibrium point. The linearized systems have the form (dropping Δ)

$$\dot{x} = Ax + Bu, \ y = Cx + Du.$$

In this chapter we study such models.

3.1 Initial-State-Response

Let us begin with the state equation forced only by the initial state—the input is set to zero:

$$\dot{x} = Ax, \ x(0) = x_0, \ A \in \mathbb{R}^{n \times n}.$$

Recall two facts:

1. If $n = 1$, i.e., A is a scalar, the unique solution of the initial-value problem is $x(t) = \mathrm{e}^{At} x_0$.
2. The Taylor series of the function e^t at $t = 0$ is

$$\mathrm{e}^t = 1 + t + \frac{t^2}{2!} + \cdots$$

   and this converges for every t. Thus

$$\mathrm{e}^{At} = 1 + At + \frac{A^2 t^2}{2!} + \cdots .$$

This second fact suggests that in the matrix case we **define** the matrix exponential e^A to be

$$\mathrm{e}^A := I + A + \frac{1}{2!} A^2 + \frac{1}{3!} A^3 + \cdots .$$

It can be proved that the right-hand series converges for every matrix A. If A is $n \times n$, so is e^A; e^A is not defined if A is not square.

Example 3.1.1

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathrm{e}^A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Notice that e^A is **not** obtained by exponentiating A componentwise. □

Example 3.1.2

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathrm{e}^A = \begin{bmatrix} \mathrm{e} & 0 \\ 0 & \mathrm{e} \end{bmatrix} = \mathrm{e}I$$

□

Example 3.1.3

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix has the property that $A^3 = 0$. Thus the power series has only finitely many nonzero terms:

$$\mathrm{e}^A = I + A + \frac{1}{2}A^2 = \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

This is an example of a **nilpotent** matrix. That is, $A^k = 0$ for some power k. □

Replacing A by At (the product of A with the scalar t) gives the matrix-valued function e^{At},

$$t \longmapsto \mathrm{e}^{At} : \mathbb{R} \longrightarrow \mathbb{R}^{n\times n},$$

defined by

$$\mathrm{e}^{At} = I + At + A^2\frac{t^2}{2!} + \cdots .$$

This function is called the **transition matrix**.

Some properties of e^{At}:

1. $\mathrm{e}^{At}\big|_{t=0} = I$
2. $\mathrm{e}^{At_1}\mathrm{e}^{At_2} = \mathrm{e}^{A(t_1+t_2)}$

   Note that $\mathrm{e}^{A_1}\mathrm{e}^{A_2} = \mathrm{e}^{A_1+A_2}$ if and only if A_1 and A_2 commute, i.e., $A_1A_2 = A_2A_1$.
3. $(\mathrm{e}^A)^{-1} = \mathrm{e}^{-A}$, so $(\mathrm{e}^{At})^{-1} = \mathrm{e}^{-At}$
4. A, e^{At} commute
5. $\dfrac{d}{dt}\mathrm{e}^{At} = A\mathrm{e}^{At}$

Now the main result:

Theorem 3.1.1 *The unique solution of the initial-value problem* $\dot{x} = Ax,\ x(0) = x_0$*, is* $x(t) = e^{At}x_0$.

This leaves us with the question of how to compute e^{At}. For hand calculations on small problems ($n = 2$ or 3), it's convenient to use Laplace transforms.

Example 3.1.4

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \ e^{At} = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

The Laplace transform of e^{At} is therefore

$$\begin{bmatrix} 1/s & 1/s^2 & 1/s^3 \\ 0 & 1/s & 1/s^2 \\ 0 & 0 & 1/s \end{bmatrix}.$$

On the other hand,

$$\begin{aligned} (sI - A)^{-1} &= \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 0 & s \end{bmatrix}^{-1} \\ &= \frac{1}{s^3} \begin{bmatrix} s^2 & s & 1 \\ 0 & s^2 & s \\ 0 & 0 & s^2 \end{bmatrix}. \end{aligned}$$

We conclude that in this example e^{At}, $(sI - A)^{-1}$ are Laplace transform pairs. This is true in general. □

If A is $n \times n$, e^{At} is an $n \times n$ matrix function of t and $(sI - A)^{-1}$ is an $n \times n$ matrix of rational functions of s.

Example 3.1.5

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ (sI - A)^{-1} &= \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}^{-1} = \frac{1}{s^2+1} \begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix} \\ \Longrightarrow e^{At} &= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \end{aligned}$$

□

Another way to compute e^{At} is via eigenvalues and eigenvectors. Instead of a general treatment, let's do two examples.

Example 3.1.6

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

The MATLAB command $[V, D] = \text{eig}\ (A)$ produces

$$V = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}.$$

The eigenvalues of A appear on the diagonal of the (always diagonal) matrix D, and the columns of V are corresponding eigenvectors. So for example

$$A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

It follows that $AV = VD$ (check this) and then that $e^{At}V = Ve^{Dt}$ (prove this). The nice thing is that e^{Dt} is trivial to determine because D is diagonal. In this case

$$e^{Dt} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}.$$

Then

$$e^{At} = Ve^{Dt}V^{-1}.$$

□

Example 3.1.5 (Cont'd)

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} j & 0 \\ 0 & -j \end{bmatrix}, \quad V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}$$

$$e^{Dt} = \begin{bmatrix} e^{jt} & 0 \\ 0 & e^{-jt} \end{bmatrix}$$

$$e^{At} = Ve^{Dt}V^{-1} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

□

The above method works when A has n linearly independent eigenvectors, so V is invertible. Otherwise the theory is more complicated and requires the so-called Jordan form of A.

3.2 Input-Response

Now we set the initial state to zero and consider the response from the input:

$$\dot{x} = Ax + Bu, \ x(0) = 0.$$

Here's a derivation of the solution: Multiply by e^{-At}:

$$e^{-At}\dot{x} = e^{-At}Ax + e^{-\mathbf{A}t}Bu.$$

Noting that

$$\frac{d}{dt}[e^{-At}x(t)] = -Ae^{-At}x(t) + e^{-At}\dot{x}(t),$$

we get

$$\frac{d}{dt}[\mathrm{e}^{-At}x(t)] = \mathrm{e}^{-At}Bu(t).$$

Integrate from 0 to t:

$$-\mathrm{e}^{At}x(t) - \underbrace{x(0)}_{=0} = \int_0^t \mathrm{e}^{-A\tau}Bu(\tau)d\tau.$$

Multiply by e^{At}:

$$x(t) = \int_0^t \mathrm{e}^{A(t-\tau)}Bu(\tau)d\tau. \tag{3.1}$$

Equation (3.1) gives the state at time t in terms of $u(\tau),\ 0 \le \tau \le t$, when the initial state equals zero.

Similarly, the output equation $y = Cx + Du$ leads to

$$y(t) = \int_0^t C\mathrm{e}^{A(t-\tau)}Bu(\tau)d\tau + Du(t).$$

Special case Suppose dim u = dim $y = 1$, i.e., the system is single-input, single-output (SISO). Then $D = D$, a scalar, and we have

$$y(t) = \int_0^t C\mathrm{e}^{A(t-\tau)}Bu(\tau)d\tau + Du(t). \tag{3.2}$$

If $u = \delta$, the unit impulse, then

$$y(t) = C\mathrm{e}^{At}B1_+(t) + D\delta(t),$$

where $1_+(t)$ denotes the unit step,

$$1_+(t) = \begin{cases} 1 & , \quad t \ge 0 \\ 0 & , \quad t < 0. \end{cases}$$

We conclude that the impulse response of the system is

$$g(t) := C\mathrm{e}^{At}B1_+(t) + D\delta(t) \tag{3.3}$$

and equation (3.2) is a convolution equation:

$$y(t) = (g * u)(t).$$

Example 3.2.1

$$\ddot{y} + \dot{y} = u$$

Take the state to be

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} := \begin{bmatrix} y \\ \dot{y} \end{bmatrix}.$$

Then

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \; D = 0.$$

The transition matrix:

$$(sI - A)^{-1} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+1)} \\ 0 & \frac{1}{s+1} \end{bmatrix}$$
$$e^{At} = \begin{bmatrix} 1 & 1 - e^{-t} \\ 0 & e^{-t} \end{bmatrix}, \; t \geq 0.$$

Impulse response:

$$\begin{aligned} g(t) &= Ce^{At}B, \quad t \geq 0 \\ &= 1 - e^{-t}, \quad t \geq 0. \end{aligned}$$

□

3.3 Total Response

Consider the state equation forced by both an initial state and an input:

$$\dot{x} = Ax + Bu, \; x(0) = x_0.$$

The system is **linear** in the sense that the state at time t equals the initial-state-response at time t plus the input-response at time t:

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau.$$

Similarly, the output $y = Cx + Du$ is given by

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t).$$

These two equations constitute a solution in the time domain.

Summary We began with an LTI system modeled by a differential equation in state form:

$$\dot{x} = Ax + Bu, \ x(0) = x_0$$
$$y = Cx + Du.$$

We solved the equations to get

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$
$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}B\mathbf{u}(\tau)d\tau + Du(t).$$

These are integral (convolution) equations giving $x(t)$ and $\mathbf{y}(t)$ explicitly in terms of x_0 and $u(\tau), \ 0 \leq \tau \leq t$. In the SISO case, if $x_0 = 0$ then

$$y = g * u, \ \text{i.e.,} \ y(t) = \int_0^t g(t-\tau)u(\tau)d\tau$$

where

$$g(t) = Ce^{At} \ B1_+(t) + D\delta(t)$$
$$1_+(t) = \text{unit step}.$$

3.4 Logic Notation

We now take a break from linear system theory to go over logic notation. Logic notation is a great aid in precision, and therefore in a clear understanding of mathematical concepts. These notes provide a brief introduction to mathematical statements using logic notation.

A **quantifier** is a mathematical symbol indicating the amount or quantity of the variable or expression that follows. There are two quantifiers:

$\exists$ denotes the **existential quantifier** meaning "there exists" (or "for some").

$\forall$ denotes the **universal quantifier** meaning "for all" or "for every."

As a simple example, here's the definition that the sequence $\{a_n\}_{n\geq 1}$ of real numbers is bounded:

$$(\exists B \geq 0)(\forall n \geq 1) \ |a_n| \leq B. \tag{3.4}$$

This statement is parsed from left to right. In words, (3.4) says this: There exists a nonnegative number B such that, for every positive integer n, the absolute value of a_n is bounded by B. Notice in (3.4) that the two quantifier phrases, $\exists B \geq 0$ and $\forall n \geq 1$, are placed in brackets and precede the term $|a_n| \leq B$. This latter term has n and B as variables in it that need to be quantified. We cannot say merely that $\{a_n\}_{n\geq 1}$ is bounded if $|a_n| \leq B$ (unless it is known or understood what the quantifiers on n and B are). In general, all variables in a statement need to be quantified.

As an example, the sequence $a_n = (-1)^n$ of alternating $+1$ and -1 is obviously bounded. Here are the steps in formally proving (3.4) for this sequence:

Take $B = 1$.

Let $n \geq 1$ be arbitrary.

Then $|a_n| = |(-1)^n| = 1$. Thus $|a_n| = B$.

The order of quantifiers is crucial. Observe that (3.4) is very different from saying

$$(\forall n \geq 1)(\exists B \geq 0)\ |a_n| \leq B, \tag{3.5}$$

which is true of **every** sequence. Let's prove, for example, that the unbounded sequence $a_n = 2^n$ satisfies (3.5):

Let $n \geq 1$ be arbitrary.

Take $B = 2^n$.

Since $|a_n| = 2^n$, so $|a_n| = B$.

As another example, here's the definition that $\{a_n\}_{n\geq 1}$ converges to 0:

$$(\forall \varepsilon > 0)(\exists N \geq 1)(\forall n \geq N)\ |a_n| < \varepsilon. \tag{3.6}$$

This says, for every positive ε there exists a positive N such that, for every $n \geq N$, $|a_n|$ is less than ε. A formal proof that $a_n = 1/n$ satisfies (3.6) goes like this:

Let $\varepsilon > 0$ be arbitrary.

Take N to be any integer greater than $1/\varepsilon$.

Let $n \geq N$ be arbitrary.

Then $|a_n| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$.

The symbol for logical negation is $\neg$. Thus $\{a_n\}_{n\geq 1}$ is not bounded if, from (3.4),

$$\neg(\exists B \geq 0)(\forall n \geq 1)\ |a_n| \leq B.$$

This is logically equivalent to

$$(\forall B \geq 0)(\exists n \geq 1)\ |a_n| > B.$$

Study how this statement is obtained term-by-term from the previous one: $\exists B \geq 0$ changes to $\forall B \geq 0$; $\forall n \geq 1$ changes to $\exists n \geq 1$; and $|a_n| \leq B$ is negated to $|a_n| > B$. The order of the variables being quantified (B then n) does not change.

Similarly, the negation of (3.6), meaning $\{a_n\}$ does not converge to 0, is

$$(\exists \varepsilon > 0)(\forall N \geq 1)(\exists n \geq N)\ |a_n| \geq \varepsilon. \tag{3.7}$$

For example, here's a proof that $a_n = (-1)^n$ satisfies (3.7):

Take $\varepsilon = 1/2$.

Let $N \geq 1$ be arbitrary.

Take $n = N$.

Then $|a_n| = 1 > \varepsilon$.

As the final example of this type, here's the definition that the function $f(x)$, $f : \mathbb{R} \longrightarrow \mathbb{R}$, is continuous at the point $x = a$:

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \text{ with } |x - a| < \delta)\ |f(x) - f(a)| < \varepsilon. \tag{3.8}$$

The negation is therefore

$$(\exists \varepsilon > 0)(\forall \delta > 0)(\exists x \text{ with } |x - a| < \delta)\ |f(x) - f(a)| \geq \varepsilon.$$

Try your hand at proving, via the last statement, that the step function

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

is not continuous at $x = 0$.

Logical conjunction (**and**) is denoted by $\wedge$ or by a comma, and logical disjunction (**or**) is denoted by $\vee$. The negation of $P \wedge Q$ is $\neg P \vee \neg Q$. The negation of $P \vee Q$ is $\neg P \wedge \neg Q$.

The final logic operation is denoted by the symbol $\Rightarrow$, which means "implies" or "if ... then." For example, here's a well-known statement about three real numbers, a, b, c:

$$b^2 - 4ac \geq 0 \Rightarrow ax^2 + bx + c \text{ has real roots.}$$

We read this as follows: If $b^2 - 4ac \geq 0$, then the polynomial $ax^2 + bx + c$ has real roots. As another example, the statement (convergence of $\{a_n\}_{n \geq 1}$ to 0)

$$(\forall \varepsilon > 0)(\exists N \geq 1)(\forall n \geq N)\ |a_n| < \varepsilon.$$

can be written alternatively as

$$(\forall \varepsilon > 0)(\exists N \geq 1)(\forall n)\ n \geq N \Rightarrow |a_n| < \varepsilon.$$

Similarly, the statement (continuity of $f(x)$ at $x = a$)

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \text{ with } |x - a| < \delta)\ |f(x) - f(a)| < \varepsilon$$

can be written alternatively as

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)\ |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon. \tag{3.9}$$

The statement $P \Rightarrow Q$ is logically equivalent to the statement $\neg Q \Rightarrow \neg P$. Example:

$$ax^2 + bx + c \text{ does not have real roots} \Rightarrow b^2 - 4ac < 0.$$

That is, if the polynomial $ax^2 + bx + c$ does not have real roots, then $b^2 - 4ac < 0$.

The truth table for the logical implication operator is

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Writing out the truth table for $P \wedge \neg Q$ will show you that it is (logically equivalent to) the negation of $P \Rightarrow Q$. So for example, the negation of (3.9) ($f(x)$ is not continuous at $x = a$) is

$$(\exists \varepsilon > 0)(\forall \delta > 0)(\exists x)\ |x - a| < \delta, |f(x) - f(a)| \geq \varepsilon.$$

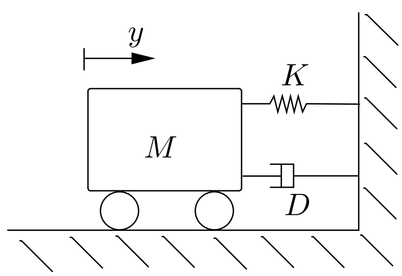
3.5 Lyapunov Stability

Stability theory of dynamical systems is an old subject, dating back several hundred years. The goal in stability theory is to draw conclusions about the qualitative behaviour of trajectories without having to solve analytically or simulate exhaustively for all possible initial conditions. The theory introduced in this section is due to the Russian mathematician A.M. Lyapunov (1892).

To get an idea of the stability question, imagine a helicopter hovering under autopilot control. Suppose the helicopter is subject to a wind gust. Will it return to its original hovering state? If so, we say the hovering state is a stable state.

Let's look at a much simpler example.

Example 3.5.1



The model with no external forces:

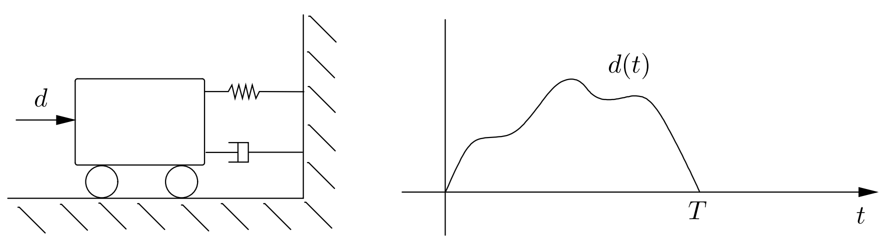
$$M\ddot{y} = -Ky - D\dot{y}$$

or

$$\dot{x} = Ax, \ A = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{D}{M} \end{bmatrix}.$$

The point $x = 0$ is an equilibrium point.

Now suppose a wind gust of finite duration is applied:



The model is

$$M\ddot{y} = d - Ky - D\dot{y},$$

or

$$\dot{x} = Ax + Ed, \ E = \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}.$$

If $x(0) = 0$, then at time $t = T$

$$x(T) = \int_0^T e^{A(T-\tau)} E d(\tau) d\tau \neq 0 \text{ in general.}$$

For $t > T$, the model is $\dot{x} = Ax$. Thus the effect of a finite-duration disturbance is to alter the initial state. In this way, the stability question concerns the qualitative behaviour of $\dot{x} = Ax$ for an arbitrary initial state; the initial time may be shifted to $t = 0$. □

We'll formulate the main concepts for the nonlinear model $\dot{x} = f(x),\ x(0) = x_0$, and then specialize to the linear one $\dot{x} = Ax,\ x(0) = x_0$, for specific results.

Assume the model under study is $\dot{x} = f(x)$ and assume $x = 0$ is an equilibrium point, i.e., $f(0) = 0$. The stability questions are

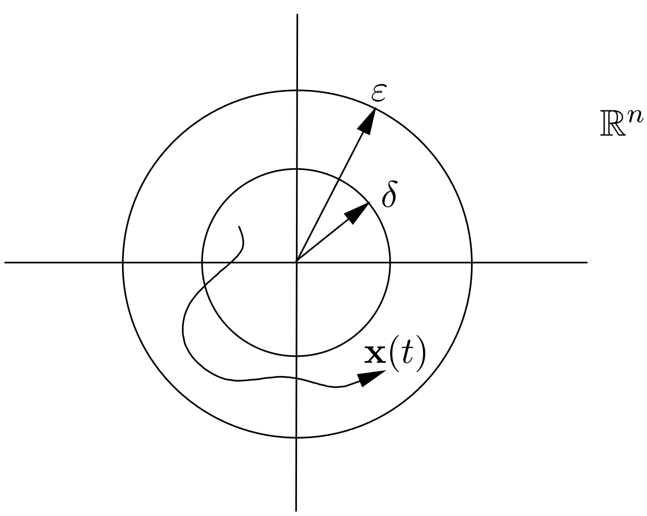
1. If $x(0)$ is near the origin, does $x(t)$ remain near the origin? This is stability.

2. Does $x(t) \to 0$ as $t \to \infty$ for every $x(0)$? This is asymptotic stability.

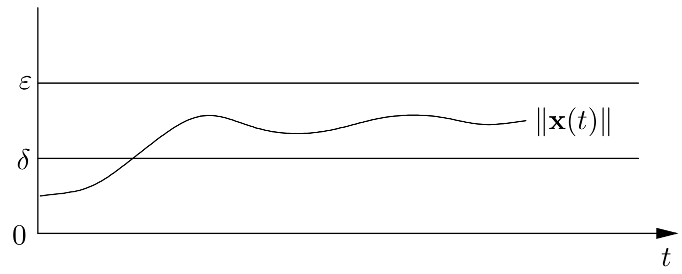
To give precise definitions to these concepts, let $\|x\|$ denote the Euclidean norm of a vector x, i.e., $\|x\| = (x^T x)^{1/2} = (\Sigma_i x_i^2)^{1/2}$. Then we say the origin is a **stable** equilibrium point if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x(0)) \|x(0)\| < \delta \Rightarrow (\forall t \geq 0) \|x(t)\| < \varepsilon.$$

In words, for every $\varepsilon > 0$ there exists $\delta > 0$ such that if the state starts in the δ-ball, it will remain in the ε-ball:



The picture with t explicit is



Another way of expressing the concept is: the trajectory will remain arbitrarily (i.e., $\forall\varepsilon$) close to the origin provided it starts sufficiently (i.e., $\exists\delta$) close to the origin.

The origin is an **asymptotically stable** equilibrium point if

(i) it is stable, and

(ii) $(\exists\varepsilon > 0)(\forall x(0))\|x(0)\| < \varepsilon \Rightarrow \lim_{t\to\infty} x(t) = \mathbf{0}$.

Clearly the second requirement is that $x(t)$ converges to the origin provided $x(0)$ is sufficiently near the origin.

If we're given the right-hand side function f, it's in general very hard to determine if the equilibrium point is stable, or asymptotically stable. Because in this course we don't have time to do the general theory, we'll look at the results only for the linear system $\dot{x} = Ax, A \in \mathbb{R}^{n\times n}$. Of course, $\mathbf{0}$ is automatically an equilibrium point. As we saw before, the trajectory is specified by $x(t) = \mathrm{e}^{At}x(0)$. So stability depends on the function $t \longmapsto \mathrm{e}^{At} : [0,\infty) \to \mathbb{R}^{n\times n}$.

Proposition 3.5.1 *For the linear system $\dot{x} = Ax$, 0 is stable iff e^{At} is a bounded function; 0 is asymptotically stable iff $\mathrm{e}^{At} \to 0$ as $t \to \infty$.*

The condition on A for e^{At} to be bounded is a little complicated (needs the Jordan form). The condition on A for $\mathrm{e}^{At} \to 0$ is pretty easy:

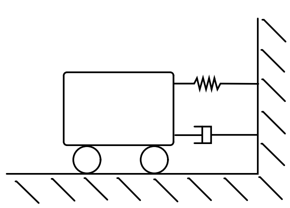
Proposition 3.5.2 $\mathrm{e}^{At} \to 0$ as $t \to \infty$ iff every eigenvalue of A has negative real part.

Example 3.5.2

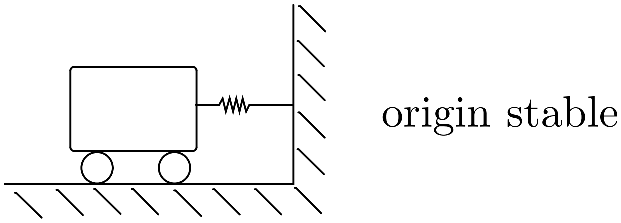
1. $\dot{x} = -x$: origin is asymptotically stable

2. $\dot{x} = 0$: origin is stable

3. $\dot{x} = x$: origin unstable

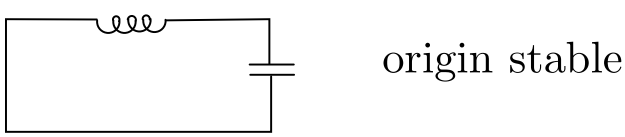
4. 

origin asymp. stable



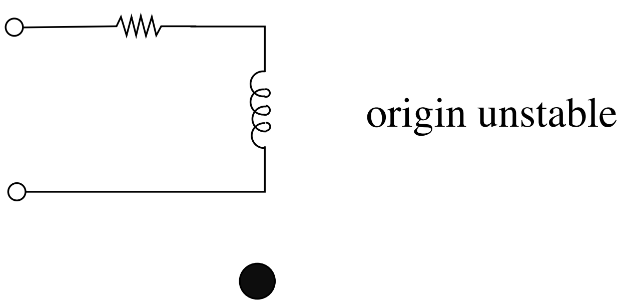
5.

6.



7.

8. maglev



$\square$

Proposition 3.5.2 is easy to prove when A has n linearly independent eigenvectors:

$$e^{At} = V e^{Dt} V^{-1}$$

$$\begin{aligned} e^{At} \to 0 &\iff e^{Dt} \to 0 \\ &\iff e^{\lambda_i t} \to 0 \ \forall i \\ &\iff \text{Re } \lambda_i < 0 \ \forall i. \end{aligned}$$

3.6 BIBO Stability

There's another stability concept, that concerns the response of a system to inputs instead of initial conditions. We'll study this concept for a restricted class of systems.

Consider an LTI system with a single input, a single output, and a strictly proper rational transfer function. The model is therefore $y = g * u$ in the time domain, or $Y(s) = G(s)U(s)$ in the s-domain. We ask the question: Does a bounded input (BI) always produce a bounded output (BO)? Note that $u(t)$ **bounded** means

$$(\exists B)(\forall t \geq 0)|u(t)| \leq B.$$

The least upper bound B is actually a norm, denoted $\|u\|_\infty$.

Example 3.6.1

1. $u(t) = 1_+(t), \ \|u\|_\infty = 1$
2. $u(t) = \sin(t), \ \|u\|_\infty = 1$
3. $u(t) = (1 - e^{-t})1_+(t), \ \|u\|_\infty = 1$
4. $u(t) = t1_+(t), \ \|u\|_\infty = \infty$, or undefined.

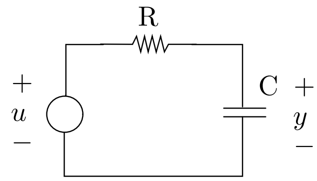
Note that in the second case, $\|u\|_\infty = |u(t)|$ for some finite t; that is, $\|u\|_\infty = \max\limits_{t \geq 0} |u(t)|$. Whereas in the third case, $\|u\|_\infty > |u(t)|$ for every finite t. $\square$

We define the system to be **BIBO stable** if every bounded u produces a bounded y.

Theorem 3.6.1 *Assume $G(s)$ is strictly proper, rational. Then the following three statements are equivalent:*

1. *The system is BIBO stable.*
2. *The impulse-response function $g(t)$ is absolutely integrable, i.e.,* $\int\limits_0^\infty |g(t)|dt < \infty$.
3. *Every pole of the transfer function $G(s)$ has negative real part.*

Example 3.6.2 RC filter



$$G(s) = \frac{1}{\mathrm{RC}s + 1}$$

$$g(t) = \frac{1}{\mathrm{RC}} \mathrm{e}^{-t/\mathrm{RC}} 1_+(t)$$

According to the theorem, every bounded u produces a bounded y. What's the relationship between $\|u\|_\infty$ and $\|y\|_\infty$? Let's see:

$$\begin{aligned} |y(t)| &= \left| \int_0^t g(t-\tau) u(\tau) d\tau \right| \\ &\leq \int_0^t |g(t-\tau)||u(\tau)| d\tau \\ &\leq \|u\|_\infty \int_0^t |g(t-\tau)| d\tau \\ &\leq \|u\|_\infty \int_0^\infty |g(t)| dt \\ &= \|u\|_\infty \int_0^\infty \frac{1}{\mathrm{RC}} \mathrm{e}^{-t/\mathrm{RC}} dt \\ &= \|u\|_\infty. \end{aligned}$$

Thus $\|y\|_\infty \leq \|u\|_\infty$ for every bounded u. □

Example 3.6.3 integrator

$$G(s) = \frac{1}{s}, \; g(t) = 1_+(t)$$

According to the theorem, the system is **not** BIBO stable. Thus there exists some bounded input that produces an unbounded output. For example

$$u(t) = 1_+(t) = \text{bounded} \Rightarrow y(t) = t1_+(t) = \text{unbounded}.$$

Notice that it is not true that every bounded input produces an unbounded output, only that some bounded input does. Example

$$u(t) = (\sin t)1_+(t) \Rightarrow y(t) \text{ bounded.}$$

□

The theorem can be extended to the case where $G(s)$ is only proper (and not strictly proper). Then write

$$G(s) = c + G_1(s), \ G_1(s) \text{ strictly proper.}$$

Then the impulse response has the form

$$g(t) = c\delta(t) + g_1(t).$$

The theorem remains true with the second statement changed to say that $g_1(t)$ is absolutely integrable.

Finally, let's connect Lyapunov stability and BIBO stability. Consider a single-input, single-output system modeled by

$$\dot{x} = Ax + Bu, \ y = Cx + Du$$

or

$$\begin{aligned} Y(s) &= G(s)U(s) \\ G(s) &= C(sI - A)^{-1}B + D \\ &= \frac{1}{\det(sI - A)} C \text{ adj } (sI - A)B + D. \end{aligned}$$

From this last expression it is clear that the poles of $G(s)$ are contained in the set of eigenvalues of A. Thus

$$\text{Lyapunov asymptotic stability} \Rightarrow \text{BIBO stability.}$$

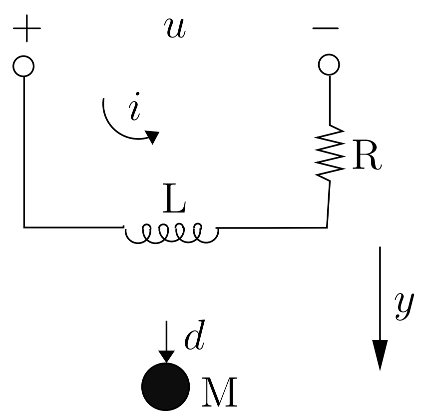
Usually, the poles of $G(s)$ are identical to the eigenvalues of A, that is, the two polynomials

$$\det(sI - A), C \text{ adj } (sI - A)B + D \det(sI - A)$$

have no common factors. In this case, the two stability concepts are equivalent. (Don't forget: We're discussing only LTI systems.)

Example 3.6.4 Maglev

Consider the problem of magnetically levitating a steel ball:



u = voltage applied to electromagnet
i = current
y = position of ball
d = possible disturbance force

The equations are

$$L\frac{di}{dt} + Ri = u, \quad M\frac{d^2y}{dt^2} = Mg + d - c\frac{i^2}{y^2}$$

where c is a constant (force of magnet on ball is proportional to i^2/y^2). The nonlinear state equations are

$$\dot{x} = f(x, u, d)$$

$$x = \begin{bmatrix} y \\ \dot{y} \\ i \end{bmatrix}, \ f = \begin{bmatrix} x_2 \\ g + \frac{d}{M} - \frac{c}{M}\frac{x_3^2}{x_1^2} \\ -\frac{R}{L}x_3 + \frac{1}{L}u \end{bmatrix}.$$

Let's linearize about $y_0 = 1,\ d_0 = 0$:

$$x_{20} = 0,\ x_{10} = 1,\ x_{30} = \sqrt{\frac{Mg}{c}},\ u_0 = R\sqrt{\frac{Mg}{c}}.$$

The linearized system is

$$\Delta\dot{x} = A\Delta x + B\Delta u + E\Delta d$$

$$\Delta y = C\Delta x$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 2g & 0 & -2\sqrt{\frac{g}{Mc}} \\ 0 & 0 & -\frac{R}{L} \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix}, \ E = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

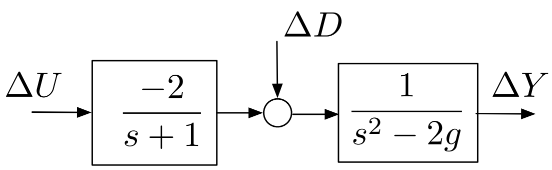
To simplify notation, let's suppose $R = L = M = 1,\ c = g$. Then

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 2g & 0 & -2 \\ 0 & 0 & -1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ E = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

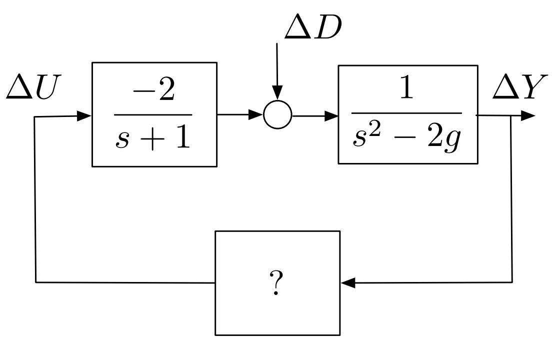
Thus

$$\begin{aligned} \Delta Y(s) &= C(sI - A)^{-1} B \Delta U(s) + C(s\mathbf{I} - A)^{-1} E \Delta D(s) \\ &= \frac{-2}{(s+1)(s^2 - 2g)} \Delta U(s) + \frac{1}{s^2 - 2g} \Delta D(s). \end{aligned}$$

The block diagram is



Note that the systems from ΔU to ΔY and ΔD to ΔY are BIBO unstable, each having a pole at $s = \sqrt{2g}$. To stabilize, we could contemplate closing the loop:



If we can design a controller (unknown box) so that the system from ΔD to ΔY is BIBO stable (all poles with Re $s < 0$), then we will have achieved a type of stability. We'll study this further in the next chapter. □

3.7 Frequency Response

Consider a single-input, single-output LTI system. It will then be modeled by

$$y = g * u \qquad \text{or} \qquad Y(s) = G(s)U(s).$$

Let us **assume** $G(s)$ is rational, proper, and has all its poles in Re $s < 0$. Then the system is BIBO stable.

The first fact we want to see is this: **Complex exponentials are eigenfunctions**.

Proof

$$u(t) = \mathrm{e}^{j\omega t},\ y(t) = \int\limits_{-\infty}^{\infty} g(t-\tau)u(\tau)d\tau$$

$$\begin{aligned}\Rightarrow y(t) &= \int\limits_{-\infty}^{\infty} g(t-\tau)\mathrm{e}^{j\omega\tau}d\tau \\ &= \int\limits_{-\infty}^{\infty} g(\tau)\mathrm{e}^{j\omega t}\mathrm{e}^{-j\omega\tau}d\tau \\ &= G(j\omega)\mathrm{e}^{j\omega t}\end{aligned}$$

□

Thus, if the input is the complex sinusoid $\mathrm{e}^{j\omega t}$, then the output is the sinusoid

$$G(j\omega)\mathrm{e}^{j\omega t} = |G(j\omega)|\mathrm{e}^{j(\omega t + \angle G(jw))}$$

which has frequency $= \omega =$ frequency of input, magnitude $= |G(j\omega)| =$ magnitude of the transfer function at $s = j\omega$, and phase $= \angle G(j\omega) =$ phase of the transfer function at $s = j\omega$.

Notice that the convolution equation for this result is

$$y(t) = \int\limits_{-\infty}^{\infty} g(t-\tau)u(\tau)d\tau,$$

that is, the sinusoidal input was applied starting at $t = -\infty$. If the time of application of the sinusoid is $t = 0$, there is a transient component in $y(t)$ too.

Next, we want to look at the special frequency response when $\omega = 0$. For this we need the final-value theorem.

Example 3.7.1 Let $y(t),\ Y(s)$ be Laplace transform pairs with $Y(s) = \dfrac{s+2}{s(s^2+s+4)}$.

This can be factored as

$$Y(s) = \frac{A}{s} + \frac{Bs+C}{s^2+s+4}.$$

Note that A equals the residue of $Y(s)$ at the pole $s = 0$:

$$A = \mathrm{Res}\,(\mathrm{Y}, 0) = \lim_{\mathrm{s}\to 0} \mathrm{s\ Y(s)} = \frac{1}{2}.$$

The inverse LT then has the form

$$Y(t) = A1_+(t) + y_1(t),$$

where $y_1(t) \to 0$ as $t \to \infty$. Thus

$$\lim_{t\to\infty} y(t) = A = \mathrm{Res}\,(\mathrm{Y}, 0).$$

□

The general result is the final-value theorem:

Theorem 3.7.1 *Suppose $Y(s)$ is rational.*

1. *If $Y(s)$ has no poles in $\Re s \geq 0$, then $y(t)$ converges to 0 as $t \to \infty$.*
2. *If $Y(s)$ has no poles in $\Re s \geq 0$ except a simple pole at $s = 0$, then $y(t)$ converges as $t \to \infty$ and $\lim_{t\to\infty} y(t)$ equals the residue of $Y(s)$ at the pole $s = 0$.*
3. *If $Y(s)$ has a repeated pole at $s = 0$, then $y(t)$ doesn't converge as $t \to \infty$.*
4. *If $Y(s)$ has a pole at $\Re s \geq 0, s \neq 0$, then $y(t)$ doesn't converge as $t \to \infty$.*

Some examples: $Y(s) = \dfrac{1}{s+1}$: final value equals 0; $Y(s) = \dfrac{2}{s(s+1)}$: final value equals 2; $Y(s) = \dfrac{1}{s^2+1}$: no final value. Remember that you have to *know* that $y(t)$ has a final value, by examining the poles of $Y(s)$, before you calculate the residue of $Y(s)$ at the pole $s = 0$ and claim that that residue equals the final value.

Return now to the setup

$$Y(s) = G(s)U(s),\ G(s) \text{ proper, no poles in Re } s \geq 0.$$

Let the input be the unit step, $u(t) = 1_+(t)$, i.e., $U(s) = \dfrac{1}{s}$. Then $Y(s) = G(s)\dfrac{1}{s}$. The final-value theorem applies to this $Y(s)$, and we get $\lim\limits_{t\to\infty} y(t) = G(0)$. For this reason, $G(0)$ is called the **DC gain** of the system.

Example 3.7.2 Using MATLAB, plot the step responses of

$$G_1(s) = \frac{20}{s^2 + 0.9s + 50}, \quad G_2(s) = \frac{-20s + 20}{s^2 + 0.9s + 50}.$$

They have the same DC gains and the same poles, but notice the big difference in transient response.

□

3.8 Problems

1. Find the transition matrix for

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

by two different methods (you may use Scilab or MATLAB).

2. Consider the system $\dot{x} = Ax$, $x(0) = x_0$. Let T be a positive sampling period. The sampled state sequence is

$$x(0), x(T), x(2T), \ldots.$$

Derive an iterative equation for obtaining the state at time $(k+1)T$ from the state at time kT.

3. Consider a system modeled by

$$\dot{x} = Ax + Bu, \quad y = Cx + Du,$$

where $\dim u = \dim y = 1$, $\dim x = 2$.

(a) Given the initial-state-responses

$$x(0) = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \implies y(t) = \mathrm{e}^{-t} - 0.5\mathrm{e}^{-2t}$$

$$x(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \implies y(t) = -0.5\mathrm{e}^{-t} - \mathrm{e}^{-2t},$$

find the initial-state-response for $x(0) = \begin{bmatrix} 2 \\ 0.5 \end{bmatrix}$.

(b) Now suppose

$$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ u \text{ a unit step} \implies y(t) = 0.5 - 0.5\mathrm{e}^{-t} + \mathrm{e}^{-2t}$$

$$x(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \ u \text{ a unit step} \implies y(t) = 0.5 - \mathrm{e}^{-t} + 1.5\mathrm{e}^{-2t}.$$

Find $y(t)$ when u is a unit step and $x(0) = 0$.

4. This problem requires formal logic.

(a) Write the definition that the function $f : \mathbb{R} \to \mathbb{R}$ is not continuous at $t = 0$.

(b) Prove that the unit step $1_+(t)$ satisfies the previous definition.

(c) Write the definition that the equilibrium 0 of the system $\dot{x} = f(x)$ is not stable.

(d) Prove that the equilibrium point 0 of the system $\dot{x} = x$ satisfies the previous definition.

5. Consider $\dot{x} = Ax$ with

$$A = \begin{bmatrix} 0 & -2 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Is the origin asymptotically stable? Find an $x(0) \neq 0$ such that $x(t) \to 0$ as $t \to \infty$.

6. Consider $\dot{x} = Ax$.

(a) Prove that the origin is stable if

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

(b) Prove that the origin is unstable if

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

7. Consider the cart-spring system. Its equation of motion is

$$M\ddot{y} + Ky = 0,$$

where $M > 0$, $K > 0$. Take the natural state and prove that the origin is stable. (For an arbitrary ε, you must give an explicit δ.)

8. Consider the convolution equation $y = g \star u$, where $g(t)$ is absolutely integrable, that is, the system is BIBO stable.. The inequality

$$\|y\|_\infty \leq \|u\|_\infty \int_0^\infty |g(t)|dt$$

is derived in the course notes. Show that this is the best bound; that is, construct a nonzero input $u(t)$ for which the inequality is an equality.

9. Consider the maglev example, Example 3.6.4. Is there a constant gain to put in the unknown box so that the system from ΔD to ΔY is BIBO stable?

10. Give an example of a continuously differentiable, bounded, real-valued function defined on the open interval $(0, 1)$ whose derivative is not bounded on that interval. What can you conclude about BIBO stability of the differentiator? Discuss stability of the differentiator in the sense of Lyapunov.

11. The transfer function of an LC circuit is $G(s) = 1/(LCs^2 + 1)$.

    (a) Is the output bounded if the input is the unit step?

    (b) Prove that the circuit is not a BIBO stable system.

12. A rubber ball is tossed straight into the air, rises, then falls and bounces from the floor, rises, falls, and bounces again, and so on. Let c denote the coefficient of restitution, that is, the ratio of the velocity just after a bounce to the velocity just before the bounce. Thus $0 < c < 1$. Neglecting air resistance, show that there are an infinite number of bounces in a finite time interval.

    Hint: Assume the ball is a point mass. Let $x(t)$ denote the height of the ball above the floor at time t. Then $x(0) = 0$, $\dot{x}(0) > 0$. Model the system before the first bounce and calculate the time of the first bounce. Then specify the values of $x, \dot{x}$ just after the first bounce. And so on.

13. The linear system $\dot{x} = Ax$ can have more than one equilibrium point.

    (a) Characterize the set of equilibrium points. Give an example A for which there's more than one.

    (b) Prove that if one equilibrium point is stable, they all are.

Chapter 4

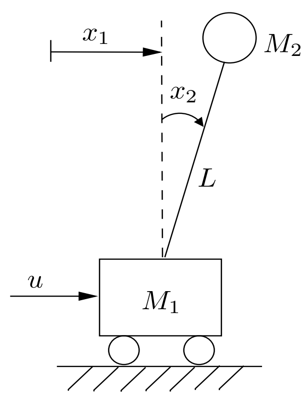
Feedback Control Theory

Feedback is a miraculous invention. In this chapter we'll see why.

4.1 Closing the Loop

As usual, we start with an example.

Example 4.1.1 linearized cart-pendulum



The figure defines x_1, x_2. Now define $x_3 = \dot{x}_1,\ x_4 = \dot{x}_2$. Take $M_1 = 1$ Kgm, $M_2 = 2$ Kgm, $L = 1$ m, $g = 9.8$ m/s^2. Then the state model is

$$\dot{x} = Ax + Bu,\ A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -19.6 & 0 & 0 \\ 0 & 29.4 & 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

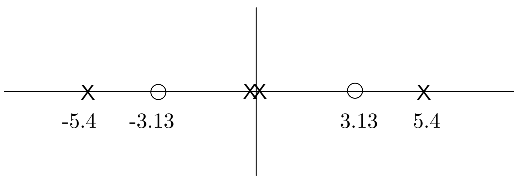
Let's suppose we measure the cart position only: $y = x_1$. Then

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}.$$

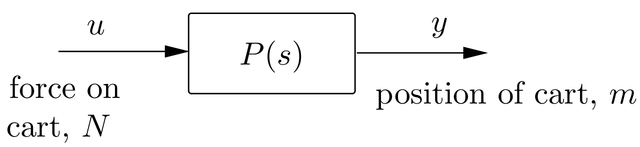
The transfer function from u to y is

$$P(s) = \frac{s^2 - 9.8}{s^2(s^2 - 29.4)}.$$

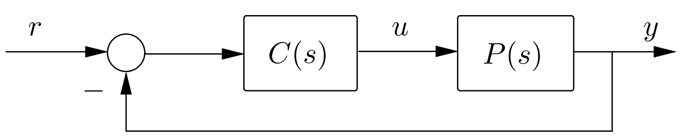
The poles and zeros of $P(s)$ are



Having three poles in Re $s \geq 0$, the plant is quite unstable. The right half-plane zero doesn't contribute to the degree of instability, but, as we shall see, it does make the plant quite difficult to control. The block diagram of the plant by itself is



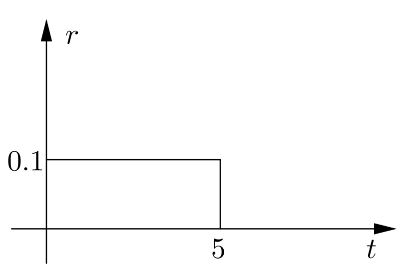
Let us try to stabilize the plant by feedback:



Here r is the reference position of the cart and $C(s)$ is the transfer function of the controller to be designed. One controller that does in fact stabilize is

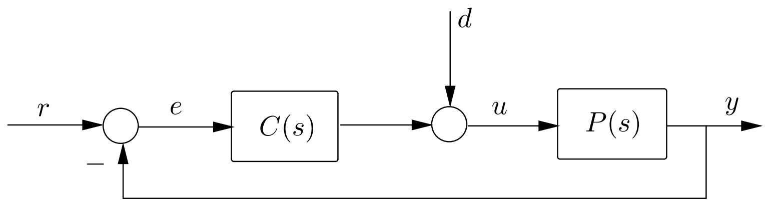
$$C(s) = \frac{10395s^3 + 54126s^2 - 13375s - 6687}{s^4 + 32s^3 + 477s^2 - 5870s - 22170}.$$

You're invited to simulate the closed-loop system; for example, let r be the input



which corresponds to a command that the cart move right 0.1 m for 5 seconds, then return to its original position. Plot x_1 and x_2, the cart and pendulum positions. □

Our objective in this section is to define what it means for the following feedback system to be stable:



The notation is this:

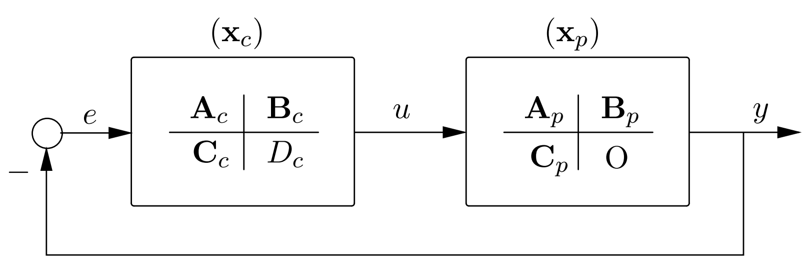
systems : $P(s)$, plant transfer function
$C(s)$, controller transfer function

signals : $r(t)$, reference (or command) input
$e(t)$, tracking error
$d(t)$, disturbance
$u(t)$, plant input
$y(t)$, plant output.

We shall **assume throughout** that $P(s),\ C(s)$ are rational, $C(s)$ is proper, and $P(s)$ is strictly proper.

Internal Stability

For this concept, set $r = d = 0$ and bring in state models for P and C:



The closed-loop equations are

$$\begin{aligned} \dot{x}_p &= A_p x_p + B_p u \\ u &= C_c x_c + D_c e \\ \dot{x}_c &= A_c x_c + B_c e \\ e &= -C_p x_p. \end{aligned}$$

Defining the closed-loop state $x_{cl} = (x_p, x_x)$, we get simply

$$\dot{x}_{cl} = A_{cl}x_{cl}, \ A_{cl} = \begin{bmatrix} A_p - B_pD_cC_p & B_pC_c \\ -B_cC_p & A_c \end{bmatrix}.$$

Internal Stability is defined to mean that the origin is an asymptotically stable equilibrium point, that is, that all the eigenvalues of A_{cl} have Re $\lambda < 0$. Thus the concept means this: With no inputs applied (i.e., $r = d = 0$), the internal states $x_p(t), x_c(t)$ will converge to zero for every initial state $x_p(0), \ x_c(0)$.

Example 4.1.2

Take $C(s) = 2, \ P(s) = 1/(s-1)$. Then

$$A_p = 1, \ B_p = 1, \ C_p = 1; \ D_c = 2$$

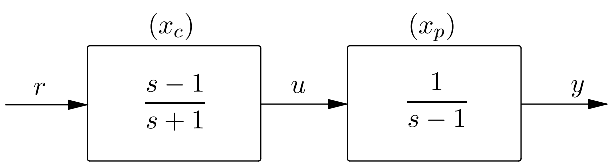
and

$$A_{cl} = A_p - B_pD_cC_p = -1.$$

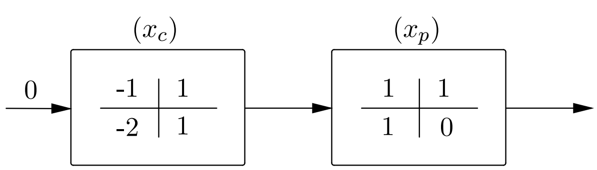
Thus the unstable plant $\dfrac{1}{s-1}$ is internally stabilized by unity feedback with the pure-gain controller 2. □

Example 4.1.3

Let's see that $1/(s-1)$ cannot be internally stabilized by cancelling the unstable pole:



The transfer function from r to y equals $\dfrac{1}{s+1}$. Hence the system from r to y **is** BIBO stable. But with $r = 0$, the state model is



Thus

$$\begin{aligned} \dot{x}_p &= x_p - 2x_c \\ \dot{x}_c &= -x_c \end{aligned}$$

$$\dot{x}_{cl} = A_{cl}x_{cl}, \ A_{cl} = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix} = \text{unstable}.$$

Clearly $x_p(t)$ doesn't converge to 0 if $x_p(0) \neq 0$. □

Input-Output Stability

Now we turn to the second way of thinking about stability of the feedback loop. First, let's solve for the outputs of the summing junctions:

$$E = R - PU$$
$$U = D + CE$$
$$\begin{bmatrix} 1 & P \\ -C & 1 \end{bmatrix} \begin{bmatrix} E \\ U \end{bmatrix} = \begin{bmatrix} R \\ D \end{bmatrix}.$$

In view of our standing assumptions (P strictly proper, C proper), the determinant of

$$\begin{bmatrix} 1 & P \\ -C & 1 \end{bmatrix}$$

is not identically zero (why?). Thus

$$\begin{bmatrix} E \\ U \end{bmatrix} = \begin{bmatrix} 1 & P \\ -C & 1 \end{bmatrix}^{-1} \begin{bmatrix} R \\ D \end{bmatrix} = \begin{bmatrix} \dfrac{1}{1+PC} & \dfrac{-P}{1+PC} \\ \dfrac{C}{1+PC} & \dfrac{1}{1+PC} \end{bmatrix} \begin{bmatrix} R \\ D \end{bmatrix}$$

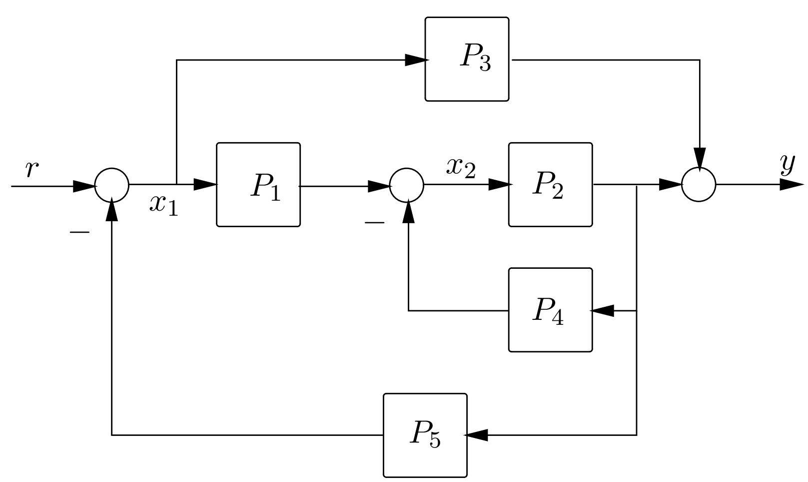
$$Y = PU = \frac{PC}{1+PC}R + \frac{P}{1+PC}D.$$

We just derived the following closed-loop transfer functions:

$$R \text{ to } E : \frac{1}{1+PC}\ , \qquad R \text{ to } U : \frac{C}{1+PC}\ , \qquad R \text{ to } Y : \frac{PC}{1+PC}$$
$$D \text{ to } E : \frac{-P}{1+PC}\ , \qquad D \text{ to } U : \frac{1}{1+PC}\ , \qquad D \text{ to } Y : \frac{P}{1+PC}\ .$$

The above method of finding closed-loop transfer functions works in general.

Example 4.1.4



Let us find the transfer function from r to y. Label the outputs of the summing junctions, as shown. Write the equations at the summing junctions:

$$\begin{aligned} X_1 &= R - P_2P_5X_2 \\ X_2 &= P_1X_1 - P_2P_4X_2 \\ Y &= P_3X_1 + P_2X_2. \end{aligned}$$

Assemble as

$$\begin{bmatrix} 1 & P_2P_5 & 0 \\ -P_1 & 1+P_2P_4 & 0 \\ -P_3 & -P_2 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ Y \end{bmatrix} = \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix}.$$

Solve for Y by Cramer's rule:

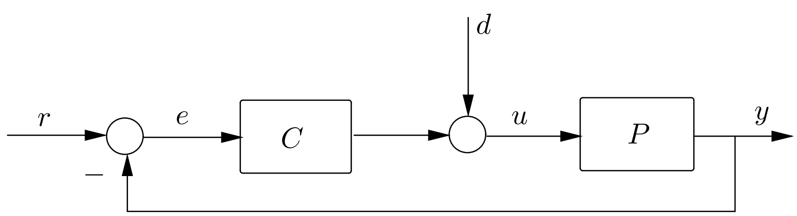
$$Y = \frac{\det \begin{bmatrix} 1 & P_2P_5 & R \\ -P_1 & 1+P_2P_4 & 0 \\ -P_3 & -P_2 & 0 \end{bmatrix}}{\det \begin{bmatrix} 1 & P_2P_5 & 0 \\ -P_1 & 1+P_2P_4 & 0 \\ -P_3 & -P_2 & 1 \end{bmatrix}}.$$

Simplify:

$$\frac{Y}{R} = \frac{P_1P_2 + P_3(1+P_2P_4)}{1+P_2P_4+P_1P_2P_5}.$$

□

Now back to



We say the feedback system is **input-output stable** provided e, u, and y are bounded signals whenever r and d are bounded signals; briefly, the system from (r,d) to (e,u,y) is BIBO stable. This is equivalent to saying that the 6 transfer funtions from (r,d) to (e,u,y) are stable, in the sense that all poles are in Re $s < 0$. It suffices to look at the 4 transfer functions from (r,d) to (e,u), namely,

$$\begin{bmatrix} \dfrac{1}{1+PC} & \dfrac{-P}{1+PC} \\ \dfrac{C}{1+PC} & \dfrac{1}{1+PC} \end{bmatrix}.$$

(Proof: If r and e are bounded, so is $y = r - e$.)

Example 4.1.5

$$P(s) = \frac{1}{s^2-1},\ C(s) = \frac{s-1}{s+1}$$

The 4 transfer functions are

$$\begin{bmatrix} E \\ U \end{bmatrix} = \begin{bmatrix} \dfrac{(s+1)^2}{s^2+2s+2} & \dfrac{s+1}{(s-1)(s^2+2s+2)} \\ \\ \dfrac{(s+1)(s-1)}{s^2+2s+2} & \dfrac{(s+1)^2}{s^2+2s+2} \end{bmatrix} \begin{bmatrix} R \\ D \end{bmatrix}.$$

Three of these are stable; the one from D to E is not. Consequently, the feedback system is **not input-output stable**. This is in spite of the fact that a bounded r produces a bounded y. Notice that the problem here is that C cancels an unstable pole of P. As we'll see, that isn't allowed.

$\square$

Example 4.1.6

$$P(s) = \frac{1}{s-1},\ C(s) = k$$

The feedback system is input-output stable iff $k > 1$ (check). $\square$

We now look at two ways to test feedback IO stability. The first is in terms of numerator and denominator polynomials:

$$P = \frac{N_p}{D_p},\ C = \frac{N_c}{D_c}.$$

We assume $(N_p,\ D_p)$ are coprime, i.e., have no common factors, and $(N_c,\ D_c)$ are coprime too. The **characteristic polynomial** of the feedback system is defined to be $N_pN_c + D_pD_c$.

Example 4.1.7

$$P(s) = \frac{1}{s^2-1},\ C(s) = \frac{s-1}{s+1}$$

The characteristic polynomial is

$$s - 1 + (s^2-1)(s+1) = (s-1)(s^2+2s+2).$$

$\square$

Theorem 4.1.1 *The feedback system is input-output stable iff the char poly has no roots in* Re $s \geq 0$.

Proof We have

$$\begin{bmatrix} \frac{1}{1+PC} & \frac{-P}{1+PC} \\ \frac{C}{1+PC} & \frac{1}{1+PC} \end{bmatrix} = \frac{1}{N_pN_c + D_pD_c} \begin{bmatrix} D_pD_c & -N_pD_c \\ N_cD_p & D_pD_c \end{bmatrix}. \tag{1}$$

($\Leftarrow$ Sufficiency) If $N_pN_c + D_pD_c$ has no roots in Re $s \geq 0$, then the four transfer functions on the left-hand side of (1) have no poles in Re $s \geq 0$, and hence they are stable.
($\Rightarrow$ Necessity) Conversely, assume the feedback system is stable, that is, the four transfer functions on the left-hand side of (1) are stable. To conclude that $N_pN_c + D_pD_c$ has no roots in Re $s \geq 0$, we must show that the polynomial $N_pN_c + D_pD_c$ does not have a common factor with all four numerators in (1), namely, $D_pD_c,\ N_pD_c,\ N_cD_p$. That is, we must show that the four polynomials

$$N_pN_c + D_pD_c,\ D_pD_c,\ N_pD_c,\ N_cD_p$$

do not have a common root. This part is left for you. □

The second way to test feedback IO stability is as follows.

Theorem 4.1.2 *The feedback system is input-output stable iff 1) the transfer function* $1 + PC$ *has no zeros in* Re $s \geq 0$*, and 2) the product* PC *has no pole-zero cancellations in* Re $s \geq 0$*.*

(Proof will be an exercise.)

Example 4.1.8

$$P(s) = \frac{1}{s^2 - 1},\ C(s) = \frac{s-1}{s+1}$$

Check that 1) holds but 2) does not. □

The Routh-Hurwitz Criterion

In practice, one checks feedback stability using MATLAB to calculate the eigenvalues of A_{cl} or the roots of the characteristic polynomial. However, it is sometimes useful, and also of historical interest, to have an easy test for simple cases.

Consider a general characteristic polynomial

$$p(s) = s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0, \quad a_i \text{ real.}$$

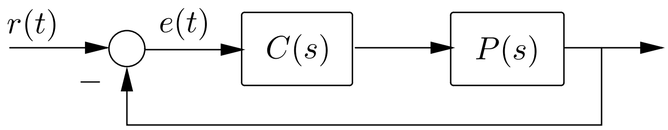
Let's say $p(s)$ is **stable** if all its roots have Re $s < 0$. The Routh-Hurwitz criterion is an algebraic test for $p(s)$ to be stable, without having to calculate the roots. Instead of studying the complete criterion, here are the results for $n = 1, 2, 3$:

1. $p(s) = s + a_0 : p(s)$ is stable (obviously) iff $a_0 > 0$
2. $p(s) = s^2 + a_1 s + a_0 : p(s)$ is stable iff $(\forall i)\ a_i > 0$
3. $p(s) = s^3 + a_2 s^2 + a_1 s + a_0 :\ p(s)$ is stable $\Longleftrightarrow$ $(\forall i)\ a_i > 0$ and $a_1 a_2 > a_0$.

4.2 The Internal Model Principle

A thermostat-controlled temperature regulator controls temperature to a prescribed setpoint. Likewise, cruise control in a car regulates the speed to a prescribed setpoint. What is the principle underlying their operation? The answer lies in the final value theorem (FVT).

Example 4.2.1



Take the controller and plant

$$C(s) = \frac{1}{s}, \ P(s) = \frac{1}{s+1}.$$

Let r be a constant, $r(t) = r_0$. Then we have

$$\begin{aligned} E(s) &= \frac{1}{1+P(s)C(s)}R(s) \\ &= \frac{s(s+1)}{s^2+s+1}\frac{r_0}{s} \\ &= \frac{s+1}{s^2+s+1}r_0 \end{aligned}$$

The FVT applies to $E(s)$, and $e(t) \to 0$ as $t \to \infty$. Thus the feedback system provides **perfect** asymtotic tracking of a step input! How it works: $C(s)$ contains an internal model of $R(s)$ (i.e., an integrator); closing the loop creates a **zero** from $R(s)$ to $E(s)$ exactly to cancel the unstable pole of $R(s)$. (This isn't illegal pole-zero cancellation.) □

Example 4.2.2

This time take

$$C(s) = \frac{1}{s}, \ P(s) = \frac{2s+1}{s(s+1)}$$

and take r to be a ramp, $r(t) = r_0 t$. Then $R(s) = r_0/s^2$ and so

$$E(s) = \frac{s+1}{s^3+s^2+2s+1}r_0.$$

Again $e(t) \to 0$; perfect tracking of a ramp. Here $C(s)$ and $P(s)$ together provide the internal model, a double integrator. □

Let's generalize:

Theorem 4.2.1 *Assume $P(s)$ is strictly proper, $C(s)$ is proper, and the feedback system is stable. If $P(s)C(s)$ contains an internal model of the unstable part of $R(s)$, then perfect asymptotic tracking occurs, i.e., $e(t) \to 0$.*

Example 4.2.3

$$R(s) = \frac{r_0}{s^2+1}, \ P(s) = \frac{1}{s+1}$$

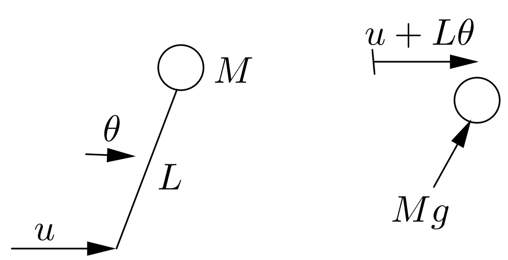
Design $C(s)$ to achieve perfect asymptotic tracking of the sinusoid $r(t)$, as follows. From the theorem, we should try something of the form

$$C(s) = \frac{1}{s^2+1}C_1(s),$$

that is, we embed an internal model in $C(s)$, and allow an additional factor to achieve feedback stability. You can check that $C_1(s) = s$ works. Notice that we have effectively created a notch filter from R to E, a notch filter with zeros at $s = \pm j$. □

Example 4.2.4

An inverted pendulum balanced on your hand.



The equation is

$$\ddot{u} + L\ddot{\theta} = Mg\theta.$$

Thus

$$s^2U + s^2L\theta = Mg\theta.$$

So the transfer function from u to θ equals

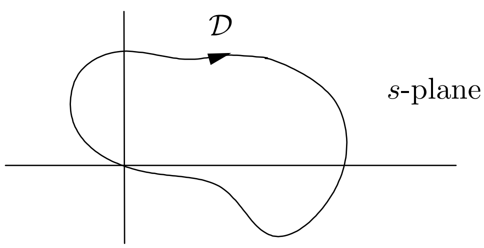
$$\frac{-s^2}{Ls^2 - Mg}.$$

Step tracking and internal stability are not simultaneously achievable. □

4.3 Principle of the Argument

This section and the following two develop the Nyquist stability criterion.

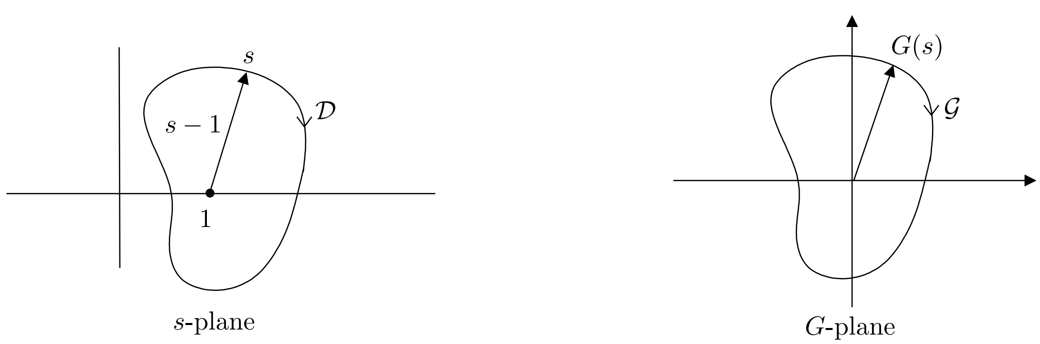
Consider a closed path $\mathcal{D}$ in the s-plane, with no self-intersections and with negative, i.e., clockwise (CW) orientation.



Let $G(s)$ be a rational function. As s goes once around $\mathcal{D}$ from any starting point, the point $G(s)$ traces out a closed curve denoted $\mathcal{G}$, the image of $\mathcal{D}$ under $G(s)$.

Example 4.3.1

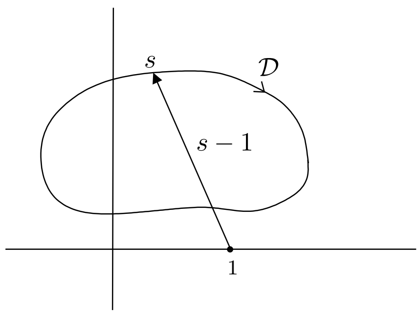
$G(s) = s - 1$

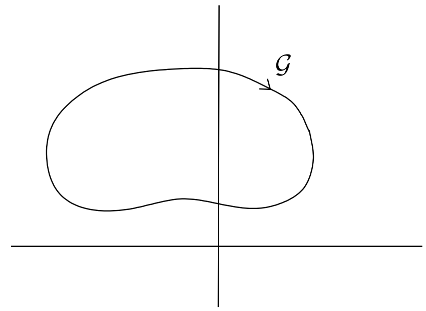


$\mathcal{D}$ encircles one zero of $G(s)$; $\mathcal{G}$ encircles the origin once CW. □

Example 4.3.2

$G(s) = s - 1$

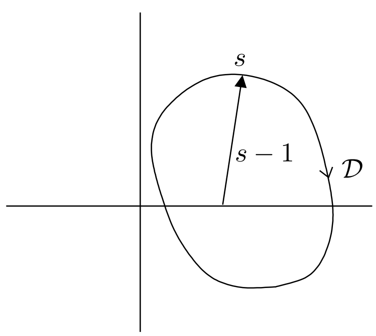


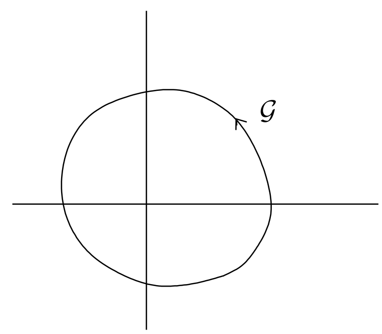


$\mathcal{D}$ encircles no zero of $G(s)$; $\mathcal{G}$ has no encirclements of the origin. $\square$

Example 4.3.3

$G(s) = \dfrac{1}{s-1}$





$\mathcal{D}$ encircles one pole; $\mathcal{G}$ encircles the origin once counterclockwise (CCW). $\square$

Theorem 4.3.1 *Suppose $G(s)$ has no poles or zeros on $\mathcal{D}$, but $\mathcal{D}$ encloses n poles and m zeros of $G(s)$. Then $\mathcal{G}$ encircles the origin exactly $n-m$ times CCW.*

Proof Write

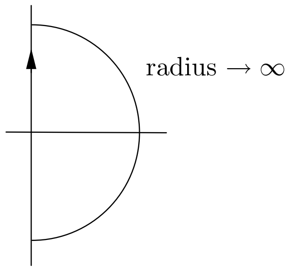
$$G(s) = K\frac{\prod_i (s-z_i)}{\prod_i (s-p_i)}$$

with K a real gain, $\{z_i\}$ the zeros, and $\{p_i\}$ the poles. Then for every s on $\mathcal{D}$

$$\arg(G(s)) = \arg(K) + \Sigma\ \arg(s-z_i) - \Sigma\ \arg(s-p_i).$$

If z_i is enclosed by $\mathcal{D}$, the net change in $\arg(s-z_i)$ is -2π; otherwise the net change is 0. Hence the net change in $\arg(G(s))$ equals $m(-2\pi) - n(-2\pi)$, which equals $(n-m)2\pi$. $\square$

The special $\mathcal{D}$ we use for the Nyquist contour is

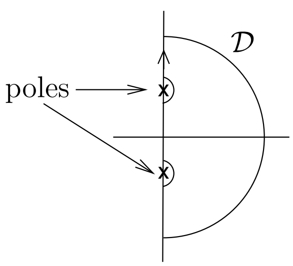


Then $\mathcal{G}$ is called the **Nyquist plot** of $G(s)$. If $G(s)$ has no poles or zeros on $\mathcal{D}$, then the Nyquist plot encircles the origin exactly $n - m$ times CCW, where n equals the number of poles of $G(s)$ in Re $s > 0$ and m equals the number of zeros of $G(s)$ in Re $s > 0$. From this follows

Theorem 4.3.2 *Suppose $G(s)$ has no poles on $\mathcal{D}$. Then $G(s)$ has no zeros in* Re $s \geq 0 \Leftrightarrow \mathcal{G}$ *doesn't pass through the origin and encircles it exactly n times CCW, where n equals the number of poles in* Re $s > 0$.

Note that $G(s)$ has no poles on $\mathcal{D}$ iff $G(s)$ is proper and $G(s)$ has no poles on the imaginary axis, and $G(s)$ has no zeros on $\mathcal{D}$ iff $G(s)$ is not strictly proper and $G(s)$ has no zeros on the imaginary axis.

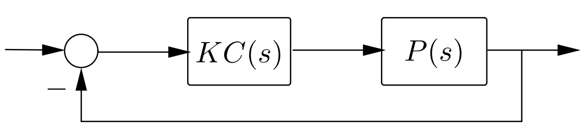
In our application, if $G(s)$ actually does have poles on the imaginary axis, we have to indent around them. You can indent either to the left or to the right; we shall always indent to the right:



4.4 Nyquist Stability Criterion (1932)

This beautiful stability criterion is due to a mathematician, Harry Nyquist, who worked at AT&T and was asked by some engineers to study the problem of stability in feedback amplifiers.

The setup is



where K is a real gain. We're after a graphical test for stability involving the Nyquist plot of $P(s)C(s)$.

The **assumptions** are that $P(s)$, $C(s)$ are proper, with $P(s)C(s)$ strictly proper, the product $P(s)C(s)$ has no pole-zero cancellations in $\text{Re } s \geq 0$, and $K \neq 0$.

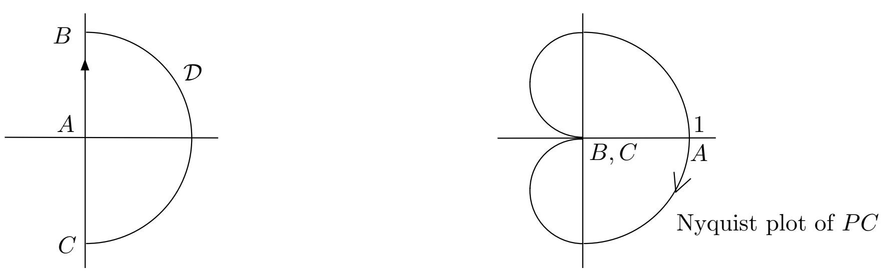
Theorem 4.4.1 *Let n denote the number of poles of $P(s)C(s)$ in $\text{Re } s > 0$. Construct the Nyquist plot of $P(s)C(s)$, indenting to the right around poles on the imaginary axis. Then the feedback system is stable iff the Nyquist plot doesn't pass through $-\frac{1}{K}$ and encircles it exactly n times CCW.*

Proof Define $G(s) = 1 + KP(s)C(s)$. By Theorem 4.3.2, the feedback system is stable iff $G(s)$ has no zeros in $\text{Re } s \geq 0$. Note that $G(s)$ and $P(s)C(s)$ have the same poles in $\text{Re } s \geq 0$, so $G(s)$ has precisely n there. Since $\mathcal{D}$ indents around poles of $G(s)$ on the imaginary axis and since $G(s)$ is proper, $G(s)$ has no poles on $\mathcal{D}$. Thus by Theorem 4.1.2, the feedback system is stable $\Leftrightarrow$ the Nyquist plot of $G(s)$ doesn't pass through 0 and encircles it exactly n times CCW. Since $P(s)C(s) = \dfrac{1}{K}G(s) - \dfrac{1}{K}$, this latter condition is equivalent to: the Nyquist plot of $P(s)C(s)$ doesn't pass through $-\frac{1}{K}$ and encircles it exactly n times CCW. $\square$

4.5 Examples

Example 4.5.1 $PC(s) = \dfrac{1}{(s+1)^2}$

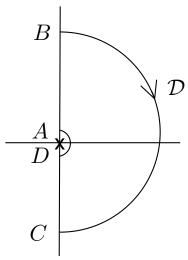
$$\text{Re } PC(j\omega) = \frac{1-\omega^2}{(1-\omega^2)^2 + (2\omega)^2}, \quad \text{Im } PC(j\omega) = \frac{-2\omega}{(1-\omega^2)^2 + (2\omega)^2}$$

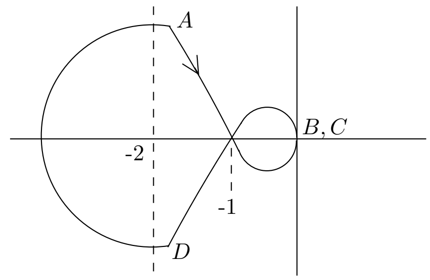


$n = 0$. Therefore the feedback system is stable iff $-1/K < 0$ or $-1/K > 1$; that is, $K > 0$ or $-1 < K < 0$; that is, $K > -1, K \neq 0$. The condition $K \neq 0$ is ruled out by assumption. But we can check now that the feedback system actually is stable for $K = 0$. So finally the condition is $K > -1$. $\square$

Example 4.5.2 $PC(s) = \dfrac{s+1}{s(s-1)}$

$$\text{Re } PC(j\omega) = -\frac{2}{\omega^2+1}, \quad \text{Im } PC(j\omega) = \frac{1-\omega^2}{\omega(\omega^2+1)}$$

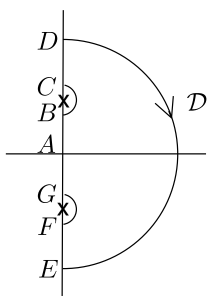


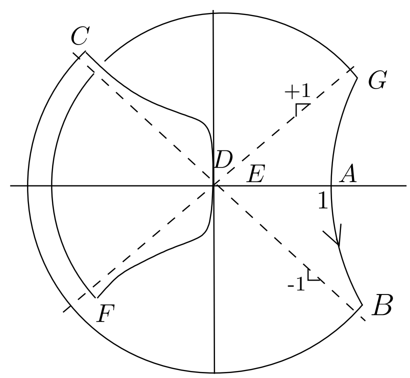


$n = 1$. Feedback stability iff $-1 < -1/K < 0$; equivalently, $K > 1$. □

Example 4.5.3 $PC(s) = \dfrac{1}{(s+1)(s^2+1)}$

$$\text{Re } PC(j\omega) = \frac{1}{1-\omega^4}, \ \text{Im } PC(j\omega) = \frac{-\omega}{1-\omega^4}$$





$n = 0$. Feedback stability iff $-1/K > 1$; equivalently, $-1 < K < 0$. □
Shortcuts:

1. Principle of conformal mapping.
2. If an indentation in $\mathcal{D}$ bypasses a pole of multiplicity k, the Nyquist plot will go through $k\pi$ radians.

4.6 Stability and Bode Plots

Control design is typically done in the frequency domain using Bode plots. For this reason it's useful to translate the Nyquist criterion into a condition on Bode plots.

First, let's review the drawing of Bode plots. (You're likely to be unfamiliar with these when the plant has right half-plane poles or zeros.) We consider only rational $G(s)$ with real coefficients. Then $G(s)$ has a numerator and denominator, each of which can be factored into terms of the following forms:

1. gain: K

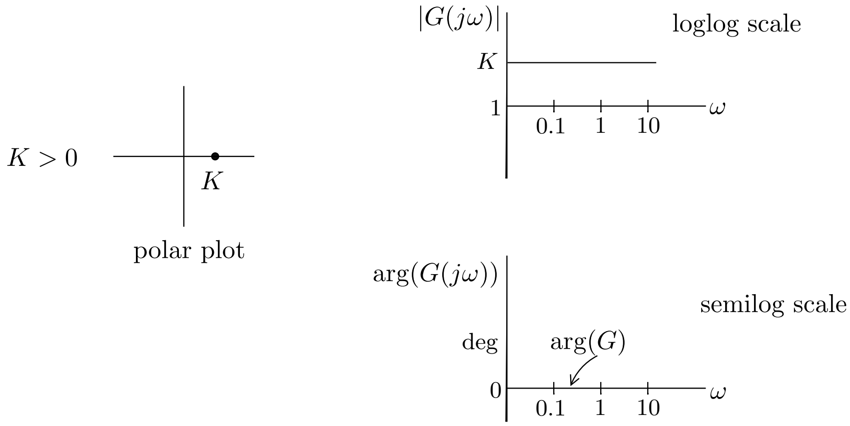
2. pole or zero at $s = 0 : s^n$
3. real nonzero pole or zero : $\tau s \pm 1$
4. complex conjugate pole or zero : $\dfrac{1}{\omega_n^2}(s^2 \pm 2\zeta\omega_n s + \omega_n^2),\ \omega_n > 0, 0 \le \zeta < 1$

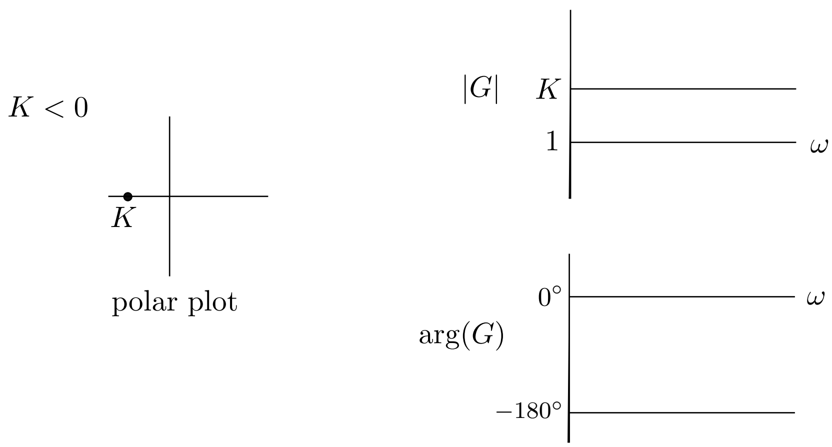
For example

$$\begin{aligned} G(s) &= \frac{40s^2(s-2)}{(s-5)(s^2+4s+100)} \\ &= \frac{40 \times 2}{5 \times 100} \frac{s^2(\frac{1}{2}s - 1)}{(\frac{1}{5}s - 1)[\frac{1}{100}(s^2+4s+100)]}. \end{aligned}$$

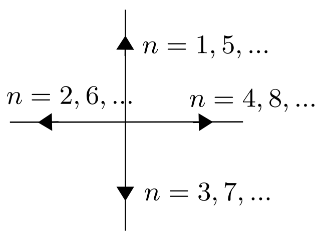
As an intermediate step, we introduce the **polar plot** : Im$G(j\omega)$ vs Re $G(j\omega)$ as $\omega : 0 \to \infty$. We now look at polar and Bode plots for each of the four terms.

1. $G(s) = K$

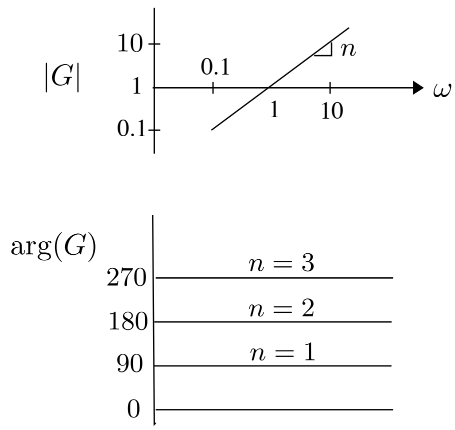

$K > 0$
K
polar plot
$|G(j\omega)|$
loglog scale
K
1
0.1
1
10
ω
$\arg(G(j\omega))$
semilog scale
deg
$\arg(G)$
0
0.1
1
10
ω


$K < 0$
K
polar plot
$|G|$
K
1
ω
$0°$
ω
$\arg(G)$
$-180°$

2. $G(s) = s^n, \ n \geq 1$



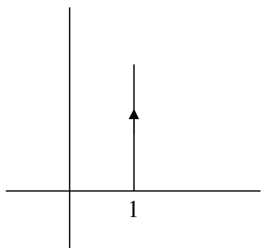
polar plot



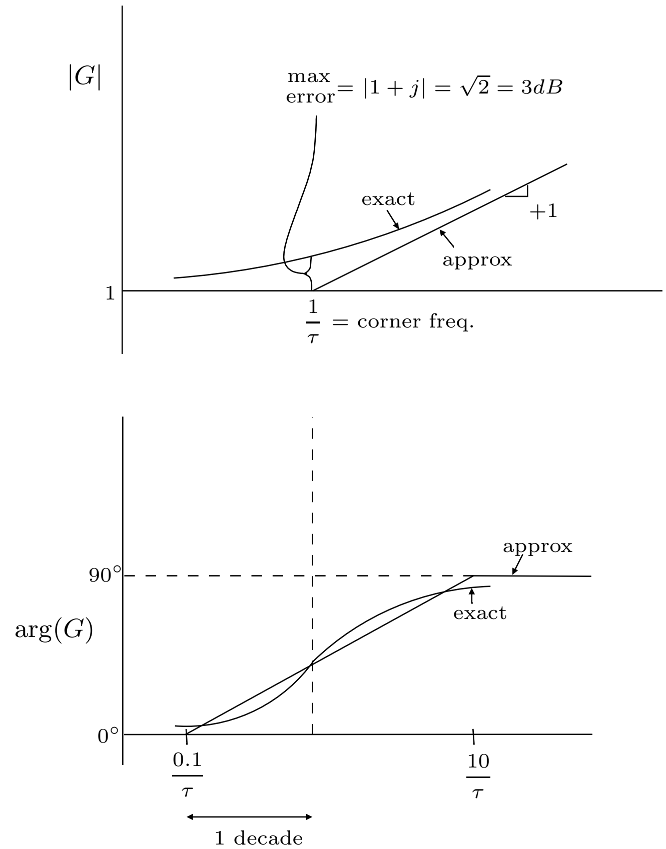
3. $G(s) = \tau s + 1,\ \tau > 0$

LHP zero

$G(j\omega) = 1 + j\tau\omega$



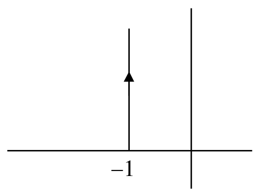
polar plot



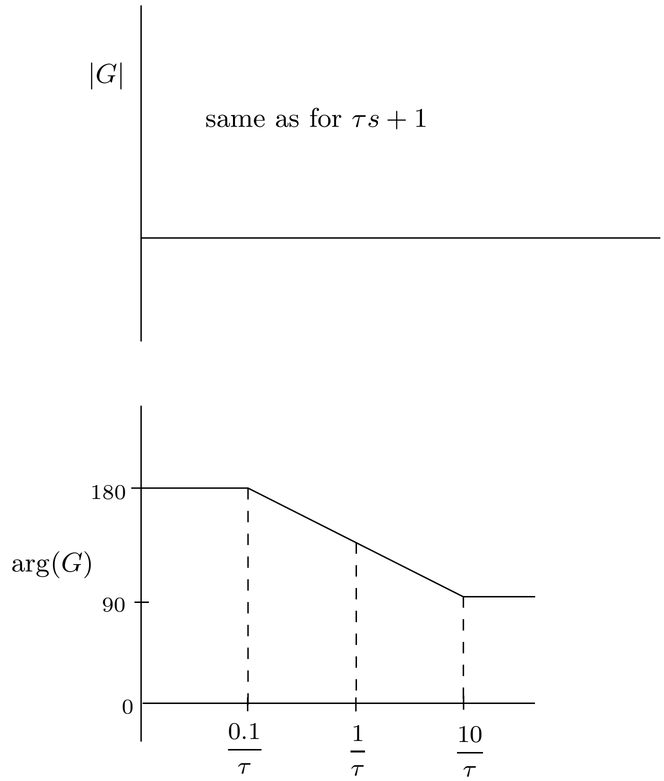
$G(s) = \tau s - 1,\ \tau > 0$

RHP zero

$G(j\omega) = -1 + j\tau\omega$

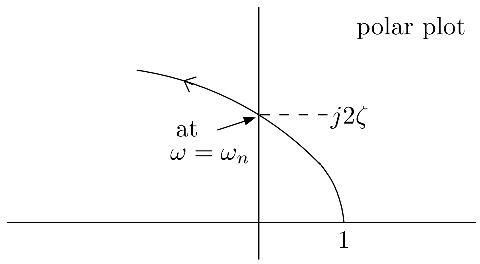


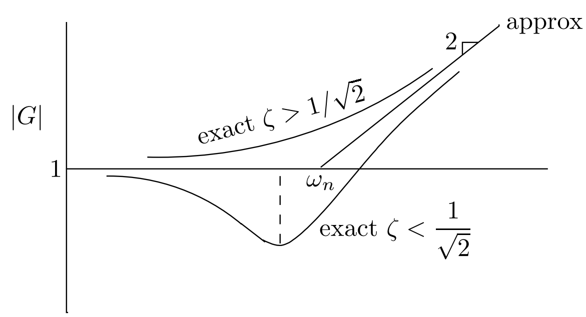
polar plot

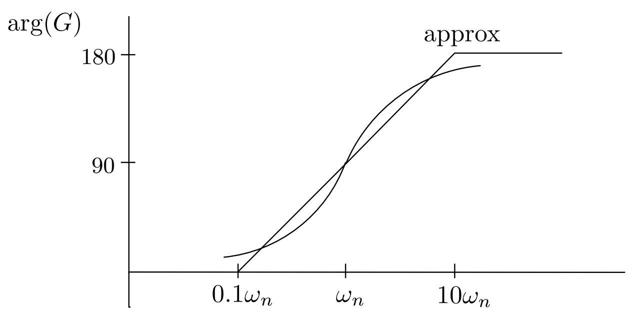


4. $G(s) = \frac{1}{\omega_n^2}\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right), \ \omega_n > 0, \ 0 < \zeta < 1$

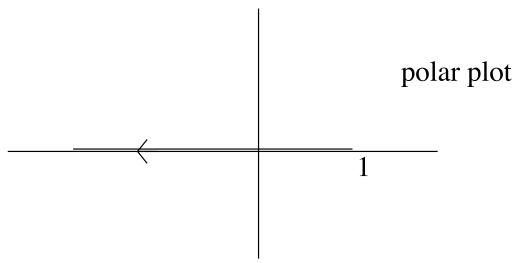
$$G(j\omega) = \frac{1}{\omega_n^2}\left[(\omega_n^2 - \omega^2) + j2\zeta\omega_n\omega\right]$$

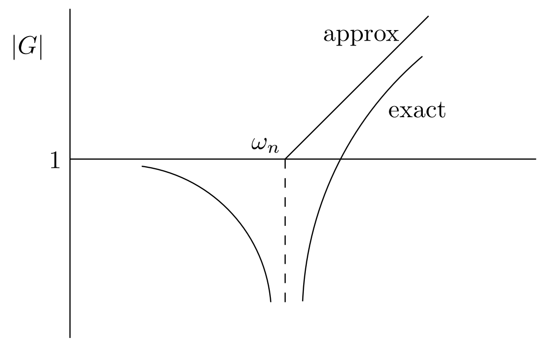


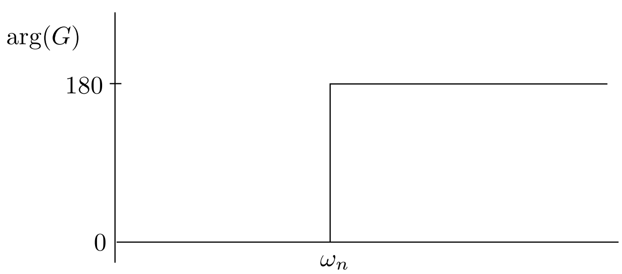




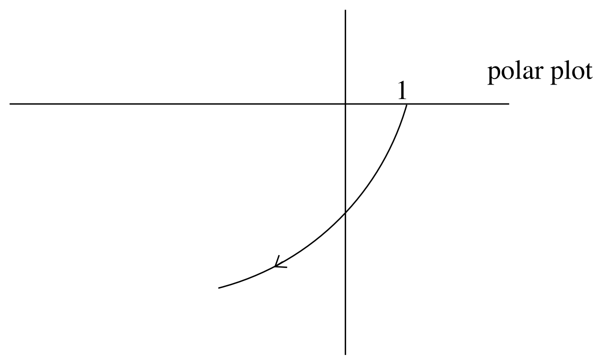
$$G(s) = \frac{1}{\omega_n^2}(s^2 + \omega_n^2),\ \omega_n > 0 \text{ (i.e.,} \zeta = 0)$$

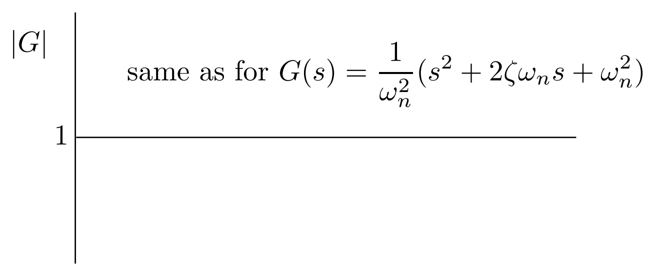


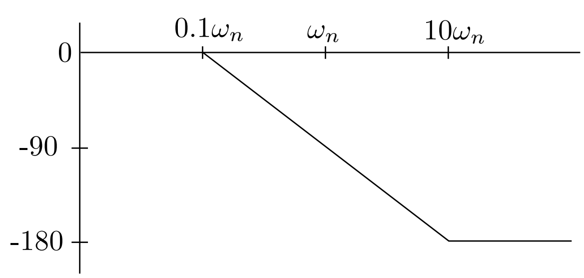




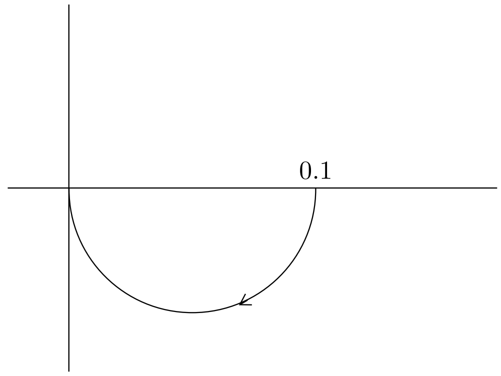
$$G(s) = \frac{1}{\omega_n^2}(s^2 - 2\zeta\omega_n s + \omega_n^2),\ \omega_n > 0,\ 0 < \zeta < 1$$





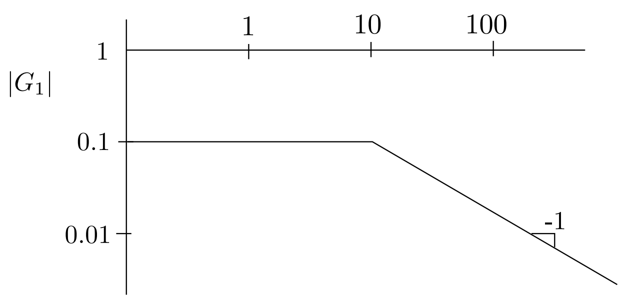


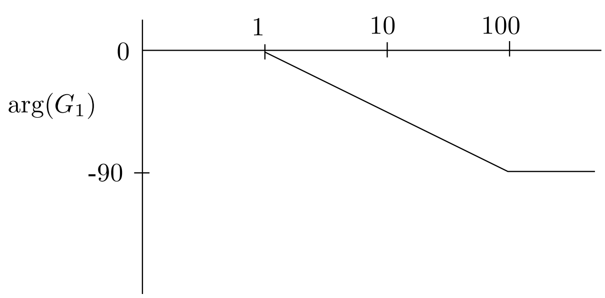
Example 4.6.1 $G_1(s) = \dfrac{1}{s+10} = 0.1\dfrac{1}{0.1s+1}$ minimum phase TF



polar plot is a semicircle:

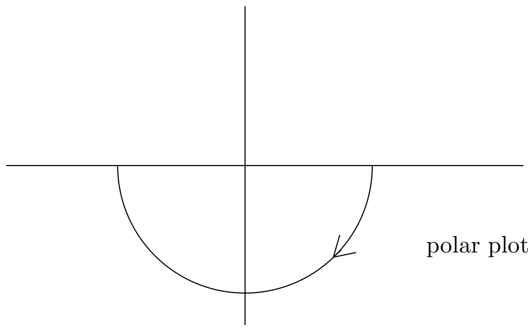
$|G(j\omega) - 0.05| = 0.05$

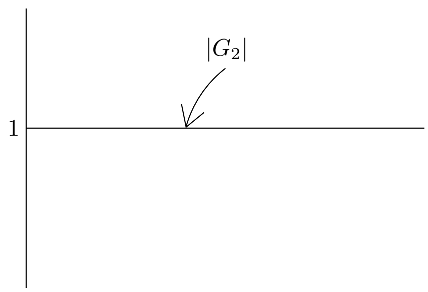


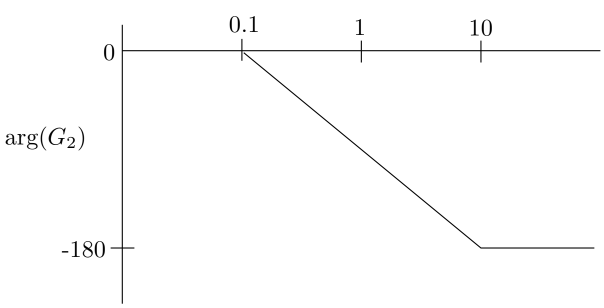


□

Example 4.6.2 $G_2(s) = \dfrac{1-s}{1+s}$ allpass TF



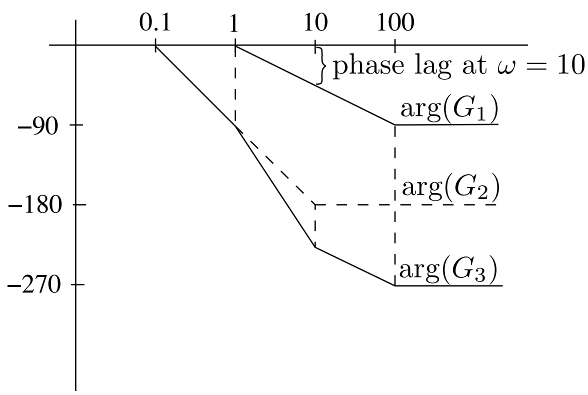




□

Example 4.6.3

$$G_3 = G_1 G_2 \qquad \text{nonminimum phase TF}$$
$$G_3(s) = \frac{1}{s+10}\frac{1-s}{1+s}$$
$$|G_3(j\omega)| = |G_1(j\omega)|$$
$$\arg(G_3) = \arg(G_1) + \arg(G_2) \le \arg(G_1)$$



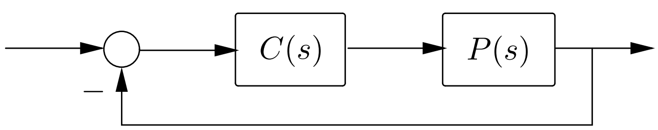
Of all TFs having the magnitude plot $|G_1|$, G_1 has the minimum phase lag at every ω. □

Using the principles developed so far, you should be able to sketch the approximation of the Bode plot of

$$\begin{aligned} G(s) &= \frac{40s^2(s-2)}{(s-5)(s^2+4s+100)} \\ &= 0.16\frac{s^2(\frac{1}{2}s-1)}{(\frac{1}{5}s-1)[\frac{1}{100}(s^2+4s+100)]}. \end{aligned}$$

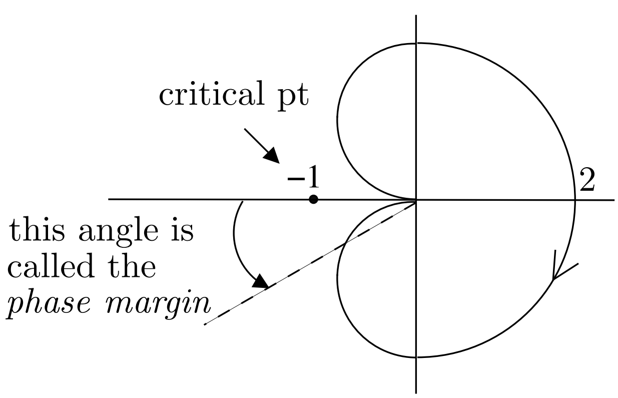
Now let's see how to deduce feedback stability from Bode plots.

Example 4.6.4

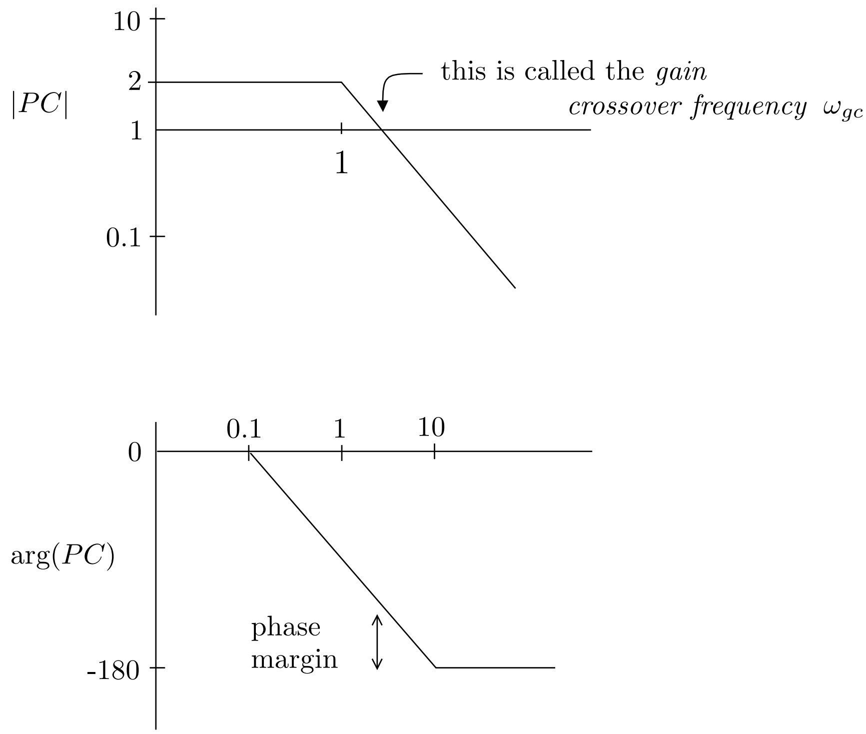


$$C(s) = 2, \quad P(s) = \frac{1}{(s+1)^2}$$

The Nyquist plot is



There are no encirclements of the critical point, so the feedback system is stable. The Bode plot of PC is

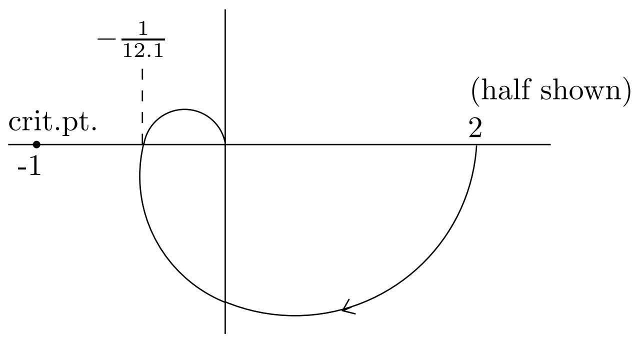

10
2
1
0.1
|PC|
1
this is called the *gain*
crossover frequency ω_{gc}
0.1
1
10
0
-180
arg(PC)
phase
margin

$\square$

Example 4.6.5

$$C(s) = 2, \quad P(s) = \frac{1}{(s+1)^2(0.1s+1)}$$

Nyquist :



Stable for $C(s) = 2K$ and $-\frac{1}{K} < -\frac{1}{12.1}$. Thus max $K = 12.1 = 21.67$ dB. See how this appears on the Bode plot. $\square$

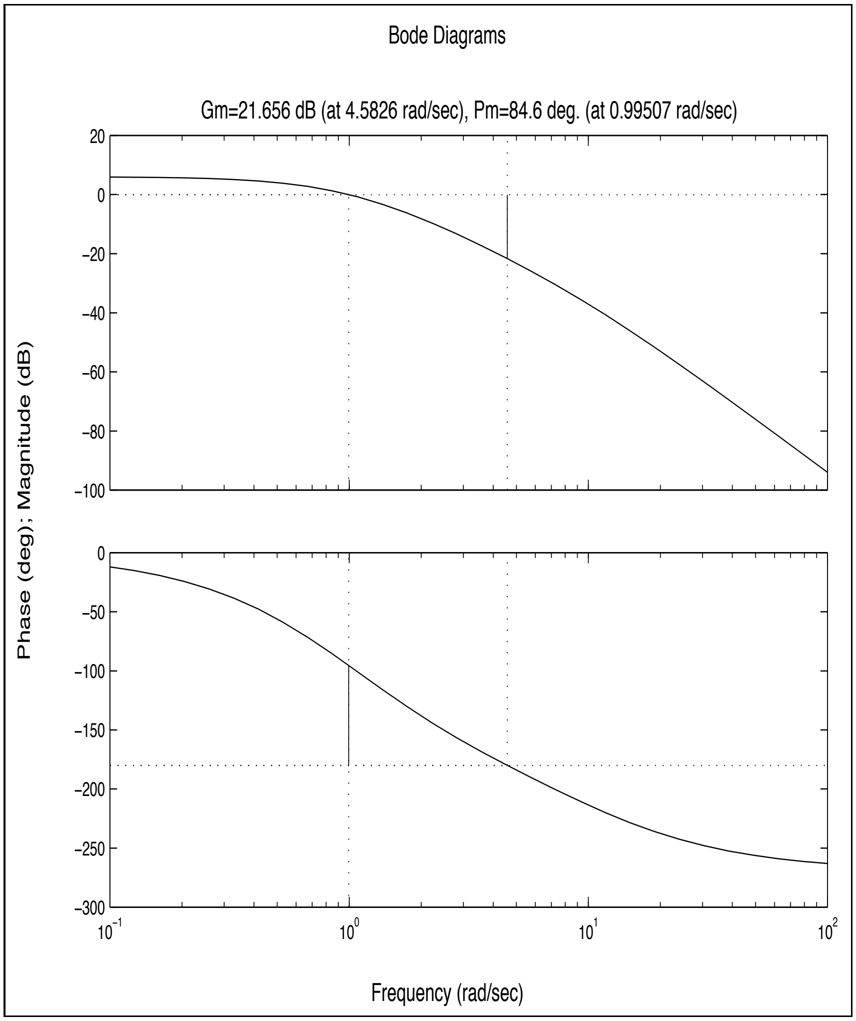
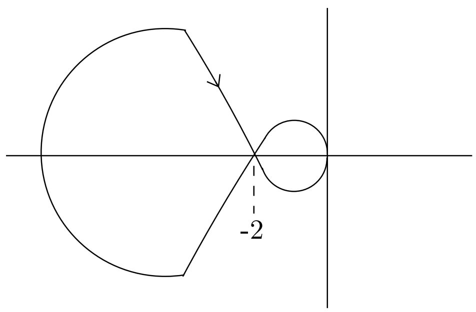


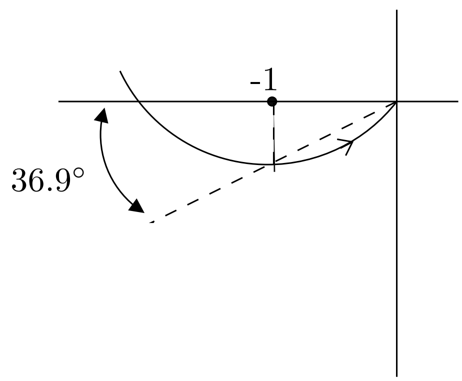
Figure 4.1: Example 4.6.5

Example 4.6.6 $C(s) = 2, \quad P(s) = \dfrac{s+1}{s(s-1)}$

The Nyquist plot of PC has the form



The critical point is -1 and we need 1 CCW encirclement, so the feedback system is stable. The phase margin on the Nyquist plot:



If $C(s) = 2K,\ K$ can be reduced until $-\dfrac{1}{K} = -2$, i.e., min $K = \dfrac{1}{2} = 6$ dB. The Bode plot is shown on the next page. MATLAB says the gain margin is negative! So MATLAB is wrong. Conclusion: We need the Nyquist plot for the correct interpretation of the stability margins.

□

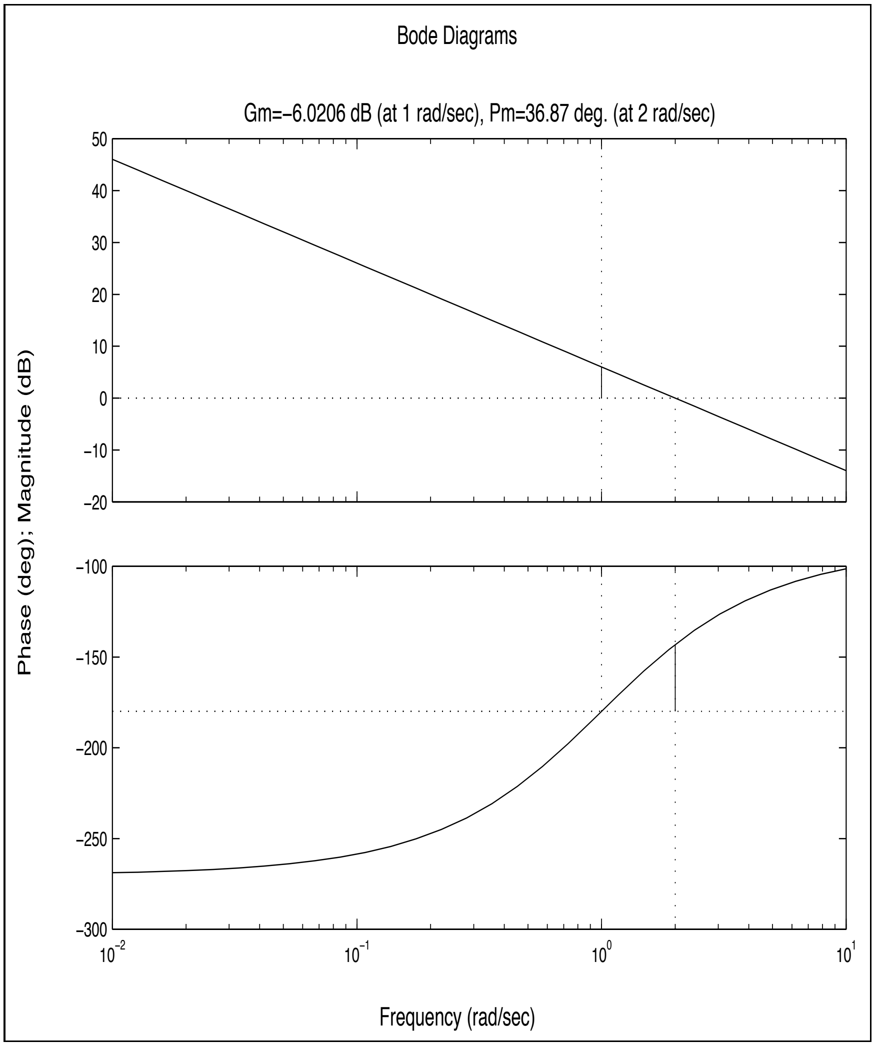
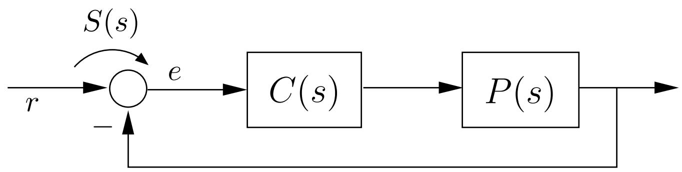


Figure 4.2: Example 4.6.6

Let's recap: The phase margin is related to the distance from the critical point to the Nyquist plot along the unit circle; the gain margin is related to the distance from the critical point to the Nyquist plot along the real axis. More generally, it makes sense to define the **stability margin** to be the distance from the critical point to the closest point on the Nyquist plot:



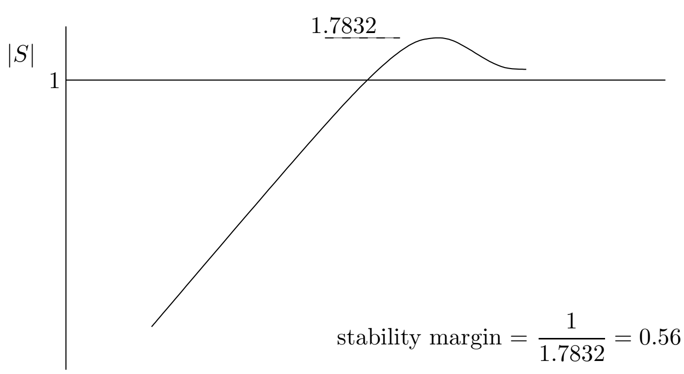
Define

$$
\begin{aligned}
S &= TF \text{ from } r \text{ to } e \\
&= \frac{1}{1+PC}.
\end{aligned}
$$

Assume feedback system is stable. Then

$$
\begin{aligned}
\text{stability margin} &= \text{dist (-1, Nyquist plot of } PC) \\
&= \min_{\omega} |-1-P(j\omega)C(j\omega)| \\
&= \min_{\omega} |1+P(j\omega)C(j\omega)| \\
&= [\max_{\omega} |S(j\omega)|]^{-1} \\
&= \text{reciprocal of peak magnitude on Bode plot of } S
\end{aligned}
$$

Example 4.6.6 (Cont'd)

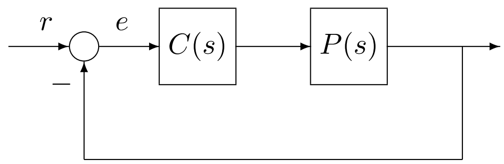


□

4.7 Problems

1. Consider the block diagram in Example 4.1.4. Let d be a disturbance entering the summing junction between P_1 and P_2. Find the transfer function from d to y.

2. Consider the feedback control system



where $P(s) = 1/(s+1)$ and $C(s) = K$.

(a) Find the minimum $K > 0$ such that the steady-state absolute error $|e(t)|$ is less than or equal to 0.01 when r is the unit step.

(b) Find the minimum $K > 0$ such that the steady-state absolute error $|e(t)|$ is less than or equal to 0.01 for all inputs of the form

$$r(t) = \cos(\omega t), \quad 0 \leq \omega \leq 4.$$

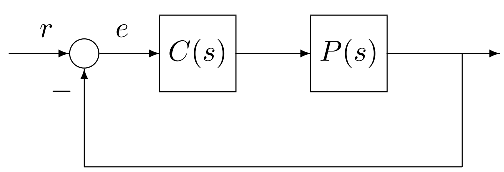
3. Same block diagram but now with

$$P(s) = \frac{1}{s^2 - 1}, \quad C(s) = \frac{s-1}{s+1}.$$

Is the feedback system internally stable?

4. Prove Theorem 4.1.2.

5. Consider the feedback control system



with

$$P(s) = \frac{5}{s+1}, \quad C(s) = K_1 + \frac{K_2}{s}.$$

It is desired to find constants K_1 and K_2 so that (i) the closed-loop poles (i.e., roots of the characteristic polynomial) lie in the half-plane Re $s < -4$ (this is for a desired speed of transient response), and (ii) when $r(t)$ is the ramp of slope 1, the final value of the absolute error $|e(t)|$ is less than or equal to 0.05. Draw the region in the (K_1, K_2)-plane for which these two specs are satisfied.

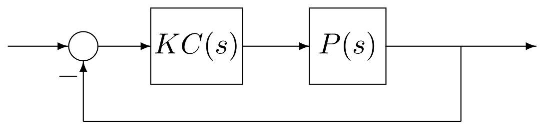
6. Same block diagram except also with a disturbance $d(t)$ entering just before the plant. Suppose $P(s) = 5/(s+1)$, $r(t) = 0$, and $d(t) = \sin(10t)1_+(t)$. Design a proper $C(s)$ so that the feedback system is stable and $e(t)$ converges to 0.

7. This problem introduces SIMULINK, a GUI running above MATLAB. An introduction to SIMULINK is appended.

   Build a SIMULINK model of the setup in Problem 5. Select K_1, K_2 as computed to achieve the desired specs. Simulate for $r(t)$ the unit ramp and display $e(t)$. Feel free to play with K_1, K_2 to see their effect.

   To be handed in: A printout of your SIMULINK diagram; A printout of your plot of $e(t)$.

8. Consider the feedback system



   with

$$P(s) = \frac{s+2}{s^2+2}, \quad C(s) = \frac{1}{s}.$$

   Sketch the Nyquist plot of PC. How many encirclements are required of the critical point for feedback stability? Determine the range of real gains K for stability of the feedback system.

9. Repeat with

$$P(s) = \frac{4s^2+1}{s(s-1)^2}, \quad C(s) = 1.$$

10. Repeat with

$$P(s) = \frac{s^2+1}{(s+1)(s^2+s+1)}, \quad C(s) = 1.$$

11. Sketch the Nyquist plot of

$$G(s) = \frac{s(4s^2+5s+4)}{(s^2+1)^2}.$$

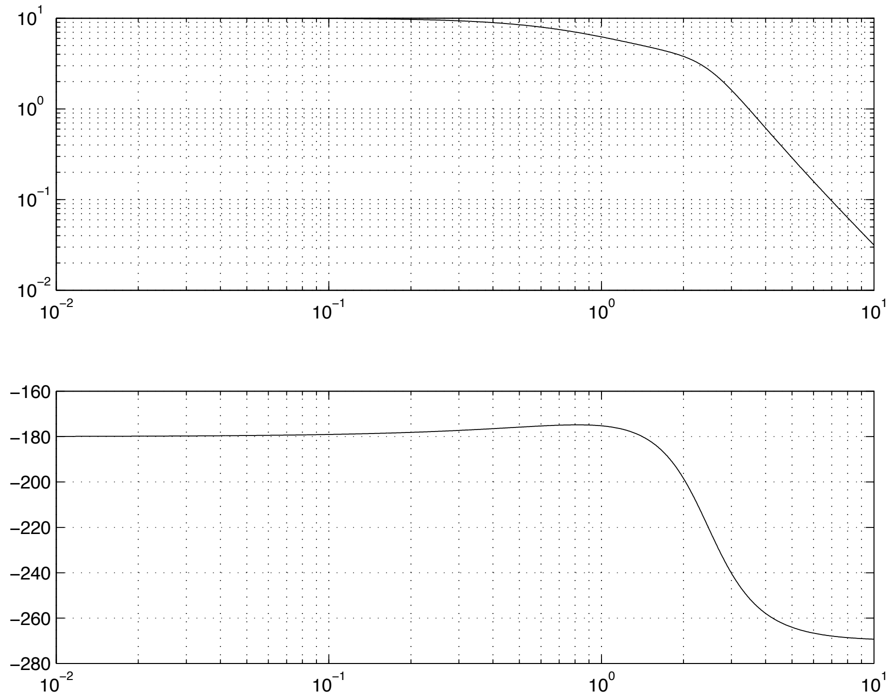
   How many times does it encircle the point $(1,0)$? What does this say about the transfer function $G(s)-1$?

12. Consider the transfer function

$$P(s) = \frac{-0.1s+1}{(s+1)(s-1)(s^2+s+4)}.$$

   Draw the piecewise-linear approximation of the Bode **phase** plot of P

13. Consider the standard feedback system with $C(s) = K$. The Bode plot of $P(s)$ is given below (magnitude in absolute units, not dB; phase in degrees). The phase starts at $-180°$ and ends at $-270°$. You are also given that $P(s)$ has exactly one pole in the right half-plane. For what range of gains K is the feedback system stable?



10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-2}
10^{-1}
10^{0}
10^{1}
−160
−180
−200
−220
−240
−260
−280
10^{-2}
10^{-1}
10^{0}
10^{1}

14. Consider the standard feedback system with

$$C(s) = 16, \quad P(s) = \frac{1}{(s+1)(30s+1)(s^2/9 + s/3 + 1)} e^{-s}.$$

This plant has a time delay, making the transfer function irrational. It is common to use a Padé approximation of the time delay. The second order Padé approximation is

$$e^{-s} = \frac{s^2 - 6s + 12}{s^2 + 6s + 12},$$

which is a rational allpass function. Using this approximation in $P(s)$, graph using MATLAB the Bode plot of $P(s)C(s)$. Can you tell from the Bode plot that the feedback system is stable? Use MATLAB to get the gain and phase margins. Finally, what is the stability margin (distance from the critical point to the Nyquist plot)?

15. Consider

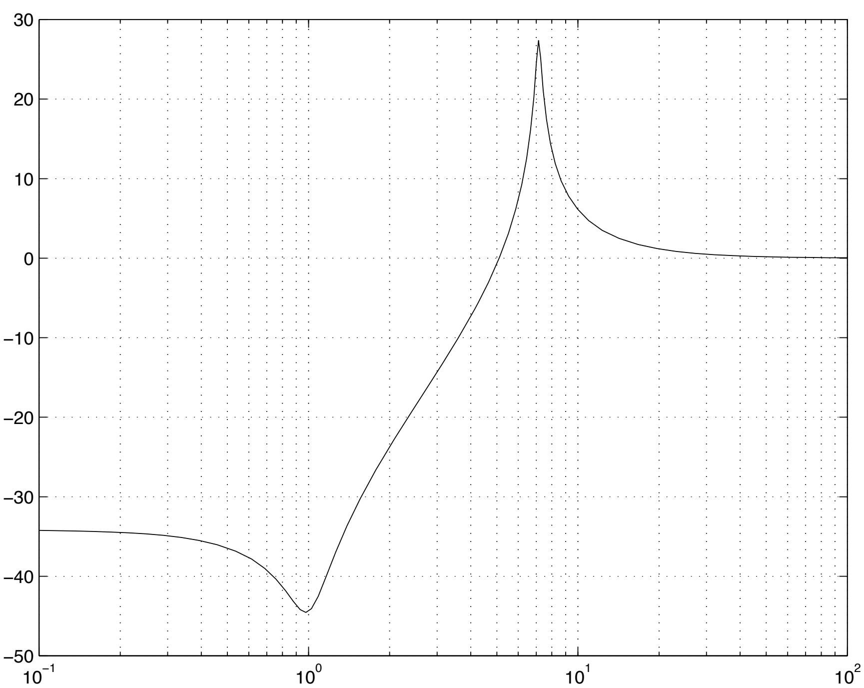
$$P(s) = \frac{10}{s^2 + 0.3s + 1}, \quad C(s) = 5.$$

The magnitude Bode plot (in dB vs rad/s) of the sensitivity function, S, is shown below.

(a) Show that the feedback system is stable by looking at the closed-loop characteristic polynomial.

(b) What is the distance from the critical point to the Nyquist plot of PC?

(c) If $r(t) = \cos(t)$, what is the steady-state amplitude of the tracking error $e(t)$?

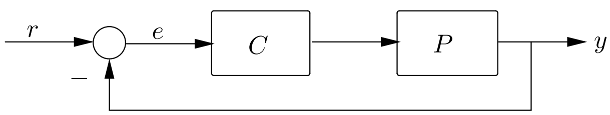


Chapter 5

Introduction to Control Design

5.1 Loopshaping

In this chpater we look at the basic technique of controller design in the frequency domain. We start with the unity feedback loop:



The design problem is this: Given P, the nominal plant transfer function, maybe some uncertainty bounds, and some performance specs, design an implementable C. The performance specs would include, as a bare minimum, stability of the feedback system. The simplest situation is where the performance can be specified in terms of the transfer function

$$S := \frac{1}{1+PC},$$

which is called the **sensitivity function**.

Aside: Here's the reason for this name. Denote by T the transfer function from r to y, namely,

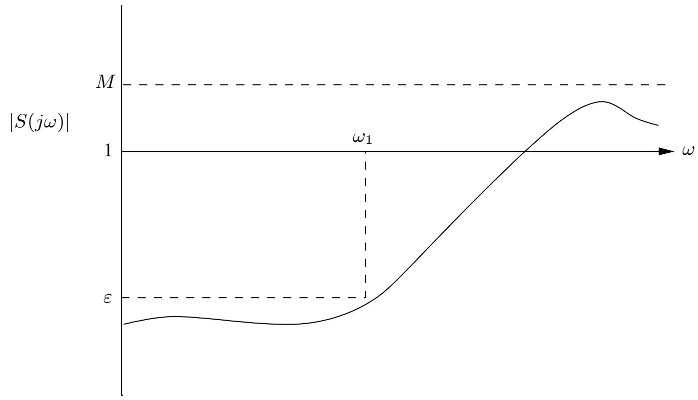
$$T = \frac{PC}{1+PC}.$$

Of relevance is the relative perturbation in T due to a relative perturbation in P:

$$\begin{aligned}
\lim_{\Delta P \to 0} \frac{\Delta T/T}{\Delta P/P} &= \lim_{\Delta P \to 0} \frac{\Delta T}{\Delta P}\frac{P}{T} \\
&= \frac{dT}{dP} \cdot \frac{P}{T} \\
&= \frac{d}{dP}\left(\frac{PC}{1+PC}\right) \cdot P \cdot \frac{1+PC}{PC} \\
&= S.
\end{aligned}$$

So S is a measure of the sensitivity of the closed-loop transfer function to variations in the plant transfer function.

For us, S is important for two reasons: 1) S is the transfer function from r to e. Thus we want $|S(j\omega)|$ to be small over the range of frequencies of r. 2) The peak magnitude of S is the reciprocal of the stability margin. Thus a typical desired magnitude plot of S is



Here ω_1 is the maximum frequency of r, ε is the maximum permitted relative tracking error, $\varepsilon < 1$, and M is the maximum peak magnitude of $|S|$, $M > 1$. If $|S|$ has this shape and the feedback system is stable, then for the input $r(t) = \cos\omega t, \omega \leq \omega_1$ we have $|e(t)| \leq \varepsilon$ in steady state, and the stability margin $1/M$. A typical value for M is 2 or 3. In these terms, the design problem can be stated as follows: Given $P,\ M,\ \varepsilon,\ \omega_1$; Design C so that the feedback system is stable and $|S|$ satisfies $|S(j\omega)| \leq \varepsilon$ for $\omega \leq \omega_1$ and $|S(j\omega)| \leq M$ for all ω.

Example 5.1.1

$$P(s) = \frac{10}{0.2s + 1}$$

This is a typical transfer function of a DC motor. Let's take a PI controller:

$$C(s) = K_1 + \frac{K_2}{s}.$$

Then any $M,\ \varepsilon,\ \omega_1$ are achievable by suitable $K_1,\ K_2$. To see this,

$$\begin{aligned} S(s) &= \frac{1}{1 + \frac{10(K_1 s + K_2)}{S(0.2s+1)}} \\ &= \frac{s(0.2s+1)}{0.2s^2 + (1 + 10K_1)s + 10K_2} \\ &= \frac{5s(0.2s+1)}{(s + K_3)^2} \end{aligned}$$

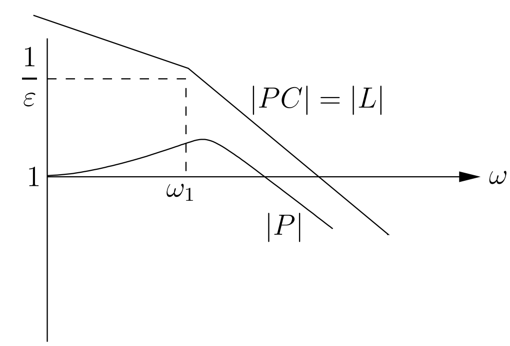
for suitable $K_1,\ K_2$, where K_3 is now freely designable. Sketch the Bode plot S and confirm that any $M > 1,\ \varepsilon < 1,\ \omega_1$ can be achieved. □

In practice it is common to combine interactively the shaping of S with a time-domain simulation.

Now, S is a nonlinear function of C. So in fact it is easier to design the loop transfer function $L := PC$ instead of $S = \dfrac{1}{1+L}$. Notice that

$$|L| >> 1 \Rightarrow |S| \approx \frac{1}{|L|}.$$

A typical desired plot is

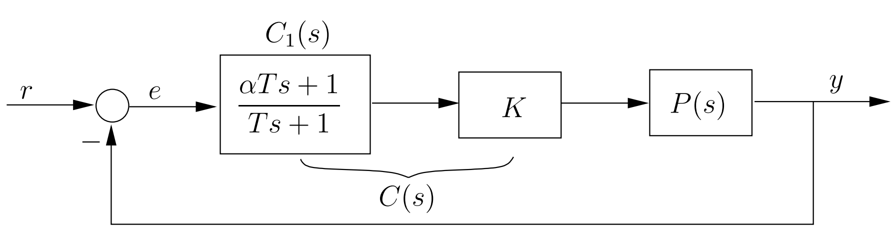


In shaping $|L|$, we don't have a direct handle on the stability margin, unfortunately. However, we do have control over the gain and phase margins, as we'll see.

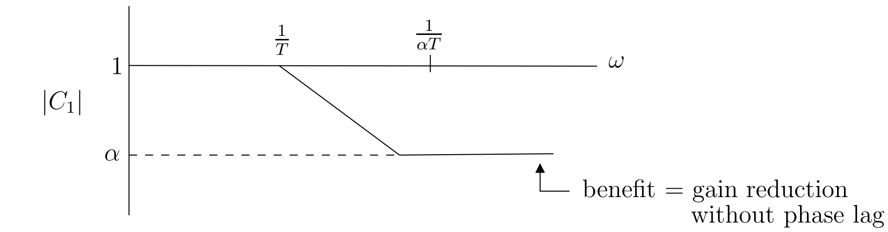
In the next two sections we present two simple loopshaping controllers.

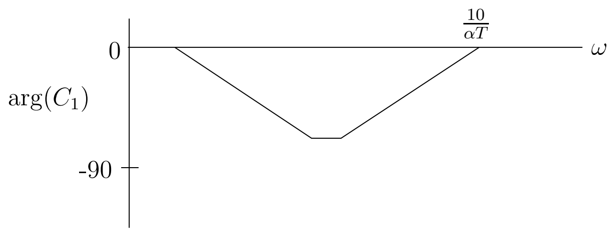
5.2 Lag Compensation

We separate the controller into two parts, K and C_1, the latter having unity DC gain:



The parameters in C are α $(0 < \alpha < 1)$, $T > 0$, $K > 0$. The approximate Bode plot of C_1 is





An example design using this type of compensator follows next.

Example 5.2.1 The plant transfer function is $P(s) = \dfrac{1}{s(s+2)}$. Let there be two specs:

1. When $r(t)$ is the unit ramp, the steady-state tracking error ≈ 5 %.
2. PM $\approx 45°$ for adequate damping in transient response.

Step 1 Choose K to get spec 1:

$$\begin{aligned}
E &= \frac{1}{1+PC} R \\
E(s) &= \frac{1}{1 + \frac{K}{s(s+2)} \frac{\alpha T s+1}{T s+1}} \frac{1}{s^2} \\
&= \frac{(s+2)(Ts+1)}{s(s+2)(Ts+1) + K(\alpha T s+1)} \frac{1}{s} \\
e(\infty) &= \frac{2}{K}
\end{aligned}$$

So spec 1 $\Leftrightarrow \frac{2}{K} = 0.05 \Leftrightarrow K = 40$. Then

$$KP(s) = \frac{40}{s(s+2)}.$$

Step 2 For KP we have $\omega_{gc} = 6$, $PM = 18°$. To increase the PM (while preserving spec 1), we'll use a lag compensator $C_1(s)$. The design is shown on the Bode plots. We want PM $\approx 45°$.

Add 4.6° for safety: 49.6°

$\arg KP = -180 + 49.6 = -130.4$ when $\omega = 1.7$

Set new $\omega_{gc} = 1.7$.

$|KP| = 19$ dB $= 8.96$ at new ω_{gc}

$\therefore$ set $\alpha = 1/8.96 = 0.111$

Set $\frac{10}{\alpha T} = 1.7 \Rightarrow T = 52.7$

Final exact PM = 180-135.4 = 44.6°

Final controller

$$C(s) = K\frac{\alpha Ts + 1}{Ts + 1},\ K = 40,\ \alpha = 0.111,\ T = 52.7$$

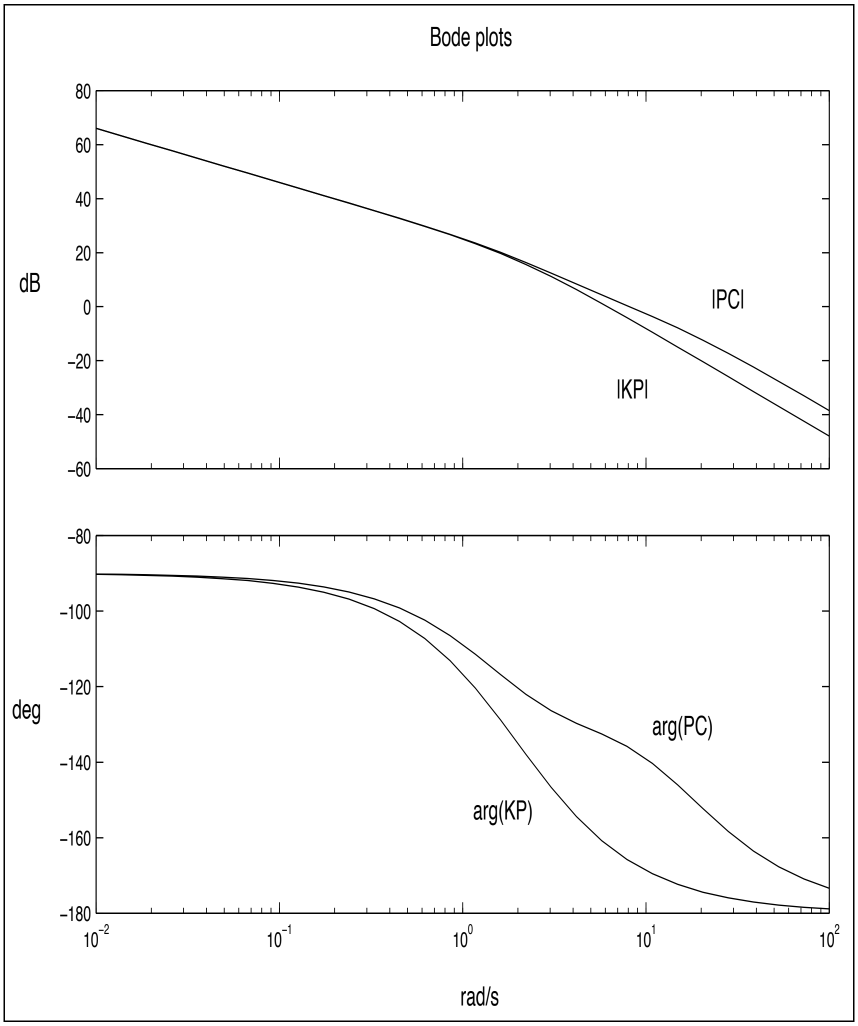


Figure 5.3: Example 5.3.1

Step responses are shown on the next plot. The step response of $KP/(1+KP)$, that is, the plant compensated only by the gain for spec 1, is fast but oscillatory. The step response of $PC/(1+PC)$ is slower but less oscillatory, which was the goal of spec 2.

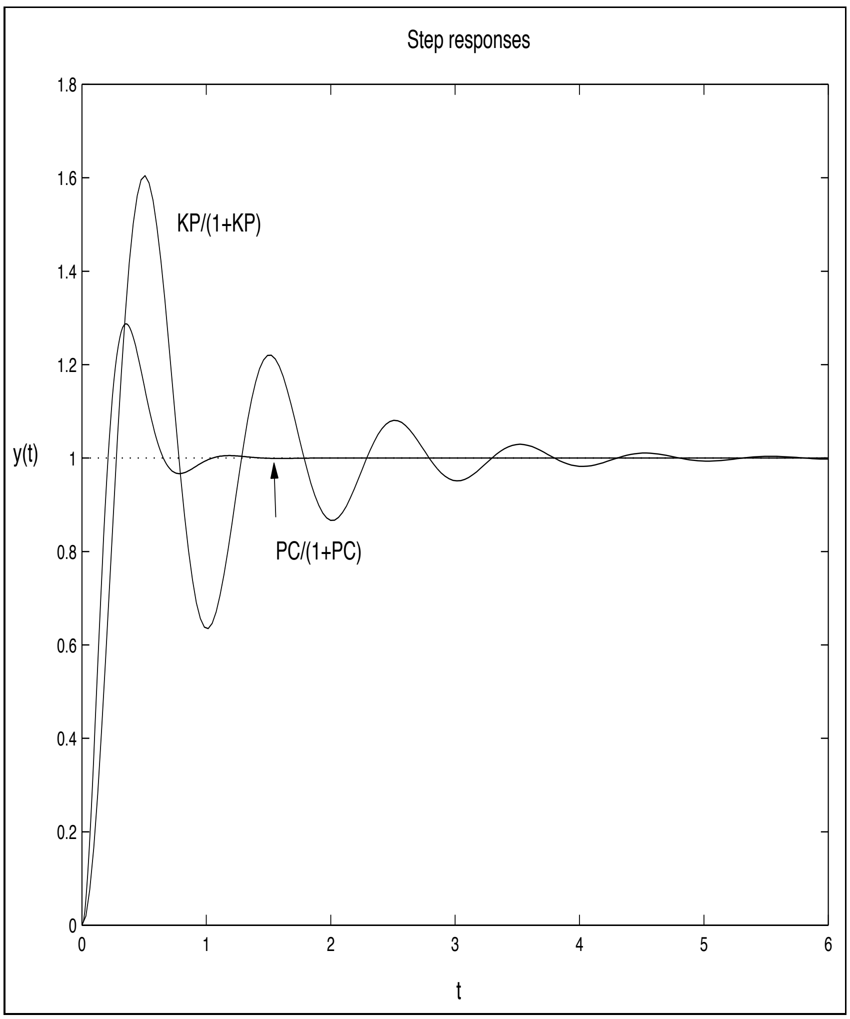
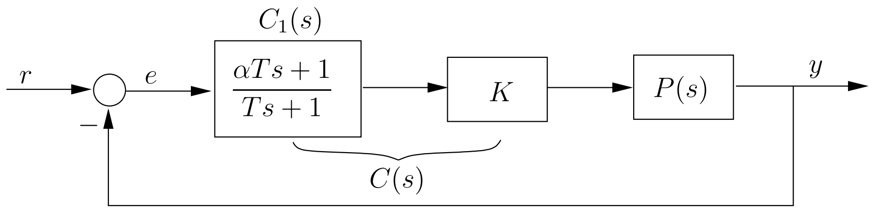
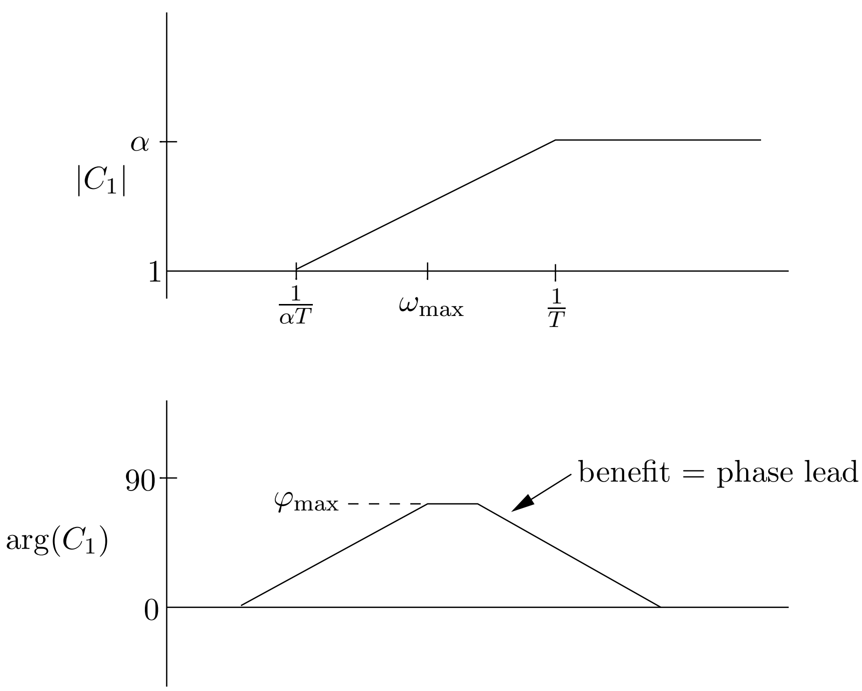


Figure 5.4: Example 5.3.1

5.3 Lead Compensation



The parameters in C are α ($\alpha > 1$), $T > 0$, $K > 0$. The approximate Bode plot of C_1 is



The angle $\varphi_{\max}$ is defined as the maximum of $\arg(C_1(j\omega))$, and $\omega_{\max}$ is defined as the frequency at which it occurs.

We'll need three formulas:

1. $\omega_{\max}$: This is the midpoint between $\frac{1}{\alpha T}$ and $\frac{1}{T}$ on the logarithmically scaled frequency axis.

Thus

$$\begin{aligned}
\log \omega_{\max} &= \frac{1}{2}\left(\log \frac{1}{\alpha T} + \log \frac{1}{T}\right) \\
&= \frac{1}{2} \log \frac{1}{\alpha T^2} \\
&= \log \frac{1}{T\sqrt{\alpha}} \\
\Rightarrow \omega_{\max} &= \frac{1}{T\sqrt{\alpha}}.
\end{aligned}$$

2. The magnitude of C_1 at $\omega_{\max}$: This is the midpoint between 1 and α on the logarithmically scaled $|C_1|$ axis. Thus

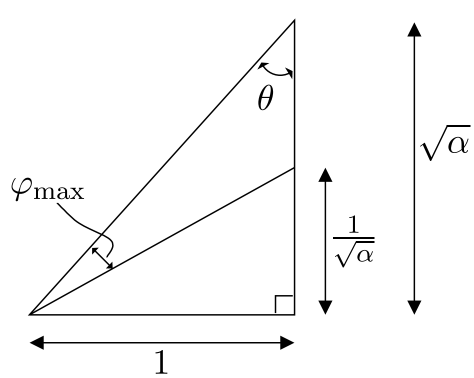
$$\begin{aligned}
\log |C_1(j\omega_{\max})| &= \frac{1}{2}(\log 1 + \log \alpha) \\
&= \log \sqrt{\alpha} \\
\Rightarrow |C_1(j\omega_{\max})| &= \sqrt{\alpha}.
\end{aligned}$$

3. $\varphi_{\max}$: This is the angle of $C_1(j\omega_{\max})$. Thus

$$\begin{aligned}
\varphi_{\max} &= \arg C_1(j\omega_{\max}) \\
&= \arg \frac{1 + \sqrt{\alpha} j}{1 + \frac{1}{\sqrt{\alpha}} j}.
\end{aligned}$$

By the sine law

$$\frac{\sin \varphi_{\max}}{\sqrt{\alpha} - \frac{1}{\sqrt{\alpha}}} = \frac{\sin \theta}{\sqrt{1 + \frac{1}{\alpha}}} :$$



But $\sin\theta = \frac{1}{\sqrt{1+\alpha}}$. Thus

$$\sin\varphi_{\max} = \left(\sqrt{\alpha} - \frac{1}{\sqrt{\alpha}}\right)\frac{1}{\sqrt{1+\alpha}\sqrt{1+\frac{1}{\alpha}}} = \frac{\alpha-1}{\alpha+1},$$

and hence

$$\varphi_{\max} = \sin^{-1}\frac{\alpha-1}{\alpha+1}.$$

Example 5.3.1

Let's do the same example as we did for lag compensation. Again, we choose $K = 40$ to achieve spec 1. The phase margin is then only 18° at a gain crossover frequency of 6 rad/s. We design a lead compensator C_1 to bump up the phase margin as follows. The problem is a little complicated because we can only guess what the new gain crossover frequency will be.

Step 1 We need at least $45 - 18 = 27°$ phase lead. Increase by 10% (a guess). Say 30°. We have

$$\sin 30° = \frac{\alpha-1}{\alpha+1} \;\Rightarrow\; \alpha = 3.$$

Step 2 We want to make $\omega_{\max}$ the new gain crossover frequency. Thus at this frequency we will be increasing the gain by $\sqrt{\alpha} = 1.732 = 4.77$ dB. Now $|KP| = \dfrac{1}{1.732} = -4.77$ dB at $\omega = 8.4$ rad/s. Thus we set

$$\omega_{\max} = 8.4 \quad\Rightarrow\quad \frac{1}{T\sqrt{\alpha}} = 8.4 \quad\Rightarrow\quad T = 0.0687.$$

The PM achieved is 44°. See the Bode plots. The controller is

$$C(s) = K\frac{\alpha T s + 1}{Ts+1},\; K = 40,\; \alpha = 3,\; T = 0.0687.$$

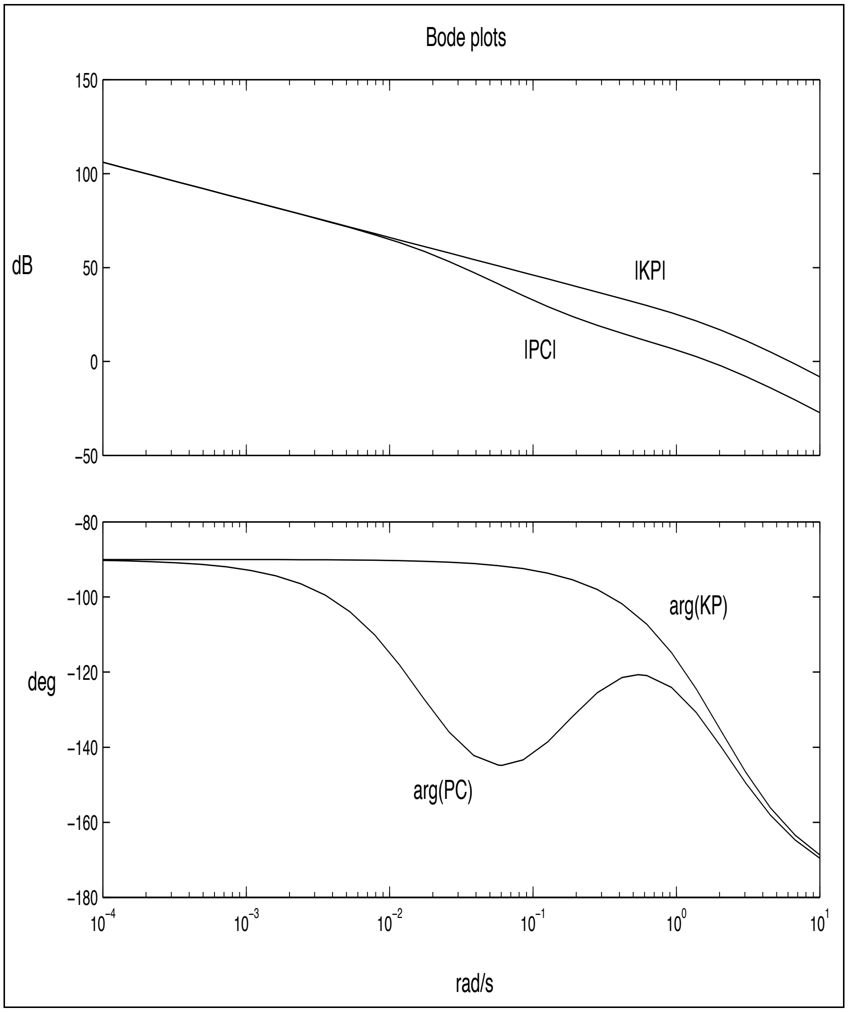


Figure 5.1: Example 5.2.1

Finally, see the closed-loop step responses.

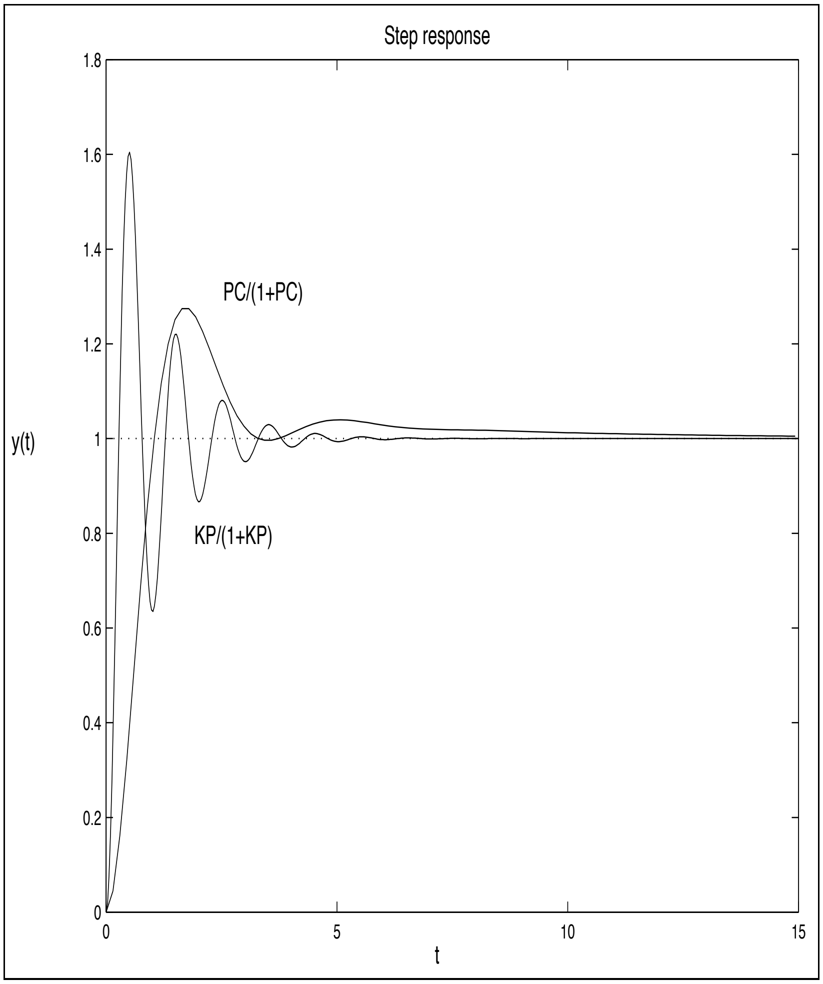


Figure 5.2: Example 5.2.1

You can compare the results of lag compensation versus lead compensation. They both increased the phase margin to the desired value, but the lead compensator produces a faster response.

5.4 Loopshaping Theory

In this section we look at some theoretical facts that we have to keep in mind while designing controllers via loopshaping.

5.4.1 Bode's phase formula

It is a fundamental fact that if $L = PC$ is stable and minimum phase and normalized so that $L(0) > 0$ (positive DC gain), then the magnitude Bode plot uniquely determines the phase Bode plot. The exact formula is rather complicated, and is derived using Cauchy's integral theorem. Let

$\omega_0 =$ any frequency

$u =$ normalized frequency $= \ln \dfrac{\omega}{\omega_0}$, i.e., $e^u = \frac{\omega}{\omega_0}$

$M(u) =$ normalized log magnitude $= \ln |L(j\omega_0 e^u)|$.

$W(u) =$ weighting function $= \ln \ \coth \dfrac{|u|}{2}$.

Recall that

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}.$$

The phase formula is

$$\arg(L(j\omega_0)) = \frac{1}{\Pi} \int_{-\infty}^{\infty} \frac{dM(u)}{du} W(u) du \text{ in radians.} \tag{5.1}$$

This shows that $\arg(L(j\omega_0))$ can be expressed as an integral involving $|L(j\omega)|$.

It turns out we may approximate the weighting function as $W(u) \approx \dfrac{\pi^2}{2}\delta(u)$. Then the phase formula (5.1) gives

$$\arg(L(j\omega_0)) \approx \frac{\pi}{2} \frac{dM(u)}{du}\bigg|_{u=0} \tag{5.2}$$

As an example, consider the situation where

$$L(j\omega) = \frac{c}{\omega^n} \text{ near } \omega = \omega_0.$$

Thus $-n$ is the slope of the magnitude Bode plot near $\omega = \omega_0$. Then

$$|L(j\omega_0 e^u)| = \frac{c}{\omega_0^n e^{nu}}$$

$$\Rightarrow M(u) = \ln |L(j\omega_0 e^u)| = \ln \frac{c}{\omega_0^n} - nu$$

$$\Rightarrow \frac{dM(u)}{du} = -n$$

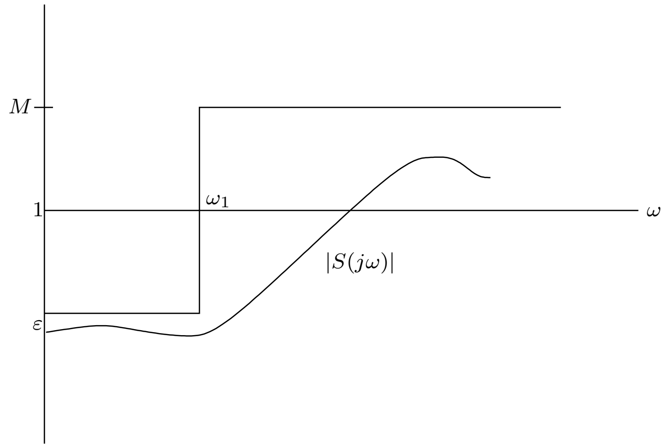
$$\Rightarrow \arg(L(j\omega_0)) = -n\frac{\pi}{2} \text{ from (5.2).}$$

Thus we arrive at the observation: If the slope of $|L(j\omega)|$ near crossover is $-n$, then $\arg(L(j\omega))$ at crossover is approximately $-n\dfrac{\pi}{2}$. **Warning** This derivation required $L(s)$ to be stable, minimum phase, positive DC gain.

What we learn from this observation is that in transforming $|P|$ to $|PC|$ via, say, lag or lead compensation, we should not attempt to roll off $|PC|$ too sharply near gain crossover. If we do, $\arg(PC)$ will be too large near crossover, resulting in a negative phase margin and hence instability.

5.4.2 The waterbed effect

This concerns the ability to achieve the following spec on the sensitivity function S:



Let us suppose $M > 1$ and $\omega_1 > 0$ are fixed. Can we make ε arbitrarily small? That is, can we get arbitrarily good tracking over a finite frequency range, while maintaining a given stability margin (1/M) ? Or is there a positive lower bound for ε? The answer is that arbitrarily good performance in this sense is achievable if and only if $P(s)$ is minimum phase. Thus, non-minimum phase plants have bounds on achievable performance: As $|S(j\omega)|$ is pushed down on one frequency range, it pops up somewhere else, like a waterbed. Here's the result:

Theorem 5.4.1 *Suppose $P(s)$ has a zero at $s = z$ with* Re *$z > 0$. Let $A(s)$ denote the allpass factor of $S(s)$. Then there are positive constants c_1, c_2, depending only on ω_1 and z, such that*

$$c_1 \log \varepsilon + c_2 \log M \geq \log |A(z)^{-1}| \geq 0.$$

Example 5.4.1 $P(s) = \dfrac{1-s}{(s+1)(s-p)}, \ p > 0, \ p \neq 1$

Let $C(s)$ be a stabilizing controller. Then

$$S = \frac{1}{1+PC} \Rightarrow S(p) = 0.$$

Thus $\dfrac{s-p}{s+p}$ is an allpass factor of S. There may be other allpass factors, so what we can say is that $A(s)$ has the form

$$A(s) = \frac{s-p}{s+p} A_1(s),$$

where $A_1(s)$ is some allpass TF (may be 1). In this example, the RHP zero of $P(s)$ is $s = 1$. Thus

$$|A(1)| = \left| \frac{1-p}{1+p} \right| |A_1(1)|.$$

Now $|A_1(1)| \leq 1$ (why?), so

$$|A(1)| \leq \left|\frac{1-p}{1+p}\right|$$

and hence

$$|A(1)^{-1}| \geq \left|\frac{1+p}{1-p}\right|.$$

The theorem gives

$$c_1 \log \varepsilon + c_2 \log M \geq \log \left|\frac{1+p}{1-p}\right|.$$

Thus, if $M > 1$ is fixed, $\log \varepsilon$ cannot be arbitrarily negative, and hence ε cannot be arbitrarily small. In fact the situation is much worse if $p \approx 1$, that is, if the RHP plant pole and zero are close. □

5.5 Problems

1. Take $P(s) = 0.1/(s^2 + 0.7s + 1)$. Design a lag compensator $C(s)$ for the following two specs:

   - The DC gain from r to e is 0.05.
   - Phase margin of 30°.

   Include all Bode plots, together with closed-loop step responses.

2. For the plant $P(s) = 1/s^2$ design a lead compensator to get a phase margin of 45° and a gain crossover frequency of 10 rad/s. Include all Bode plots, together with closed-loop step responses.

3. For the plant $P(s) = 1/s(s+1)$ design a lead compensator to get a phase margin of 50° and a gain crossover frequency of 2 rad/s. Include all Bode plots, together with closed-loop step responses.

4. Consider

   $$P(s) = 4\frac{s-2}{(s+1)^2}.$$

   Suppose $C(s)$ is a (proper) controller that stabilizes the feedback system and for which the sensitivity function satisfies $|S(j\omega)| \leq 1.5$ for all ω. Thus the stability margin (distance from the critical point -1 to the Nyquist plot) equals $1/1.5$. Suppose the operating frequency range is from 0 to 0.1 rad/s. Then we would like the error magnitude $|S(j\omega)|$ to be small over this range. Since $P(s)$ is non-minimum phase, this error magnitude cannot be arbitrarily small. Give a positive lower bound for

   $$\max_{0 \leq \omega \leq 0.1} |S(j\omega)|.$$

Epilogue

Here's what I hope you learned:

- How to model simple mechanical systems.
- What it means for a system to be linear.
- The value of graphical simulation tools like Scicos and SIMULINK.
- Why we use transfer functions and the frequency domain.
- What stability means.
- What feedback is and why it's important.
- What makes a system easy or hard to control.
- How to design a simple feedback loop.