University of Toronto
Department of Electrical and Computer Engineering
ECE410F Control Systems
Problem Set #6
Solutions

1. We can easily verify that (A, B) is stabilizable. Therefore we can first design the asymptotic part of the controller. We want $eig(\overline{A}) = eig(A + BF_1) = \{ -1, -2 \}$. Using pole placement we obtain

$$F_1 = [\ -2 \ \ -2\ ] .$$

Then we obtain

$$\overline{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} .$$

Now we design the feedforward or exact matching part of the controller. That is, we must find a pair (Π, F_2) satisfying the FBI equations

$$\begin{aligned} \Pi S &= \overline{A}\Pi + BF_2 & (1) \\ 0 &= C\Pi - C_d . & (2) \end{aligned}$$

To that end, we must find a state space model for the exosystem. We take $p(t) = t^2$ and differentiate two times to obtain the ODE $p^{(3)} = 0$. Letting $w_1 = p$, $w_2 = \dot{p}$, and $w_3 = \ddot{p}$, the exosystem model is

$$\dot{w} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} w , \qquad y_d = [\ 2\ 0\ 0]$$

where the initial condition of the exosystem is $w(0) = [\ 0\ 0\ 1\ ]^T$. Then for $\Pi : \mathbb{R}^3 \to \mathbb{R}^2$ and $F_2 : \mathbb{R}^3 \to \mathbb{R}$, (1) gives

$$\begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \\ \pi_4 & \pi_5 & \pi_6 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \\ \pi_4 & \pi_5 & \pi_6 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [\ f_1 \ \ f_2 \ \ f_3\ ] .$$

We obtain the following solutions:

$$\begin{aligned} \pi_1 &= \pi_5 = 2 \\ \pi_2 &= \pi_3 = \pi_4 = \pi_6 = 0 \\ f_1 &= 4 \\ f_2 &= 6 \\ f_3 &= 2 . \end{aligned}$$

The tracking controller is then of the form:

$$\begin{aligned} u &= F_1 x + F_2 w \\ &:= \overline{F}_2 w + F_1(x - \Pi w) \\ &= -2x_1 - 2x_2 + 4w_1 + 6w_2 + 2w_3 . \end{aligned}$$

2. Suppose only the position of the snowplow is available for measurement. That is $y = [1\ 0]x$. We want to design an observer such that the eigenvalues are at $\{-10, -10\}$. First it is easily verified that (C, A) is observable. The observer equation is

$$\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly\,.$$

Setting $det(sI - A + LC) = (s + 10)^2$ we obtain $L^T = [19\ 81]$. To obtain a transfer function for the compensator (observer + controller) we take $U(s)$ as the output and $W(s)$ and $Y(s)$ as the two inputs to the compensator. First we take the Laplace transform of the observer equation to obtain

$$\hat{X}(s) = (sI - A - BF_1 + LC)^{-1} \left(BF_2 W(s) + LY(s) \right) .$$

Then using $u = F_1\hat{x} + F_2 w$ and substituting the previous equation we obtain

$$\begin{aligned} U(s) &= F_1\hat{X}(s) + F_2W(s) \\ &= F_1\left[(sI - A - BF_1 + LC)^{-1} + I\right]F_2W(s) + F_1(sI - A - BF_1 + LC)^{-1}LY(s)\,. \end{aligned}$$

3. We must design a tracking controller so that x_2 tracks a constant value of 20m/s. The velocity dynamics are

$$\dot{x}_2 = -x_2 + u\,, \qquad y = x_2\,.$$

The exosystem is

$$\dot{w} = 0\,, \qquad w(0) = 20\,.$$

The regulator equations are

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} \Pi \\ \overline{F}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ C_d \end{bmatrix}$$

This yields the constraints

$$-\Pi + \overline{F}_2 = 0\,, \qquad \Pi = 1\,.$$

Thus, $\overline{F}_2 = 1$. We can design F_1 by pole placement. Since this is a scalar system, we obtain immediately that $F_1 = -4$. The final tracking controller is

$$\begin{aligned} u &= \overline{F}_2 w + F_1(x_2 - \Pi w) \\ &= 5w - 4x_2\,. \end{aligned}$$