University of Toronto
Department of Electrical and Computer Engineering
ECE557F – Systems Control

Experiment 4

CONTROL OF AN INVERTED PENDULUM ON A CART

1 Purpose

This lab continues the investigation of the inverted pendulum on a cart, started in Lab 3, but this time you will use the real equipment. The purpose of the pendulum experiment is to demonstrate controller and observer design techniques on a physical "difficult-to-control" system. The controller will mainly be designed via optimal control. The control objective is for the cart to track a square wave while the pendulum remains upright. Also, the controller must improve the system's robustness to external disturbances.

2 Introduction

The following figure gives an illustration of the physical system.
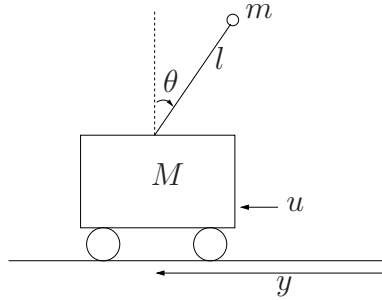
Ignoring inertia and friction effects, the simplified dynamics of the inverted pendulum is given by the following differential equations:

$$\ddot{y} = \frac{u - ml\dot{\theta}^2 \sin\theta + mg\cos\theta\sin\theta}{M + m\sin^2\theta} \tag{1}$$

$$\ddot{\theta} = \frac{(M+m)g\sin\theta + u\cos\theta - ml\dot{\theta}^2\sin\theta\cos\theta}{l(M + m\sin^2\theta)}. \tag{2}$$

The above system assumes a force input that horizontally pushes the cart. The pendulum system in our lab, however, takes a voltage input that drives the motor, which in turn drives the cart. So, to convert the former to the latter, we use the following relationship between

the driving force generated by the DC motor and the force acting on the cart through the motor pinion:

$$u = \alpha_1 \left(\frac{dy}{dt}\right) + \alpha_2 V, \tag{3}$$

where α_1 and α_2 are characteristics related to the motor, and V is the voltage input into the motor. Hence, substituting (3) into (1) and (2), we get the following nonlinear dynamic equations of the inverted pendulum system with a voltage input:

$$\ddot{y} = \frac{(\alpha_1 \dot{y} + \alpha_2 V) - ml\dot{\theta}^2 \sin\theta + mg\cos\theta\sin\theta}{M + m\sin^2\theta} \tag{4}$$

$$\ddot{\theta} = \frac{(M+m)g\sin\theta + (\alpha_1\dot{y} + \alpha_2 V)\cos\theta - ml\dot{\theta}^2\sin\theta\cos\theta}{l(M + m\sin^2\theta)}, \tag{5}$$

where V is a voltage and is now our input to the system.

For the actual inverted pendulum in the lab we have the following physical parameters:

- $M = 1.0731$ Kg

- $m = 0.2300$ Kg
- $\alpha_1 = -7.7443\frac{Ns}{m}$

- $l = 0.3302$ m
- $\alpha_2 = 1.7265\frac{N}{V}$

- $g = 9.8 \ m/s^2$

The system has two sensors which can measure the position of the cart and the angle of the pendulum.

2.1 Objectives:

The control design objectives of this experiment are:

(a) Balance the inverted pendulum about its vertical position;

(b) Control the cart to track a square wave of amplitude ± 20 mm with a frequency of 0.05 Hz;

(c) Achieve good transient response (Note: this is deliberately vague, and better transient response is often obtained by tuning control parameters); and

(d) Achieve a certain level of robustness to physical disturbances.

3   Preparation

In the preparation you will linearize the nonlinear model of the pendulum and cart and use this linear model to design a pole placement controller, full order observer, and an optimal controller. You will simulate both the pole placement controller and optimal controller on the nonlinear model. Note that there are some differences between the model you linearized in Lab 3 and the model here.

(a) Assume that the state vector is $x = [y \ \theta \ \dot{y} \ \dot{\theta}]^T$ (note the order is different from Lab 3) and derive a nonlinear state equation for x with input V. Then, determine the equilibrium point(s) of the system, $(\overline{y}, \overline{\theta}, \overline{\dot{y}}, \overline{\dot{\theta}})$ and $\overline{V}$. Next, compute the linearized model

$$\begin{aligned} \dot{\tilde{x}} &= A\tilde{x} + B\tilde{V} \\ \tilde{z} &= C\tilde{x} \end{aligned}$$

where $\tilde{x} = x - \overline{x}$, $\tilde{z} = z - \overline{z}$, $\tilde{V} = V - \overline{V}$. In this step of the prep, leave the model parameters M, m, l, and g as variables, i.e. do not plug in the numerical values.

(b) Write a Matlab script that sets the proper values of the model parameters and the equilibrium point $(\overline{y}, 0, 0, 0)$, and defines the (A, B, C, D) matrices in terms of these parameters. Use this script to carry out the following steps, which should be done using MATLAB and SIMULINK.

(c) Assume that the state of the pendulum system is fully observed. Design a state feedback law, $u = K\tilde{x}$, so that the closed loop poles are located at $\{-2, -4, -10, -20\}$.

(d) Now assume that only the cart position, y, and the pendulum angle, θ, are measured. Design a full-order observer to estimate the states by computing the full-order observer gain, L, so that observer poles are located at $\{-40, -40, -50, -50\}$.

(e) Before proceeding to add your controller and observer to the nonlinear system to obtain a closed-loop system, recall that the controller and observer are designed based on the "twiddle" variables $\tilde{x}$, $\tilde{z}$, and $\tilde{V}$. When you connect back to the nonlinear model, you must "un-twiddle" the variables by setting $x = \tilde{x} + \overline{x}$, $z = \tilde{z} + \overline{z}$, and $V = \tilde{V} + \overline{V}$. An example is shown in Figure 1. In particular, you can see that $\hat{x}$ is an estimate of $\tilde{x}$, not x.

(f) Now open the SIMULINK model `Lab4_prelab.slx` and modify the compensator block to include your controller and observer to give the output feedback control law, $u = K\hat{x}$. Make sure the model makes sense to you, in light of the comments in part (e).

Next, run the `init4.m` file in MATLAB to define the system parameters, and then simulate the response of the nonlinear pendulum system in SIMULINK. The initial conditions of the plant are, $(y_o, \theta_o, \dot{y}_o, \dot{\theta}_o) = (0, 10° \times \frac{\pi}{180°}, 0, 0)$ (i.e. the inverted pendulum
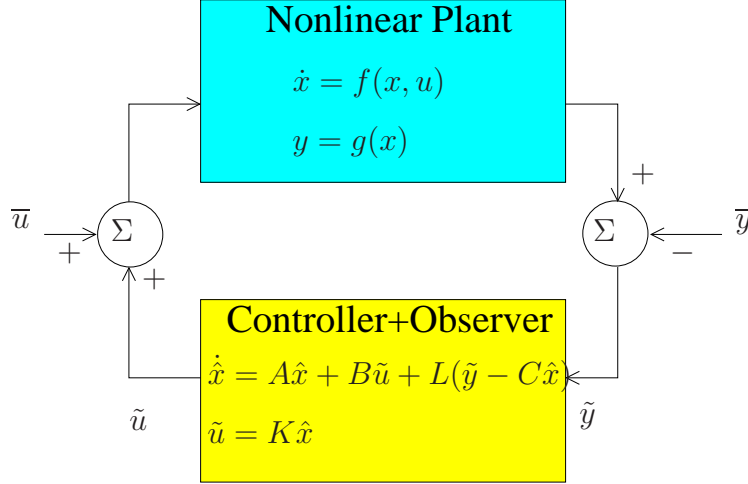
Figure 1: Block diagram of closed-loop nonlinear system

starts with an initial angle of $10°$). Also, set the observer initial conditions to zero. Plot your results on 4 plots, each plot containing one state component and its observer estimate. For the plot with y and $\hat{y} + \bar{y}$, also include the square wave input, $\bar{y}$.

Note that the nonlinear model used in this file actually includes inertia and friction terms excluded from our modeling in (4) and (5). This nonlinear model better approximates the real physical system. Try to make sense of the results you obtain, i.e. does the cart track a square wave, why or why not?

Feel free to tune the poles of the controller and see how it affects the response.

(g) Next you will design an optimal controller by selecting appropriate values for Q and R. Let

$$Q = \begin{bmatrix} q_1 & 0 & 0 & 0 \\ 0 & q_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad R = r$$

The parameters q_1 and q_2 will influence the speed of response of the cart position and the pendulum angle, respectively, while r will affect the control signal, V.

Design an optimal controller with $q_1 = 1$, $q_2 = 4$ and $r = 0.0005$. This time, plot on 2 plots only the cart position response and the angle response. Likewise, for the plot with the cart position response, y, also include the square wave input, $\bar{y}$.

Again, feel free to play with q_1, q_2 and r to see how it affects the response.

Each student must hand in their preparation in the beginning of the lab. The preparation should include:

1. The nonlinear state equations, the equilibrium point(s), and the linearized model as a function of the model parameters.

2. A MATLAB script for the pre-lab.

3. The controller gain (K) and observer gain (L) obtained in parts (c),(d), and (e).

4. Four plots for the response with the pole placement controller and observer.

5. Two plots for the response with the optimal controller and observer.

4   Experiment

Assuming that in your pre-lab you have tested out your optimal controller in simulation and obtained satisfactory response, it is now time to try the real thing.

CAUTION: Before running your experiment

- Please ensure that the cart is positioned at the middle of the track!

- Do not change any of the calibration gains in the SIMULINK model, as they are placed there for safety!

- Do not change the frequency and gain of the square input, unless you have permission from your TA!

- When running the experiment ensure that one student from the group is firmly holding the track!

- If for any reason your model goes unstable, turn off the experiment right away!

(a) Copy the lab files in `Share (S:) Courses \ECE557 \LAB4` to your working directory. Make sure MATLAB is also set to your working directory.

(b) Open the SIMULINK file `Lab4_Part1.slx`. Study the diagram and try to make sense of the blocks. Modify the compensator subsystem to implement the controller and observer you designed in the pre-lab (you may use q_1, q_2, and r given in the pre-lab, or you may use your own as long as you have verified that they work and achieve satisfactory response in simulation). Get a TA to verify your diagram and controller/observer design.

(c) Now it's time to build.

   1. Run the initialization files (i.e. `init4.m` and your pre-lab script) to define the system parameters, the controller and observer gain.

2. In SIMULINK, select the QUARC menu → **Clean All**. (Answer "yes" if it asks if it is OK to remove any directories or files; this clears any previous compiler files.)

   In SIMULINK, select the QUARC menu → **Build**. (This will compile your file - MATLAB will display it's progress.)

3. Wait for the model to finish compiling. Then go back to the SIMULINK model window and click the **Connect to Target** button that is just below the **Analysis** menu. The **run** button on it's right should turn green.

(d) To start the system

1. First let the pendulum stick hang down (i.e. in the 'gantry' position).

2. Then press the green **Run** button.

3. Then slowly bring the pendulum stick to the upright inverted position. Once the stick reaches 180° from the hanging down position, the controller will 'kick in'. At that point gently release the stick. The system should now be stable.

   Caution: the transition when the controller 'kicks in' may be violent if you don't let go of the stick in time or continue to grip it when the controller is working, so beware.

4.1   Tuning the Controller for Better Tracking

The system should now be stable, and the inverted pendulum is balanced in the upright position while the cart tracks a square wave. However, note that the cart is not tracking the square wave very well. We now need to tune the values of q_1, q_2, and r to try to get a better response.

(a) Firstly, using the q_1, q_2, and r obtained in the pre-lab, save the real-time data to a file and plot on two graphs the actual and the simulated position and angle responses. For the plot with position response, also include the reference square wave. How well does the cart track the square wave?

Also, plot on two graphs the cart velocity, $\dot{y}$, and the pendulum velocity, $\dot{\theta}$, along with their estimated states from the observer, $\dot{\hat{y}}$ and $\dot{\hat{\theta}}$. How well is the observer estimating the states?

(b) Change (increase and decrease) the values of q_1, q_2, and r individually and observe how the resulting response varies. Give a brief description (i.e. tracking, oscillations, overshoot, setting time, etc.).

Note: to change the values in real-time (i.e. while the system is still running), update the values in the MATLAB command prompt, and then type **'wc_update'** to update the corresponding variables in SIMULINK.

(c) Your TA will provide you a set of control specifications. By tuning, find values q_1, q_2, and r that meet the specifications. Provide a plot of the position and angle response, along with the reference signal, and highlight the features of the plot demonstrating that your controller meets the specifications.

4.2   Testing Robustness to Disturbances

In this part of the lab, we will balance the inverted pendulum without tracking, and tune our controller to improve the system's robustness to external disturbances.

(a) Open the SIMULINK file `Lab4_Part2.slx`. Modify the compensator block to implement your controller and observer from 4.1(c).

(b) Build and start the system as before.

(c) When the system is stable, gently tap the pendulum stick and observe the response. Save the data to a file and plot the position and angle response of the 'tap'. Briefly explain how that system reacts to the tap to recover stability.

(d) Change the values of q_1, q_2, and r and tap again. Does the system have a smoother recovery, or is the system less robust? Try a few more different values. (no plots required).

(e) Finally, try to find a set of q_1, q_2, and r (if possible) that makes the system more robust to disturbances. Plot the position and angle response.

5   Report

Please follow the report format instructions for Lab 4 provided by your TA.