University of Toronto
Department of Electrical and Computer Engineering
ECE557F Systems Control
Problem Set #7

1. (i) Since we are using a feedforward controller, we first solve for the steady state values of x^* and u^*. For this problem, they satisfy

$$\begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x^* \\ u^* \end{bmatrix} = \begin{bmatrix} 0 \\ y_d \end{bmatrix}$$

This gives $x_2^* = 0$, $x_1^* = y_d$, and $u^* = -2x_1^* = -2y_d$ Placing poles at -1 and -2 corresponds to a desired characteristic polynomial $r(s) = s^2 + 3s + 2$. Since the system is in controllable canonical form, we can immediately write down

$$K = [4\ 4]$$

The controller is given by

$$u = u^* + Kx^* - Kx = 2y_d - Kx$$

The transfer function from y_d to y is

$$\begin{aligned} y(s) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} 2y_d(s) \\ &= \frac{1}{s^2 + 3s + 2} 2y_d(s) \end{aligned}$$

so that the DC-gain is 1. Hence asymptotic tracking is achieved.

(ii) The observer equation for x_2 can be written down right away.

$$\dot{\hat{x}}_2 = \hat{x}_2 + u + 2y + l(\dot{y} - \hat{x}_2)$$

To place the observer pole at 4, we choose $l = 5$, giving

$$\dot{\hat{x}}_2 = -4\hat{x}_2 + u + 2y + 5\dot{y}$$

Let $z = \hat{x}_2 - 5y$. We get

$$\dot{z} = -4z + u - 18y$$

with

$$\hat{x} = \begin{bmatrix} y \\ z + 5y \end{bmatrix}$$

(iii) Using the separation principle, we can write the output feedback control law as

$$u = 2y_d - K\hat{x} = 2y_d - 4x_1 - 4\hat{x}_2 = 2y_d - 4x_1 - 4(z + 5x_1) = 2y_d - 24x_1 - 4z$$

The observer equation becomes

$$\dot{z} = -4z - 18x_1 + 2y_d - 24x_1 - 4z = -8z - 42x_1 + 2y_d$$

The closed loop system can be written as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 2x_1 + x_2 + 2y_d - 24x_1 - 4z = -22x_1 + x_2 - 4z + 2y_d \\ \dot{z} &= -8z - 42x_1 + 2y_d \end{aligned}$$

which can now be written as

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ -22 & 1 & -4 \\ -42 & 0 & -8 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} 2y_d \\ y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \end{aligned}$$

Thus the closed loop transfer function from y_d to y is given by

$$\begin{aligned} \frac{y(s)}{y_d(s)} &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} s & -1 & 0 \\ 22 & s-1 & 4 \\ 42 & 0 & s+8 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} 2 \\ &= \frac{2(s+4)}{s^3 + 7s^2 + 14s + 8} \\ &= \frac{2(s+4)}{(s+1)(s+2)(s+4)} \\ &= \frac{2}{(s+1)(s+2)} \end{aligned}$$

Since the DC-gain is 1, and the system is stable, we see that asymptotic tracking is indeed achieved.

2. (i) The augmented system is of the form

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} y_d$$

For any $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$ and K_I,

$$u = -Kx - K_I\xi$$

gives the following closed loop system matrix

$$A_c = \begin{bmatrix} 0 & 1 & 0 \\ 2 - K_1 & 1 - K_2 & -K_I \\ 1 & 0 & 0 \end{bmatrix}$$

which has the characteristic polynomial

$$\begin{aligned} det(sI - A_c) &= \begin{bmatrix} s & -1 & 0 \\ K_1 - 2 & s + K_2 - 1 & K_I \\ -1 & 0 & s \end{bmatrix} \\ &= s^2(s + K_2 - 1) + (K_1 - 2)s + K_I \\ &= s^3 + (K_2 - 1)s^2 + (K_1 - 2)s + K_I \end{aligned}$$

The desired characteristic polynomial is $r(s) = (s+1)^2(s+2) = s^3 + 4s^2 + 5s + 2$. This gives the gain values

$$K_1 = 7 \;\; K_2 = 5 \;\; K_I = 2$$

The closed loop system is therefore given by

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ -5 & -4 & -2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} y_d \\ y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} \end{aligned}$$

The transfer function from y_d to y is easily determined to be

$$\frac{y(s)}{y_d(s)} = \frac{2}{s^3 + 4s^2 + 5s + 2}$$

Since the transfer function is stable and its DC-gain is 1, asymptotic tracking of y_d is achieved.

(ii) If the plant system matrix is

$$A_p = \begin{bmatrix} 0 & 1 \\ 2.1 & 1 \end{bmatrix}$$

and the same control law is used, inspection of the previous step shows that the closed loop system matrix will now be given by

$$A_c = \begin{bmatrix} 0 & 1 & 0 \\ -4.9 & -4 & -2 \\ 1 & 0 & 0 \end{bmatrix}$$

The resulting transfer function from y_d to y is found to be

$$\frac{y(s)}{y_d(s)} = \frac{2}{s^3 + 4s^2 + 4.9s + 2}$$

Thus, by the same reasoning, asymptotic tracking is achieved.

(iii) The minimal order observer was found to be given by

$$\dot{z} = -4z + u - 18y$$

with

$$\hat{x} = \begin{bmatrix} y \\ z + 5y \end{bmatrix}$$

Using the separation principle, we set

$$u = -7y - 5\hat{x}_2 - 2\xi = -32y - 5z - 2\xi$$

Substituting into the observer equation gives

$$\dot{z} = -9z - 50y - 2\xi$$

The closed loop equations are given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 2x_1 + x_2 - 32x_1 - 5z - 2\xi = -30x_1 + x_2 - 5z - 2\xi \end{aligned}$$

Combining, we get the system

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\xi} \\ \dot{z} \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -30 & 1 & -2 & -5 \\ 1 & 0 & 0 & 0 \\ -50 & 0 & -2 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \xi \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} y_d \\ y &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \xi \\ z \end{bmatrix} \end{aligned}$$

The transfer function from y_d to y is found to be

$$\begin{aligned} \frac{y(s)}{y_d(s)} &= \frac{2(s+4)}{s^4 + 8s^3 + 21s^2 + 22s + 8} \\ &= \frac{2(s+4)}{(s+4)(s+1)^2(s+2)} \\ &= \frac{2}{(s+1)^2(s+2)} \end{aligned}$$

so that again asymptotic tracking is achieved.

3. (i) The augmented system is given by

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ -1 \end{bmatrix} y_d$$

Using the feedback $u = -Kx - K_I\xi$, we get the closed loop system matrix

$$A_c = \begin{bmatrix} 2-K & -K_I \\ 1 & 0 \end{bmatrix}$$

The desired characteristic polynomial is $r(s) = s^2 + 3s + 2$, giving

$$K = 5 \quad K_I = 2$$

to give

$$A_c = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}$$

(ii) If we add the feedforward term, we get the following closed loop system

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} N \\ -1 \end{bmatrix} y_d$$

The transfer function from y_d to y is given by

$$\begin{aligned} \frac{y(s)}{y_d(s)} &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+3 & 2 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} N \\ -1 \end{bmatrix} \\ &= \frac{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -2 \\ 1 & s+3 \end{bmatrix} \begin{bmatrix} N \\ -1 \end{bmatrix}}{s^2+3s+2} \\ &= \frac{Ns+2}{s^2+3s+2} \end{aligned}$$

This shows that for any N, asymptotic tracking is achieved. The presence of N introduces a zero in the numerator of the transfer function, which can be tuned to improve transient response.

4. We can easily verify that (A, B) is stabilizable. Therefore we can first design the asymptotic part of the controller. We want $eig(\overline{A}) = eig(A + BF_1) = \{ -1, -2 \}$. Using pole placement we obtain

$$F_1 = [\ -2 \ \ -2 \ ] .$$

Then we obtain

$$\overline{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} .$$

Now we design the feedforward or exact matching part of the controller. That is, we must find a pair (Π, F_2) satisfying the FBI equations

$$\begin{aligned} \Pi S &= \overline{A}\Pi + BF_2 &(1) \\ 0 &= C\Pi - C_d . &(2) \end{aligned}$$

To that end, we must find a state space model for the exosystem. We take $p(t) = t^2$ and differentiate three times to obtain the ODE $p^{(3)} = 0$. Now we put this ODE in state space form. Letting $w_1 = p$, $w_2 = \dot{p}$, and $w_3 = \ddot{p}$, the exosystem model is

$$\dot{w} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} w , \qquad y_d = [\ 2 \ 0 \ 0] w := C_d w ,$$

where the initial condition of the exosystem is $w(0) = [\ 0 \ 0 \ 1 \ ]^T$. Then for $\Pi : \mathbb{R}^3 \to \mathbb{R}^2$ and $F_2 : \mathbb{R}^3 \to \mathbb{R}$, (1) gives

$$\begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \\ \pi_4 & \pi_5 & \pi_6 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \\ \pi_4 & \pi_5 & \pi_6 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix} .$$

We obtain the following solutions:

$$\begin{aligned}
\pi_1 &= \pi_5 = 2 \\
\pi_2 &= \pi_3 = \pi_4 = \pi_6 = 0 \\
f_1 &= 4 \\
f_2 &= 6 \\
f_3 &= 2\,.
\end{aligned}$$

The tracking controller is then of the form:

$$\begin{aligned}
u &= F_1 x + F_2 w \\
&:= \overline{F}_2 w + F_1(x - \Pi w) \\
&= -2x_1 - 2x_2 + 4w_1 + 6w_2 + 2w_3\,.
\end{aligned}$$

5. Suppose only the position of the snowplow is available for measurement. That is $y = [1\ 0]x$. We want to design an observer such that the eigenvalues are at $\{-10, -10\}$. First it is easily verified that (C, A) is observable. The observer equation is

$$\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly\,.$$

Setting $det(sI - A + LC) = (s + 10)^2$ we obtain $L^T = [19\ 81]$. The final equation for the compensator (observer + controller) is

$$\begin{aligned}
U(s) &= F_1 \hat{X}(s) + F_2 W(s) \\
&= F_1(sI - A + LC - BF_1)^{-1}(BF_2 w + Ly) + F_2 W(s)\,.
\end{aligned}$$

6. We must design a tracking controller so that x_2 tracks a constant value of 20m/s. The velocity dynamics are

$$\dot{x}_2 = -x_2 + u\,, \qquad y = x_2\,.$$

The exosystem is

$$\dot{w} = 0\,, \qquad w(0) = 20\,, \qquad y_d = w\,.$$

The regulator equations are

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} \Pi \\ \overline{F}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ C_d \end{bmatrix}$$

This yields the constraints

$$-\Pi + \overline{F}_2 = 0\,, \qquad \Pi = 1\,.$$

Thus, $\overline{F}_2 = 1$. We can design F_1 by pole placement. Since this is a scalar system, we obtain immediately that $F_1 = -4$. The final tracking controller is

$$\begin{aligned}
u &= \overline{F}_2 w + F_1(x_2 - \Pi w) \\
&= 5w - 4x_2\,.
\end{aligned}$$