ECE356 - LINEAR SYSTEMS AND CONTROL
MIDTERM SOLUTIONS

1. Consider a model of fisheries management. State x_1 is the population level of a prey species, x_2 is the population level of a predator species, and x_3 is the effort expended by humans in fishing the predator species. The model is

$$\begin{aligned}
\dot{x}_1 &= (r_1 - x_2)x_1 \\
\dot{x}_2 &= (r_2 - x_3)x_2 \\
\dot{x}_3 &= u \\
y &= x_2
\end{aligned}$$

where u is the input, y is the output, and $r_1 > 0$ and $r_2 > 0$.

(a) Find all equilibrium pairs $(\overline{x}, \overline{u})$ assuming the prey species does not die out.

(b) Linearize the nonlinear system about the equilibrium pair $(\overline{x}, \overline{u})$ found in part (a).

(c) Find the transfer function of the linearized state model from part (b).

(a) We are told that $\overline{x}_1 \neq 0$, so $\overline{x}_2 = r_1$, $\overline{x}_3 = r_2$, and $\overline{u} = 0$.

(b) Let $\tilde{x} = x - \overline{x}$, $\tilde{u} = u - \overline{u}$, and $\tilde{y} = x_2 - \overline{x}_2$. The linearization is

$$\dot{\tilde{x}} = \begin{bmatrix} 0 & -\overline{x}_1 & 0 \\ 0 & 0 & -r_1 \\ 0 & 0 & 0 \end{bmatrix} \tilde{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tilde{u}\,.$$

$$\tilde{y} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \tilde{x}\,.$$

(c)

$$\frac{Y(s)}{R(s)} = C(sI - A)^{-1}B = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s} & -\frac{\overline{x}_1}{s^2} & \frac{r_1\overline{x}_1}{s^3} \\ 0 & \frac{1}{s} & -\frac{r_1}{s^2} \\ 0 & 0 & \frac{1}{s} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = -\frac{r_1}{s^2}\,.$$

2. Consider the LTI control system

$$\dot{x} = Ax + Bu\,.$$

Let $T > 0$ be the sampling period. The sampled state sequence is $x(0), x(T), x(2T), \ldots$. Let $u(t)$ be an input signal, and suppose it is also sampled at times $T, 2T, \ldots$ and is held constant between sample times. Derive an iterative equation for computing the state $x((k+1)T)$ from $x(kT)$ and $u(kT)$. (You may assume A^{-1} exists).

We have

$$\begin{aligned} x((k+1)T) &= \mathrm{e}^{AT}x(kT) + \int_0^T \mathrm{e}^{A(T-\tau)}Bu(kT)d\tau \\ &= \mathrm{e}^{AT}x(kT) + \mathrm{e}^{AT}\int_0^T \mathrm{e}^{-A\tau}d\tau \;\; Bu(kT)\,. \end{aligned}$$

This can be simplified using the fact that A^{-1} exists. We know that

$$\frac{d}{dt}\mathrm{e}^{At} = A\mathrm{e}^{At}\,.$$

Thus,

$$\mathrm{e}^{At} = \int A\mathrm{e}^{At}dt = A\int \mathrm{e}^{At}dt\,.$$

Therefore

$$\int \mathrm{e}^{At}dt = A^{-1}\mathrm{e}^{At}\,.$$

Applying above,

$$\begin{aligned} x((k+1)T) &= \mathrm{e}^{AT}x(kT) + \mathrm{e}^{AT}\int_0^T \mathrm{e}^{-A\tau}d\tau \;\; Bu(kT) \\ &= \mathrm{e}^{AT}x(kT) + \mathrm{e}^{AT}A^{-1}\left[I - \mathrm{e}^{-AT}\right] \; Bu(kT) \\ &= \mathrm{e}^{AT}x(kT) + A^{-1}\left[\mathrm{e}^{AT} - I\right] \; Bu(kT)\,. \end{aligned}$$

3. (i) Write the logic definition of "the equilibrium $x = 0$ is stable i.s.L.".

(ii) Write the logic definition of "the equilibrium $x = 0$ is not asymptotically stable.".

(iii) State necessary and sufficient conditions for an equilibrium point $x = 0$ of $\dot{x} = Ax$ to be asymptotically stable.

(iv) Consider the system

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} x\,.$$

Using part (i), prove that $x = 0$ is stable i.s.L. (Be sure to separate the preparation part from the main proof.)

(i) $(\forall \epsilon > 0)(\exists \delta > 0)(\forall t \geq 0)(\forall x(0)) \|x(0)\| < \delta \implies \|x(t)\| < \epsilon$.

(ii) We are given a system $\dot{x} = f(x)$ and a point $x_0 \in \mathbb{R}^n$ such that $f(x_0) = 0$. We say x_0 is *asymptotically stable* if it is stable i.s.L. and it is attractive.

We say x_0 is *not asymptotically stable* if it is not stable i.s.L. OR it is not attractive. In logic format we have

$$(\exists \epsilon > 0)(\forall \delta > 0)(\exists t \geq 0)(\exists x(0)) \quad \|x(0)\| < \delta \;\wedge\; \|x(t)\| \geq \epsilon$$

OR

$$(\forall \delta > 0)(\exists x(0)) \quad \|x(0)\| < \delta \;\wedge\; \lim_{t \to \infty} x(t) \neq 0\,.$$

Note that the "wedge" symbol represents the logical AND.

(iii)

Theorem 1. *The equilibrium $x = 0$ of $\dot{x} = Ax$ is asymptotically stable if and only if the eigenvalues of A lie in the open left-half complex plane (OLHP).*

(iv) **Preparation:** We compute

$$x(t) = \mathrm{e}^{At}x(0) = \begin{bmatrix} \mathrm{e}^{-t} & 0 \\ 0 & 1 \end{bmatrix} x(0)\,.$$

Therefore

$$\|x(t)\| = \sqrt{\mathrm{e}^{-2t}x_1(0)^2 + x_2(0)^2}\,.$$

Proof:

- Let $\epsilon > 0$ be arbitrary.
- Choose $\delta = \epsilon$.
- Suppose $\|x(0)\| < \delta$. Then we have

$$\|x(t)\| = \sqrt{\mathrm{e}^{-2t}x_1(0)^2 + x_2(0)^2} \leq \sqrt{x_1(0)^2 + x_2(0)^2} = \|x(0)\| < \delta = \epsilon$$

as desired.

4. (i) Give three features of a control problem that indicate the need for a feedback approach rather than an open-loop approach.

   (ii) Give necessary and sufficient conditions such that the standard unity feedback system with open loop system $P(s)C(s)$ is asymptotically stable.

   (iii) Explain clearly, in mathematical terms, and using a block diagram the idea behind Black's negative feedback amplifier.

(i) Any of the following answers is correct:

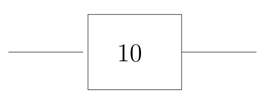
- presence of a dynamic system
- disturbances
- model uncertainty
- open-loop unstable system
- right half-plane zeros
- inability to construct open-loop inputs
- unacceptable open-loop transient response

(ii)

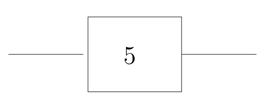
Theorem 2. *The standard unity feedback system is asymptotically stable if and only if*

- *The poles of the closed-loop system are in the open left half plane (OLHP).*
- *$P(s)C(s)$ has no pole-zero cancellations in the closed right half plane (CRHP).*

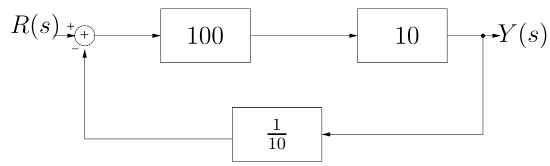
(iii) H.S. Black was an electrical engineer at Bell Laboratories who in 1927 invented the negative feedback amplifier. The invention made possible long distance telephone communications. As his wiki page states: *...his invention is considered the most important breakthrough of the twentieth century in the field of electronics.* Despite this, the US patent office resisted to issue the patent; it took more than nine years to obtain. The main idea of his design is extremely easy to understand, and it beautifully exploits the fundamental principle of *feedback*. Consider an amplifier with a gain of 10:



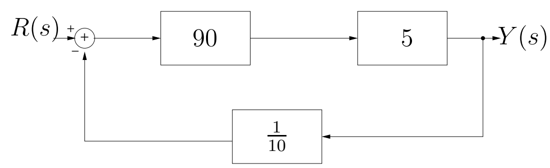
The device degrades because of aging, changes in load, temperature, power supply voltages, and nonlinearities. Suppose it degrades by 50%:



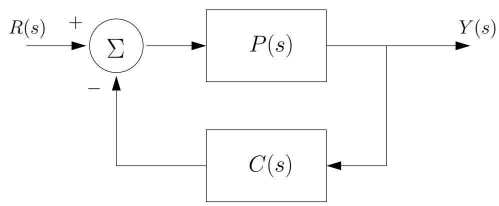
Consider instead Black's design. In the feedforward path he placed a high gain pre-amplifier and in the feedback path he put a highly reliable attentuation block (which can be built use a resistor bridge) with a gain of $\frac{1}{10}$. The new block diagram is:



The overall gain of this system is $\frac{Y(s)}{R(s)} = \frac{1000}{101} \simeq 10$. Now if both the pre-amp and main amplifier degrade as in this figure:



then the gain is $\frac{450}{1+45} \simeq 10$, which is only a minor degradation.
Another way to explain the main idea is using the notion of *loop gain*. Consider the block diagram below



Here $P(s)$ is an unreliable, possibly nonlinear *plant* and $C(s)$ is a highly reliable *compensator*. We define the *loop gain* to be

$$|P(s)C(s)| .$$

The transfer function of the closed-loop system is

$$\frac{Y(s)}{R(s)} = \frac{P(s)}{1 + P(s)C(s)} .$$

We observe that if $|P(s)C(s)| >> 1$, then

$$\frac{Y(s)}{R(s)} \simeq \frac{1}{C(s)} ,$$

which is a highly reliable transfer function. Instead, if $|P(s)C(s)| << 1$, then

$$\frac{Y(s)}{R(s)} \simeq P(s) ,$$

which is an unreliable transfer function. The conclusion is that the effect of unreliability in $P(s)$ can be "masked" by using high loop gain.

5. Consider the standard unity feedback system with plant $P(s) = \frac{1}{s+2}$ and compensator $C(s) = K$. Suppose that a constant disturbance d_0 enters at the plant input. Assume that the reference signal is $R(s) = \frac{1}{s}$. Show that this unity feedback system rejects disturbances using high loop gain.

The block diagram of the system is depicted on page 7. The state model of the plant is

$$\begin{aligned} \dot{x} &= -2x + u \\ y &= x\,. \end{aligned}$$

Also,

$$u = d_0 + K(r - y) = d_0 + K(1 - x)\,.$$

We get the closed-loop state model

$$\dot{x} = -(2 + K)x + K + d_0\,.$$

The output response is

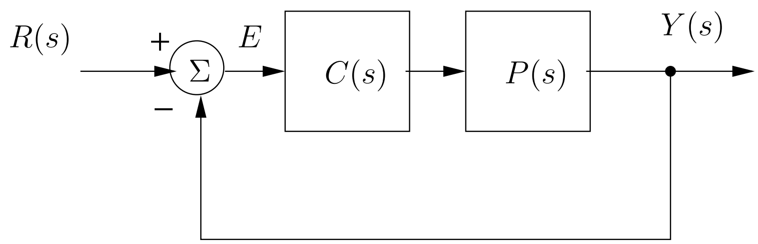
$$\begin{aligned} y(t) &= \mathrm{e}^{-(2+K)t} y(0) + \int_0^t \mathrm{e}^{-(2+K)(t-\tau)}(K + d_0)d\tau \\ &= \mathrm{e}^{-(2+K)t} y(0) - \frac{1}{2+K}\mathrm{e}^{-(2+K)t)}(K + d_0) + \frac{K}{2+K} - \frac{1}{2+K}d_0\,. \end{aligned}$$

Now we analyze the terms. The first two terms are transient terms, and these will die out assuming $K > -2$. The third term $\frac{K}{2+K}$ recovers the reference step $\frac{1}{s}$ that the system must track, modulo a small steady-state error, assuming high loop gain K.

Finally, the last term $\frac{1}{2+K}d_0$ shows the effect of the disturbance on the system output. We see that this disturbance term can be made arbitrarily small by making K sufficiently large. This shows that the unity feedback system rejects disturbances using high loop gain.

6. (i) State the final value theorem.

(ii) Consider the system



Prove that if the closed-loop transfer function $\frac{Y(s)}{R(s)}$ has a pole-zero cancellation, then $P(s)C(s)$ has a pole-zero cancellation.

(i)

Theorem 3. *Let $y(t)$ be a function with a Laplace transform $Y(s)$. If the poles of $sY(s)$ are in the open left half plane (OLHP), then*

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s).$$

(ii) *Proof.* Let

$$P(s) = \frac{N(s)}{D(s)}, \qquad C(s) = \frac{N_c(s)}{D_c(s)}$$

where we assume without loss of generality that $\{N(s), D(s)\}$ are coprime (i.e. they have no common factors) and $\{N_c(s), D_c(s)\}$ are coprime. The closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{P(s)C(s)}{1 + P(s)C(s)} = \frac{N_c(s)N(s)}{N_c(s)N(s) + D_c(s)D(s)} =: H(s)\,.$$

Suppose there is a pole-zero cancellation in $H(s)$. That is $H(s)$ has the form

$$H(s) = \frac{\widehat{N}(s)(s+a)}{\widehat{D}(s)(s+a)}$$

for some polynomials $\widehat{N}(s)$ and $\widehat{D}(s)$ and for some $a \in \mathbb{C}$. Comparing with the previous formula for $H(s)$ we get

$$H(s) = \frac{\widehat{N}(s)(s+a)}{\widehat{N}(s)(s+a) + D_c(s)D(s)}\,.$$

Comparing the denominators we get

$$D_c(s)D(s) = (\widehat{D}(s) - \widehat{N}(s))(s+a)\,.$$

Since

$$N_c(s)N(s) = \widehat{N}(s)(s+a)$$

it follows that $P(s)C(s)$ has a pole-zero cancellation of the factor $(s+a)$. □