**University of Toronto**
**Department of Electrical and Computer Engineering**
**ECE557F Systems Control**
**Problem Set #3**

1. $Q_c = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

   $\mathcal{R}(Q_c) = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ so that the system is not controllable. Now $\begin{bmatrix} 4 \\ 1 \\ 4 \end{bmatrix}$ is not in $\mathcal{R}(Q_c)$ so that the vector is not reachable from the origin.

2. $Q_c = \begin{bmatrix} 1 & 1 & 4 & 0 & \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 1 & 1 & \\ 1 & 0 & 1 & 1 & \end{bmatrix}$

   The 1st 4 columns of $Q_c$ are linearly independent so that $(A, B)$ is controllable.

3. $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$

$$
\begin{aligned}
\det(sI - A) &= \det \begin{bmatrix} s & -1 & -1 \\ -1 & s+1 & 0 \\ 0 & -2 & s \end{bmatrix} \\
&= s(s^2 + s - 1) - 2 = s^3 + s^2 - s - 2 \\
V &= [A^2B \;\; AB \;\; B] \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \quad \text{(Hence } (A, B) \text{ is controllable)} \\
&= \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \\
V^{-1} &= \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}
\end{aligned}
$$

The desired char. poly. $r(s) = (s+1)^3 = s^3 + 3s^2 + 3s + 1$. Thus

$$
\begin{aligned}
K &= [-3 \quad -4 \quad -2]V^{-1} \\
&= [-3 \quad -4 \quad -2]\begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} = [-1 \quad -1 \quad -1] \\
A + BK &= \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 2 & 0 \end{bmatrix} + \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & -1 \end{bmatrix}
\end{aligned}
$$

4. We have the following differential equations

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 + 2x_2 &= \dot{x}_3 + x_3 \\
2\dot{x}_3 + x_3 &= u
\end{aligned}
$$

$$
\begin{aligned}
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\
&= \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} u
\end{aligned}
$$

$$
\begin{aligned}
\det(sI - A) &= \det \begin{bmatrix} s & -1 & 0 \\ 0 & s+2 & -\frac{1}{2} \\ 0 & 0 & s+\frac{1}{2} \end{bmatrix} = s(s+2)(s+\frac{1}{2}) \\
&= s^3 + \frac{5}{2}s^2 + s \\
V &= [A^2B \quad AB \quad B] \begin{bmatrix} 1 & 0 & 0 \\ \frac{5}{2} & 1 & 0 \\ 1 & \frac{5}{2} & 1 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{3}{4} & \frac{1}{2} & 0 \\ \frac{11}{8} & -\frac{3}{4} & \frac{1}{2} \\ \frac{1}{8} & -\frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \frac{5}{2} & 1 & 0 \\ 1 & \frac{5}{2} & 1 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} -6 & 4 & 0 \\ 11 & -6 & 4 \\ 1 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \frac{5}{2} & 1 & 0 \\ 1 & \frac{5}{2} & 1 \end{bmatrix} \\
&= \frac{1}{8} \begin{bmatrix} 4 & 4 & 0 \\ 0 & 4 & 4 \\ 0 & 8 & 4 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 1 & 0.5 \end{bmatrix} \\
\therefore \quad V^{-1} &= \begin{bmatrix} 2 & 2 & -2 \\ 0 & -2 & 2 \\ 0 & 4 & -2 \end{bmatrix}
\end{aligned}
$$

Desired $r(s) = (s+1)[(s+3)^2 + 1] = (s+1)(s^2 + 6s + 10) = s^3 + 7s^2 + 16s + 10$

$$
\begin{aligned}
k &= [p_0 - r_0 \quad p_1 - r_1 \quad p_2 - r_2]V^{-1} \\
&= [-10 \quad -15 \quad -\frac{9}{2}] \begin{bmatrix} 2 & 2 & -2 \\ 0 & -2 & 2 \\ 0 & 4 & -2 \end{bmatrix} = [-20 \quad -8 \quad -1]
\end{aligned}
$$

5. We prove this using the PBH test and Theorem 4.3.1.

($\Rightarrow$) Suppose $(A, B)$ is controllable. Suppose, by way of contradiction, that it is not the case that $(A + BK, B)$ is controllable for all $K$. So there exists $\overline{K}$ such that $(A + B\overline{K}, B)$ is not controllable. By the PBH test, this implies there exists an eigenvalue $\lambda \in \mathbb{C}$ of $A + B\overline{K}$ such that

$$
rank \begin{bmatrix} A + B\overline{K} - \lambda I & B \end{bmatrix} < n
$$

Thus, there exists a complex vector $v$ such that

$$
v^* \begin{bmatrix} A + B\overline{K} - \lambda I & B \end{bmatrix} = 0
$$

where $v^*$ denotes the conjugate transpose of $v$. But then for any $F$,

$$
v^*(A + B\overline{K} - \lambda I + B(-\overline{K} + F)) = v^*(A + BF - \lambda I) = 0\,.
$$

This implies $\lambda$ is an eigenvalue of $A + BF$ for any $F$. By the Pole Placement Theorem 4.3.1, this implies $(A, B)$ is not controllable, a contradiction.

($\Leftarrow$) Suppose $(A + BK, B)$ is controllable for all $K$. Set $K = 0$ to obtain the result.