1. Consider the system $\dot{x} = Ax$, $y = Cx$ with $x \in \mathbb{R}^2$ and $y \in \mathbb{R}$. Suppose that the matrix A is partially specified as

$$A = \begin{bmatrix} \alpha & -2 \\ 1 & 0 \end{bmatrix}.$$

Assume that the last component of each eigenvector of A is 1.

(a) Find the matrix A and the initial state $x(0)$ which gives an output response

$$y(t) = 4e^{-t} \qquad \text{when} \qquad C = [-1 \quad 1].$$

[10 marks]

(b) Find the output matrix C for which the output response is

$$y(t) = 4e^{-t} \qquad \text{when} \qquad x(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

[10 marks]

Sol'n

Find α s.t. one eigenvalue is -1.

$\det(sI-A) = 0 \iff s^2 - \alpha s + 2 = 0$, $\lambda_1 = -1$ is a root $\Rightarrow \alpha + 3 = 0$ $\boxed{\alpha = -3}$

[1 mark]

Find λ_2: $s^2 + 3s + 2 = 0 \Rightarrow \boxed{\lambda_2 = -2}$

[5 marks]

Initial state $x(0)$ will have to be along the e-vector for $\lambda_1 = -1$.

Find e-vectors: $V_1 = \begin{bmatrix} V_{11} \\ 1 \end{bmatrix}$; $V_2 = \begin{bmatrix} V_{21} \\ 1 \end{bmatrix}$ for $\lambda_1 = -1$, $\lambda_2 = -2$

$\Rightarrow \boxed{V_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} ; V_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$

Let: $x_0 = \xi_1 V_1 + \xi_2 V_2$ [1 mark]

[5 marks]

Then $y(t) = C e^{At} x_0 = C \cdot V_1 \cdot e^{-t} \xi_1 + C V_2 e^{-2t} \xi_2$

$= 2 e^{-t} \xi_1 + 3 e^{-2t} \xi_2$

so that $\xi_1 = 2$ and $\boxed{\xi_2 = 0} \Rightarrow \boxed{x(0) = 2V_1}$

b) Take $C = [c_1 \quad c_2]$

$$y(t) = CV_1 e^{-t} \cdot \xi_1 + CV_2 \cdot e^{-2t} \xi_2$$

$$= (c_2 - c_1) e^{-t} \xi_1 + (c_2 - 2c_1) e^{-2t} \xi_2$$

Need. $\Downarrow$

$$① \begin{cases} (c_2 - c_1)\xi_1 = 4 \\ (c_2 - 2c_1)\xi_2 = 0 \end{cases}$$

such that $y(t) = 4e^{-t}$,

Also: $X(0) = \xi_1 V_1 + \xi_2 V_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\Downarrow$

$$\begin{cases} -\xi_1 - \cancel{2}\xi_2 = 1 \\ \xi_1 + \xi_2 = -2 \end{cases} \Rightarrow \begin{cases} \xi_1 = -3 \\ \xi_2 = 1 \end{cases}$$

Use ξ_1, ξ_2 into ① $\Rightarrow \begin{cases} c_2 - c_1 = -4/3 \\ c_2 - 2c_1 = 0 \end{cases}$

$$\Rightarrow c_1 = -\frac{4}{3}; \; c_2 = -\frac{8}{3}$$

$$C = \left[-\frac{4}{3} \quad -\frac{8}{3}\right]$$

<u>then</u>: $u = K_1 x + e_1 F x$

$$= (K_1 + e_1 F) x = K x \text{ where}$$

$$K = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} -192 & -896 & 22 \end{bmatrix}$$

$$= \begin{bmatrix} -192 & -896 & 22 \\ 0 & 0 & -1 \end{bmatrix}$$

2. Find examples of two systems $\dot{x} = A_1 x$ and $\dot{x} = A_2 x$, $x \in \mathbb{R}^2$, such that

simultaneously

(a) $eig(A_1) = eig(A_2)$.

(b) The system $\dot{x} = A_1 x$ has a solution $x(t)$ whose magnitude $\|x(t)\| = \sqrt{x_1^2(t) + x_2^2(t)}$ is unbounded as $t \to \infty$.

(c) All solutions of the system $\dot{x} = A_2 x$ are bounded.

Justify your answer. [15 marks]

- In order that all solutions of $\dot{X} = A_2 X$ are bounded, we must have $eig(A_2) \subset \{ l.h.c.p. \}$. [5 marks]

- In order that at least one solution of $\dot{X} = A_1 X$ is unbounded, we must have $eig(A_1) \subset \{ r.h.c.p \}$. [5 marks]

- Since $eig(A_1) = eig(A_2)$, the only possibility is $eig(A_1) = eig(A_2) \subset \{ j\omega \text{ axis} \}$. [2 marks]

- Suppose $eig(A_1) = \{ \pm j\omega \}$ with $\omega \neq 0$. Then the solutions are of the form $X(t) = \cos(\omega t) V_1 + \sin(\omega t) V_2$ for some $V_1, V_2 \in \mathbb{R}^2$, and such solutions are never unbounded. Therefore $\omega = 0$. [2 marks]

- We have $eig(A_1) = eig(A_2) = \{0, 0\}$. There are two Jordan forms: [1 mark]

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Solutions for $\dot{x} = A_1 x$ are

$x(t) =$ constant $\Rightarrow$ bounded.

Solutions for $\dot{x} = A_2 x$ are

$$x(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x_0$$

Pick $x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$x(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}$ so $\|x(t)\| = \sqrt{t^2+1}$

unbounded as $t \to \infty$.