University of Toronto
Department of Electrical and Computer Engineering
ECE410F   Control Systems
Problem Set #2
Selected Solutions

1. The solution of the system is:

$$
\begin{aligned}
x(t) &= e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\
y(t) &= Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t).
\end{aligned}
$$

In this case we get: $0 < t \le T$:

$$
\begin{aligned}
x(t) &= e^{At}x_0 + \int_0^t e^{A(t-\tau)}B\bar{u}d\tau \\
y(t) &= Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}B\bar{u}d\tau + D\bar{u}.
\end{aligned}
$$

$t > T$:

$$
\begin{aligned}
x(t) &= e^{At}x_0 + \int_0^T e^{A(t-\tau)}B\bar{u}d\tau \\
y(t) &= Ce^{At}x_0 + \int_0^T Ce^{A(t-\tau)}B\bar{u}d\tau.
\end{aligned}
$$

2. First compute the eigenvalues and eigenvectors of A. We get $eig(A) = \{1, 4\}$ and the corresponding eigenvectors are $v_1 = [\ -1\ \ 2\ ]^T$ and $v_2 = [\ 1\ 1\ ]^T$. Then we can write

$$
x_0 = \alpha_1 v_1 + \alpha_2 v_2.
$$

Solving for the α's we get $\alpha_1 = -\frac{1}{3}$, and $\alpha_2 = \frac{8}{3}$. Hence the modal decomposition of $x(t)$ is

$$
x(t) = -\frac{1}{3}e^t \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \frac{8}{3}e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
$$

3. Using MATLAB (command `eig`), we find that the modal form (or Jordan form) of A is diagonal, so the columns of the transforming matrix P are the eigenvectors of A:

$$
P = \begin{bmatrix} 0 & 0.1826 & -0.3482 \\ 0 & -0.3651 & -0.3482 \\ 1.0000 & -0.9129 & -0.8704 \end{bmatrix},
$$

$$
\Lambda = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix},
$$

The new state equation in modal form is $\Lambda = P^{-1}AP$ which we obtained above and $\tilde{B} = P^{-1}B$. Again with the help of MATLAB we get

$$\tilde{B} = P^{-1}B = \begin{bmatrix} -0.0000 \\ 1.8257 \\ -1.9149 \end{bmatrix},$$

Also, $\tilde{C} = CP$, or

$$\tilde{C} = \begin{bmatrix} 0 & 0.1826 & -0.3482 \end{bmatrix}.$$

The model representation can be expressed as

$$\dot{\bar{x}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \bar{x} + \begin{bmatrix} -0.0000 \\ 1.8257 \\ -1.9149 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 0 & 0.1826 & -0.3482 \end{bmatrix} \bar{x}.$$

5. First, we construct a controllability matrix for each system. Second, we find out the rank of the controllability matrix. The system is completely controllable iff Q_c is full rank.

   (a)

   $$Q_c = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$$
   $$= \begin{bmatrix} 0 & 3 & 11 \\ 0 & 4 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

   Clearly, the rank of Q_c is equal to 3, so the system is controllable.

   (b)

   $$Q_c = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$$
   $$= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

   Clearly, the rank of Q_c is equal to 1, so the system is not controllable.

6. We need to compute Q_c in terms of k_1 and k_2. First we get

$$AB = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} k_1 - k_2 \\ -(k_1 - k_2) \end{bmatrix}.$$

Then the controllability matrix is

$$Q_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} k_1 & k_1 - k_2 \\ k_2 & -(k_1 - k_2) \end{bmatrix}.$$

The system is completely controllable iff Q_c has full rank, i.e.,

$$|k_1| \neq |k_2|.$$

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7. At first sight, one will be thrown off by the question due to the ugliness of the numbers. However, we can see that A has a block structure. That is

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

with $A_{12} = 0_{2\times 2}$. Therefore, the eigenvalues can be found from $eig(A_{11})$ and $eig(A_{22})$. Noticing that $A_{22} = \begin{bmatrix} 0 & 13 \\ 0 & 0 \end{bmatrix}$ has two eigenvalues (poles) at the origin we can conclude that the system is not asymptotically stable.

For the second part of the question, let's first check if the system is actually stabilizable with a state feedback controller (i.e. with a controller of the form $u = Kx$, where K has no zero terms). By checking the rank of the controllability matrix Q_c, we notice that $rank(Q_c) = 4$ (use MATLAB if you like):

$$Q_c = \begin{bmatrix} -0.2000 & -0.1700 & 0.0343 & -0.0066 \\ 0.0300 & -0.0043 & 0.0008 & -0.0002 \\ 0 & -0.2000 & 0.2200 & -0.0216 \\ 0 & 0.0300 & -0.0043 & 0.0008 \end{bmatrix}.$$

Therefore, since the system is controllable it is automatically stabilizable. Notice that if the system were not stabilizable with an arbitrary state feedback controller then clearly it would not be stabilizable with the given controller.

Next, we're told that the input must be of the form $u = -[k_1 \quad 0 \quad k_3 \quad 0]x$. Also, notice that the matrix $\tilde{A}_{11} = \begin{bmatrix} 0 & -6 \\ 0 & 0 \end{bmatrix}$ is a good approximation of A_{11}. Similarly, we can set $\tilde{B} = [-20 \quad 3 \quad 0 \quad 0]^T$. There is no loss of generality for changing the B matrix, as we can simply let the scaling factor be distributed to k_1 and k_3. The above approximation allows us to work with a much nicer system:

$$\dot{\bar{x}} = \begin{bmatrix} 0 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 13 \\ 0 & 1 & 0 & 0 \end{bmatrix} \bar{x} + \begin{bmatrix} -20 \\ 3 \\ 0 \\ 0 \end{bmatrix} \bar{u}$$

The next step is to find the characteristic polynomial of $(\tilde{A} - \tilde{B} K)$ and use for instance the Routh-Hurwitz criterion to determine if one can shift the poles to the left half plane. After some manipulation, you'll notice that all the poles can be shifted. For instance, by setting $\bar{u} = -[-0.2 \quad 0 \quad 0.1 \quad 0]\bar{x}$, or $u = -[-20 \quad 0 \quad 10 \quad 0]x$ for the original system. With the chosen u we get that the $eig(A - BK) = -3.7354, \ -0.1407, \ -0.1619 \pm 0.6279i$, which are all in the left half complex plane. So the answer is, yes, we can stabilize the system with $u = -[k_1 \quad 0 \quad k_3 \quad 0]x$.

Two comments:

(a) You don't have to approximate the system, but it is much nicer to work with the approximation rather than the original. Also, whenever approximations are used, one must go back and actually check if the results are true for the original system.

(b) The choice of k_1 and k_3 is not unique.