University of Toronto
Department of Electrical and Computer Engineering
ECE410F Control Systems
Problem Set #5
Solutions

1. This is a standard LQR problem, with $Q = I_{2\times 2}$ and $R = \epsilon > 0$. The pair (A, B) is controllable, hence stabilizable, which guarantees this problem to be solvable. The optimal control is $K^* = -R^{-1}B^TP$, where P is the positive definite solution of (ARE):

$$A^TP + PA - PBR^{-1}B^TP + Q = 0\,,$$

i.e.

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} P + P \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \frac{1}{\epsilon} P \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} P + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0\,.$$

Solving the above equation, we obtain

$$P = \begin{bmatrix} \sqrt{1+2\sqrt{\epsilon}} & \sqrt{\epsilon} \\ \sqrt{\epsilon} & \sqrt{(1+2\sqrt{\epsilon})\epsilon} \end{bmatrix}.$$

Therefore, the associated control law is

$$\begin{aligned} u^* &= K^*x \\ K^* &= \begin{bmatrix} -\frac{1}{\sqrt{\epsilon}} & -\sqrt{\frac{1+2\sqrt{\epsilon}}{\epsilon}} \end{bmatrix}. \end{aligned}$$

The magnitude of u at $t = 0$, $\|u(0)\|$, is

$$\begin{aligned} u(0) &= \|K^*x(0)\| \\ &= \frac{1}{\sqrt{\epsilon}} + \sqrt{\frac{1+2\sqrt{\epsilon}}{\epsilon}}\,. \end{aligned}$$

In Figure 1, we observe that as $\epsilon \longrightarrow 0$, the control effort at the initial time goes to ∞, as expected.

2. The state-space model of the system is

$$\begin{aligned} \dot{x} &= \lambda x + u \\ y &= x\,. \end{aligned}$$

Stabilizability and detectability are easily verified so the problem is solvable. We solve

$$A^TP + PA - PBR^{-1}B^TP + C^TQC = 0$$

to obtain $2\lambda P - \frac{P^2}{\epsilon} + 1 = 0$. Solving for P and keeping in mind that $P > 0$, we get

$$P = \epsilon\lambda + \sqrt{\lambda^2\epsilon^2 + \epsilon}\,.$$

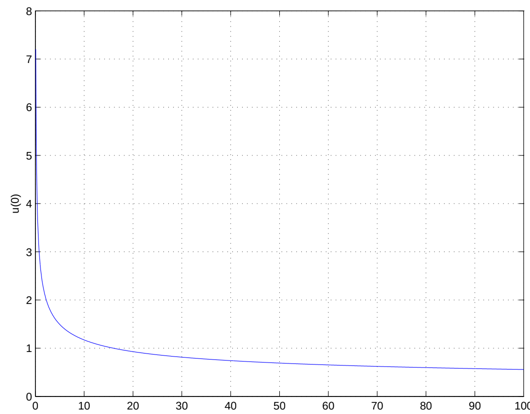

Figure 1: u(0)

and

$$u^* = \left(-\lambda - \sqrt{\lambda^2 + \frac{1}{\epsilon}}\right) x\,.$$

Next we consider the closed-loop poles of the system

$$eig(A + BK^*) = -\sqrt{\lambda^2 + \frac{1}{\epsilon}}\,.$$

We see that regardless of whether $\lambda < 0$ or $\lambda > 0$, if $\epsilon \to 0$ then the closed-loop poles approach $-\infty$, whereas if $\epsilon \to \infty$ then the poles approach $-|\lambda|$.

3. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$. It is easily verified that (C, A, G) is stabilizable and detectable. Therefore, the Kalman filter can be computed as

$$\dot{\hat{x}} = (A - KC)\hat{x} + Ky + Bu\,.$$

The optimal Kalman gain K is obtained by

$$K = PC^T V^{-1},$$

where P is the symmetric, positive definite solution of

$$0 = AP + PA^T + GWG^T - PC^T V^{-1} CP\,.$$

Solving the above equation, we obtain

$$P = V \begin{bmatrix} -1 + \sqrt{1 + 2\beta} & 1 + \beta - \sqrt{1 + 2\beta} \\ 1 + \beta - \sqrt{1 + 2\beta} & -1 - 2\beta + (1 + \beta)\sqrt{1 + 2\beta} \end{bmatrix},$$

and the optimal Kalman gain is

$$K = P \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{V} = \begin{bmatrix} -1 + \sqrt{1 + 2\beta} \\ 1 + \beta - \sqrt{1 + 2\beta} \end{bmatrix}.$$

The poles of the Kalman filter are

$$eig(A - KC) = \frac{1}{2}\left(-\sqrt{1+2\beta} \pm \sqrt{1-2\beta}\right).$$

When $W = 1$ and $V \longrightarrow 0$, then $\beta \longrightarrow \infty$, and

$$\frac{1}{2}(-\sqrt{1+2\beta} \pm \sqrt{1-2\beta}) \longrightarrow \sqrt{\frac{\beta}{2}}(-1 \pm j).$$

In other words, as $\beta \longrightarrow \infty$ the filter acts faster.