University of Toronto
Department of Electrical and Computer Engineering
ECE410F Control Systems
Problem Set #1
Selected Solutions

1.
$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\theta & 0 \\ 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \\ 0 & h_3 \end{bmatrix} u\,.$$
$$y = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} x\,.$$

2. Let $\dot{x} = f(x,u)$. Suppose this system can be linearized to

$$\begin{aligned} \triangle\dot{x} &= A\triangle x + B\triangle u, \\ \triangle y &= C\triangle x. \end{aligned}$$

An equilibrium point $x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$, u_0 satisfies $f(x_0,u_0) = 0$, i.e.,

$$\begin{aligned} 0 &= x_{10}(-\alpha_1 + \sin x_{20}) + x_{20}\sin x_{20} + u_0, \\ 0 &= x_{10}\sin x_{10} + x_{20}(-\alpha_2 + \sin x_{20}) + u_0. \end{aligned}$$

Taking one equation minus the other, we get

$$-\alpha_1 x_{10} + \alpha_2 x_{20} + x_{10}\sin x_{20} - x_{10}\sin x_{10} = 0$$

Now let's take $x_{10} = x_{20} = 0$, which means $u_0 = 0$. Then we compute the Jacobian matrices:

$$\begin{aligned} A &= \left.\frac{\partial f}{\partial x}\right|_{(x_0,u_0)} \\ &= \begin{bmatrix} -\alpha_1 + \sin x_{20} & x_{10}\cos x_{20} + \sin x_{20} + x_{20}\cos x_{20} \\ \sin x_{10} + x_{10}\cos x_{10} & -\alpha_2 + \sin x_{20} + x_{20}\cos x_{20} \end{bmatrix}_{(x_0,u_0)} \\ &= \begin{bmatrix} -\alpha_1 & 0 \\ 0 & -\alpha_2 \end{bmatrix}, \\ B &= \left.\frac{\partial f}{\partial u}\right|_{(x_0,u_0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ C &= \begin{bmatrix} 1 & 0 \end{bmatrix}. \end{aligned}$$

The next part is to determine the response to a step input $e^{-3}\overline{u}(t)$. First, we calculate the state transition matrix:

$$
\begin{aligned}
(sI - A)^{-1} &= \begin{bmatrix} \frac{1}{s+\alpha_1} & 0 \\ 0 & \frac{1}{s+\alpha_2} \end{bmatrix}, \\
e^{At} &= \begin{bmatrix} e^{-\alpha_1 t} & 0 \\ 0 & e^{-\alpha_2 t} \end{bmatrix}, t \geq 0
\end{aligned}
$$

Since the initial state is zero and $D = 0$, an approximate solution corresponding to the linearized model can be expressed as

$$
\begin{aligned}
y(t) &= \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau \\
&= \int_0^t \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} e^{-\alpha_1(t-\tau)} & 0 \\ 0 & e^{-\alpha_2(t-\tau)} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3} d\tau \\
&= \frac{1}{\alpha_1} e^{-3}(1 - e^{-\alpha_1 t}).
\end{aligned}
$$

3. We consider each of the input-output pairs. Let $Y_1(s) = V_1(s) + V_2(s)$ and $Y_2(s) = V_3(s)$ where $V_1(s) = \frac{1}{s}U_1(s)$, $V_2(s) = \frac{1}{s(s+2)}U_2(s)$, and $V_3(s) = \frac{1}{s+2}U_2(s)$. These transfer functions can be written as differential equations and are given as follow:

$$
\begin{aligned}
\dot{v}_1 &= u_1 \\
\ddot{v}_2 + 2\dot{v}_2 &= u_2 \\
\dot{v}_3 + 2v_3 &= u_2
\end{aligned}
$$

The first and third differential equations are first order while the second differential equation is second order. Hence, this is a fourth order system so we introduce four state variables: $x_1 = v_1$, $x_2 = v_2$, $x_3 = \dot{v}_2$, and $x_4 = v_3$. The state space model of the system is shown below:

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
$$

$$
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
$$

4. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\dot{x} = f(x, u)$. Suppose this system can be linearized to

$$
\triangle \dot{x} = A \triangle x + B \triangle u.
$$

An equilibrium point $x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$, $u_0 = \begin{bmatrix} u_{10} \\ u_{20} \end{bmatrix} = 0$ satisfies $f(x_0, u_0) = 0$, i.e.,

$$
\begin{aligned}
0 &= c_1 - \frac{c_2 x_{20}}{c_3 + x_{20}} - c_4 x_{10} - c_5 x_{10} x_{20}, \\
0 &= \frac{c_6 x_{20}}{c_7 + x_{20}} - c_7 x_{10} x_{20}.
\end{aligned}
$$

From the second equation, we can get

$$
0 = \left(\frac{c_6}{c_7 + x_{20}} - c_7 x_{10}\right) x_{20},
$$

such that $x_{20} = 0$. Therefore, $x_{10} = \frac{c_1}{c_4}$. Then the Jacobians evaluate to

$$
\begin{aligned}
A &= \left.\frac{\partial f}{\partial x}\right|_{(x_0,u_0)} \\
&= \begin{bmatrix} -c_4 - c_5 x_{20} + u_{10} & -\frac{c_2 c_3}{(c_3+x_{20})^2} - c_5 x_{10} \\ -c_7 x_{20} & \frac{c_6 c_7 (1-u_{20})}{(c_7+x_{20})^2} - c_7 x_{10} \end{bmatrix}_{(x_0,u_0)} \\
&= \begin{bmatrix} -c_4 & -\frac{c_2}{c_3} - \frac{c_1 c_5}{c_4} \\ 0 & \frac{c_6}{c_7} - \frac{c_1 c_7}{c_4} \end{bmatrix}, \\
B &= \left.\frac{\partial f}{\partial u}\right|_{(x_0,u_0)} \\
&= \begin{bmatrix} x_{10} & 0 \\ 0 & -\frac{c_6 x_{20}}{c_7+x_{20}} \end{bmatrix}_{(x_0,u_0)} \\
&= \begin{bmatrix} \frac{c_1}{c_4} & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned}
$$

When $c_i = 1$ for $i = 1, \ldots, 7$, the eigenvalues of the resultant system are $\{-1, 0\}$, so that this system is not asymptotically stable.

5. (a)

$$
\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B = \frac{1 + cs}{s^2 + as + b}
$$

Also,

$$
\begin{aligned}
x_1 &= \dot{y} + ay + cu \\
x_2 &= y\,.
\end{aligned}
$$

(b)

$$
\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B = \frac{1 + cs}{s^2 + as + b}
$$

Also,

$$
\begin{aligned}
x_1 &= y \\
x_2 &= \dot{y} - cu\,.
\end{aligned}
$$

(c)

6. (a) Recall

$$\begin{aligned}\frac{Y(s)}{U(s)} &= C[sI-A]^{-1}B+D \\ &= \tfrac{1}{s^2-1}[0 \quad 1]\begin{bmatrix} s & 1 \\ 1 & s\end{bmatrix}\begin{bmatrix} -1 \\ 1\end{bmatrix} \\ &= \tfrac{s-1}{s^2-1} \\ &= \tfrac{1}{s+1}\end{aligned}$$

This transfer function has a single pole -1 in the left-half plane, clearly implying stability. Notice that anytime we have a pole/zero cancellation in the right half plane, then we might run into problems, as is illustrated in part (b).

(b) Our goal is to show that the output grows exponentially. Recall from class that

$$y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)}Bu(\tau)\, d\tau\,.$$

Lets first find $e^{At} = \mathcal{L}^{-1}\{(sI-A)^{-1})\}$ (you are not restricted to use the Laplace method to find e^{At}). We already have $(sI-A)^{-1}$ from part (a), and by manipulation we get that

$$e^{At} = \frac{1}{2}\begin{bmatrix} e^t+e^{-t} & e^t-e^{-t} \\ e^t-e^{-t} & e^t+e^{-t}\end{bmatrix}$$

and

$$Ce^{At}x_0 = \frac{1}{2}\begin{bmatrix} e^t-e^{-t} & e^t+e^{-t}\end{bmatrix} x_0\,.$$

Normally we would also compute the second term of $y(t)$, but since the question asks us to show that in general $y(t)$ will grow exponentially, we don't need to. We can see that as $t\to\infty$ the term $e^t\to\infty$ and so will $y(t)$, depending on the initial condition. In particular, for all $x_0 \neq [k \quad -k]^T$, with $k\in\mathbb{R}$, y is unstable. (Why? If you're unsure, come to the tutorial. Think of eigenvectors).

7.

$$J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2.7321 & 0 \\ 0 & 0 & .7321\end{bmatrix}$$

8. Consider the autonomous system $\dot{x} = Ax$. In these problems we are asked to find the Jordan form of A. Generally this would require computing the eigenvectors and generalized eigenvectors of A. You have not been taught about generalized eigenvectors, but you are provided with extra data about A that enables you to solve these problems.

(a) Suppose that $eigs(A) = \{-1,-3,-3,-1+j2,-1-j2\}$. Also, suppose the rank of $(A-\lambda I)_{\lambda=-3}$ is 4. Now we know that the Jordan form must look like one of the following two cases. Either

$$\Lambda = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -1+2j & 0 \\ 0 & 0 & 0 & 0 & -1-2j\end{bmatrix}$$

or

$$\Lambda = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -1+2j & 0 \\ 0 & 0 & 0 & 0 & -1-2j \end{bmatrix}.$$

Basically you need to determine how many 1's are on the upper diagonal corresponding to the Jordan block for eigenvalue -3, which is the only eigenvalue that is repeated. This can be determined directly from the rank information provided. We are told that the rank of $(A - \lambda I)_{\lambda=-3}$ is 4. Now the relationship between A and Λ is by way of a similarity transformation $A = P\Lambda P^{-1}$ and this does not change the rank. In other words $\text{rank}(A - \lambda I)_{\lambda=-3} = \text{rank}(\Lambda - \lambda I)_{\lambda=-3}$. Knowing that $\text{rank}(\Lambda - \lambda I)_{\lambda=-3} =$ rules out the first choice for Λ above since we would get two columns zeroed out, resulting in a rank of 3. Therefore, we obtain the solution

$$\Lambda = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -1+2j & 0 \\ 0 & 0 & 0 & 0 & -1-2j \end{bmatrix}.$$

(b) Suppose that $eigs(A) = \{-1, -2, -2, -2\}$. Also, suppose the rank of $(A - \lambda I)_{\lambda=-2}$ is 3. Following the same procedure as described above,

$$\Lambda = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

(c) Suppose that $eigs(A) = \{-1, -2, -2, -2, -3\}$. Also, suppose the rank of $(A - \lambda I)_{\lambda=-2}$ is 3. Then,

$$\Lambda = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}.$$