Chapter 7

Output Regulation

In this chapter we extend the pole placement, observer-based output feedback design to solve tracking problems. By tracking we mean that the output is commanded to track asymptotically a desired reference trajectory. We will examine several variants of the problem. The simplest case is when there is full state information and there are no disturbances affecting the system. The next case is when there is partial state information, but the system is observable, and there are no disturbances. Applying the separation principle, we can solve this problem using observer-based output feedback design. Also, we study a special case when the reference trajectory to be tracked is a non-zero constant. Finally, the case when there are disturbances will be addressed in the next chapter.

First, we look at several examples as motivation and to illustrate two issues that will be faced in designing tracking controllers.

Example 7.0.1.

Suppose we want a small mobile robot to track a curve on the floor. We model the robot as a "kinematic unicycle", which is the simplest vehicle model that captures the no-sideslip constraint of wheeled vehicles. If the curve is a circle whose radius is not too small, intuition suggests that it should be feasible to design steering and velocity inputs to achieve the circular path exactly. If the initial condition of the robot is not aligned with the circle, then one would design a feedback controller that makes the robot approach the circular path asymptotically. Suppose instead that the path is a circle, combined with a straight line path that emanates from the circle at a right angle. If we require the robot to follow this path with unit speed, then because no vehicle can turn a right corner instantaneously, this path is infeasible. While exact tracking is not possible, one may be able to find a controller that keeps the trajectory of the robot as near as possible to the desired path. In summary, there are two issues in designing a tracking controller: feasibility of *exact tracking* of the desired output and *asymptotic convergence* to the desired output.

Example 7.0.2.

Consider the linear system

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \,,$$

and suppose we want this system to track a desired trajectory $x_d = (\mathrm{e}^{-t}, 0)$ which starts at initial condition $(1, 0)$ and ends at the origin. The first question to ask is: can the system track the desired trajectory assuming that the initial condition is $x(0) = (1, 0)$? Second: can we design an asymptotic controller that tracks the desired trajectory even if the initial condition is not $(1, 0)$? It is not difficult to see that the system cannot achieve exact tracking of the desired trajectory because the closed loop system can never point to the left along the positive x_1 axis for any control value. We would like to mathematically formalize this observation to obtain checkable conditions on the feasibility of exact tracking. Once it is determined that exact tracking can be achieved, we want a systematic procedure to design asymptotic tracking controllers.

One case in which exact tracking is trivial is SISO systems in controllable canonical form. Consider the system in Section 3.5. If we set

$$u = \alpha_0 x_1 + \alpha_1 x_2 + \cdots + \alpha_{n-1} x_n + v$$

where $v \in \mathbb{R}$ is a new input, then the system is converted to a chain of integrators with input v and output y. In order to achieve exact tracking of a desired output $y_d(t)$ we simply require that $v = y_d^{(n)}(t)$, i.e. v is the nth time derivative of the desired output signal. Also initial conditions must match exactly; namely, $x_i(0) = y_d^{(i-1)}(0)$, $i = 1, \ldots, n$.

7.1 Output Regulation with Full State Information

Consider the linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \,. \end{aligned} \tag{7.1}$$

Let y_d denote the desired signal for the output $y(t)$ to track asymptotically. We assume $y_d \neq 0$ and we assume it is generated as the output of an *exosystem* given by

$$\begin{aligned} \dot{w} &= Sw \\ y_d &= C_d w \,. \end{aligned}$$

The vector $w \in \mathbb{R}^q$ is the state of the exosystem. The tracking error is

$$e = y - y_d = Cx - C_d w \,.$$

The control objective is to design a feedback law $u(t) = F_1 x + F_2 w$ such that

(AS) $(A + BF_1)$ is stable, and

(R) $e(t) \to 0$ as $t \to \infty$.

The first requirement is that the closed loop system be stable when $w = 0$. The second requirement is that regulation of the output is achieved. A controller that satisfies the above objectives is called a *regulator*. In this section we make the following two assumptions:

- (A, B) is stabilizable.
- Both x and w are measurable.

7.1.1 Exact Tracking

First we consider the question of feasibility of the exact tracking problem. This problem can be stated as follows. Given the system (7.1) and a desired output $y_d(t)$ find a control $u(t)$ and an initial condition $x(0)$ such that $e(t) = 0$, $\forall t \geq 0$. We are interesting in studying this problem for theoretical interest and not practical motivations. This problem can be seen as a necessary first step in solving the asymptotic tracking problem.

Lemma 7.1.1. *Suppose there exists $u = \overline{F}_2 w$, where $\overline{F}_2 \in \mathbb{R}^{m\times q}$, and a map $\Pi : \mathbb{R}^q \to \mathbb{R}^n$ such that*

$$\Pi S = A\Pi + B\overline{F}_2 \tag{7.2}$$
$$C\Pi = C_d\,. \tag{7.3}$$

If $x(0) = \Pi w(0)$, then $e(t) = 0$ for all $t \geq 0$.

Proof. Suppose there exists $u = \overline{F}_2 w$ and Π such that (7.2)-(7.3) hold. Then define $z = x - \Pi w$. We have

$$\begin{aligned} \dot z &= \dot x - \Pi \dot w \\ &= Ax + B\overline{F}_2 w - \Pi S w \\ &= Ax - A\Pi w + A\Pi w + B\overline{F}_2 w - \Pi S w \\ &= Az \qquad\qquad \text{by (7.2)}\,. \end{aligned}$$

Since $z(0) = 0$ by assumption, the unique solution of $\dot z = Az$ is $z(t) = 0$, for all $t \geq 0$. That is, $x = \Pi w$ for all $t \geq 0$. Then, using (7.3) we obtain

$$e(t) = Cx(t) - C_d w(t) = C\Pi z(t) + C\Pi w(t) - C_d w(t) = 0\,.$$

□

The equations (7.2)-(7.3) are known as the *regulator* or *FBI* equations (after B. Francis, C. Byrnes, and A. Isidori).

Based on the proof of Lemma 7.1.1 we define the *tracking subspace* $\mathcal{T} \subset \mathbb{R}^{q+n}$

$$\mathcal{T} = \{(x, w) \mid x - \Pi w = 0\}.$$

If the regulator equations hold, then the tracking subspace is invariant under the closed loop dynamics. That is, $\mathcal{T}$ is A_{cl}-invariant where

$$A_{cl} = \begin{bmatrix} A & B\overline{F}_2 \\ 0 & S \end{bmatrix}.$$

Lemma 7.1.1 shows that the regulator equations and proper choice of initial conditions are sufficient for exact tracking. To what extent are the regulator equations also necessary for exact tracking? For this we require some extra conditions.

Lemma 7.1.2. *Assume that (C, A) is observable and $eig(S) \cap eig(A) = \emptyset$. Also assume that $(S, w(0))$ is controllable. Suppose that there exists an initial condition $x(0)$ and a control $u = \overline{F}_2 w$, where $\overline{F}_2 \in \mathbb{R}^{m\times q}$, such that the closed-loop system satisfies $e(t) = 0$ for all $t \geq 0$. Then there exists a map $\Pi : \mathbb{R}^q \to \mathbb{R}^n$ such that* (7.2)-(7.3) *hold. Moreover $x(0) = \Pi w(0)$.*

Proof. Since $eig(S) \cap eig(A) = \emptyset$, by Sylvester's Theorem (Gantmacher, *Theory of Matrices, vol. 1*, p. 225) there exists a unique solution Π of (7.2). Define $z(t) = x(t) - \Pi w(t)$. Using (7.2), we obtain $\dot z = Az$. Now consider

$$e(t) = Cz(t) + (C\Pi - C_d)w(t) = 0\,, \qquad \forall t \geq 0\,.$$

Since $eig(S) \cap eig(A) = \emptyset$, we know $Cz(t) = 0$ and $(C\Pi - C_d)w(t) = 0$, for all $t \geq 0$. Since (C, A) is observable, the first equation $C\mathrm{e}^{At}z(0) = 0$ yields $z(0) = 0$, or $x(0) = \Pi w(0)$, as desired. Next consider

$$(C\Pi - C_d)\mathrm{e}^{St}w(0) = 0\,, \qquad \forall t \geq 0\,.$$

By controllability of $(S, w(0))$, we obtain $C\Pi = C_d$, which gives (7.3). □

Example 7.1.1.

Consider again the example 7.0.2. We want to determine if the reference signal $y_d(t) = [\ \mathrm{e}^{-t}\ 0\ ]^T$ is feasible. First we observe that the signal is generated by an exosystem

$$\dot{w} = -w\,, \qquad w(0) = 1\,, \qquad y_d = \begin{bmatrix} 1 \\ 0 \end{bmatrix} w = C_d w\,.$$

Next we check the regulator equations:

$$\begin{bmatrix} \Pi_1 \\ \Pi_2 \end{bmatrix}(-1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \Pi_1 \\ \Pi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\overline{f}_2\,.$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \Pi_1 \\ \Pi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Simplifying, we obtain the equations

$$\begin{aligned} \Pi_1 &= \Pi_2 \\ -\Pi_2 &= \Pi_1 + \overline{f}_2 \\ \Pi_1 &= 1 \\ \Pi_2 &= 0\,, \end{aligned}$$

for which there is no solution. Finally, we can verify that (C, A) is observable and $(S, w(0))$ is controllable. From Lemma 7.1.2 we conclude that the exact tracking problem is infeasible, as expected.

7.1.2 Asymptotic Tracking

The regulator equations tell us a relationship between x and w, namely $x = \Pi w$ and a feedforward (open-loop) control $u = \overline{F}_2 w$ for exacting tracking of the desired output. What if $x(0) \neq \Pi w(0)$? To deal with a mismatch in initial conditions we need a feedback correction term in the control. The modified control for asymptotic tracking is:

$$u = \overline{F}_2 w + F_1(x - \Pi w) \triangleq F_1 x + F_2 w\,. \tag{7.4}$$

Let $\mathbb{C}^+$ denote the closed right-half complex plane.

Lemma 7.1.3. *Suppose that $eig(S) \in \mathbb{C}^+$ and $\overline{A} = A + BF_1$ is Hurwitz. A regulator $u = \overline{F}_2 w + F_1(x - \Pi w)$ exists if and only if there exist maps $\Pi : \mathbb{R}^q \to \mathbb{R}^n$ and $\overline{F}_2 : \mathbb{R}^q \to \mathbb{R}^m$ satifying* (7.2)-(7.3).

Proof. ($\Longleftarrow$) Suppose there exist $(\Pi, \overline{F}_2)$ satisfying the regulator equations. Define $z = x - \Pi w$.

$$\begin{aligned}\dot{z} &= \dot{x} - \Pi\dot{w} \\ &= \overline{A}z + [A\Pi + B\overline{F}_2 - \Pi S]w \\ &= \overline{A}z\,.\end{aligned}$$

Since $\overline{A}$ is Hurwitz, $z(t) \to 0$. Then

$$\begin{aligned}y(t) = Cx &= Cx - C\Pi w + C\Pi w \\ &= Cz + y_d\,.\end{aligned}$$

Hence, $y(t) - y_d(t) = Cz \to 0$ as $t \to \infty$.

($\Longrightarrow$) Since $eig(S) \cap eig(\overline{A} = \emptyset$, by Sylvester's Theorem (Gantmacher, *Theory of Matrices, vol. 1*, p. 225), there exists a unique solution Π satisfying

$$\Pi S - \overline{A}\Pi = BF_2\,.$$

Letting $\overline{F}_2 = F_2 + F_1\Pi$, we obtain (7.2). As shown above, $\dot{z} = \overline{A}z$. Hence $z(t) \to 0$ as $t \to \infty$. Also as above, $e(t) = Cz + (C\Pi - C_d)w$. By assumption $e(t) \to 0$ and since $z(t) \to 0$, it must be that $(C\Pi - C_d)w \to 0$ for all initial conditions $w(0)$. Since $eig(S) \in \mathbb{C}^+$, $w(t) \not\to 0$. Hence, $C\Pi = C_d$. □

Example 7.1.2.

Consider again the example 7.0.2, but with a less ambitious tracking problem. Suppose that $y = x_1$ and $y_d(t) = \mathrm{e}^{-t}$. The exosystem is

$$\dot{w} = -w\,, \qquad w(0) = 1\,, \qquad y_d = w\,.$$

$$\begin{bmatrix} \Pi_1 \\ \Pi_2 \end{bmatrix}(-1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \Pi_1 \\ \Pi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\overline{f}_2\,.$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \Pi_1 \\ \Pi_2 \end{bmatrix} = 1\,.$$

We obtain $\Pi_1 = 1$, $\Pi_2 = 1$, and $\overline{f}_2 = -2$. For exact tracking we use $u = -2w$, and the initial conditions are $x_1(0) = \Pi_1 w(0) = 1$ and $x_2(0) = \Pi_2 w(0) = 1$.

Next we design an asymptotic controller. Let $u = -2w + F_1(x - \Pi w)$, where $F_1 \in \mathbb{R}^{1\times 2}$. Since (A, B) is controllable we can design F_1 such that $A + BF_1$ has any desired closed-loop eigenvalues. If we want the eigenvalues to be $-1, -1$ then $F_1 = [0 - 2]$.

7.2 Special Case - Constant Reference Signals

While the tracking problem has been solved for much more general classes of reference inputs, in this section we focus on constant step reference inputs. This is an important special case, since the most common tracking problem is that of set point tracking. For convenience, we use $y_d(t) = y_d$ to denote the constant desired reference trajectory. Also, for simplicity, we shall assume that the number of inputs is equal to the number of outputs. Since $y_d \neq 0$, the steady state value of $x(t)$ cannot be 0. Also the exosystem is $\dot{w} = 0$, with $w(0) = 1$, so the regulator equations are

$$\begin{aligned} 0 &= A\Pi + B\overline{F}_2 \\ C\Pi &= y_d\,. \end{aligned}$$

Rearranging this equation we have

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} \Pi \\ \overline{F}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ y_d \end{bmatrix}. \tag{7.5}$$

By assumption of equal number of inputs and outputs, $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ is a square matrix. Since we would like to track any set point changes, y_d is arbitrary. Equation (7.5) can be solved uniquely for $\begin{bmatrix} \Pi \\ \overline{F}_2 \end{bmatrix}$ if and only if

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \quad \text{is nonsingular.} \tag{7.6}$$

Let us give an interpretation to condition (7.6). Suppose the input to (7.1) is given by $e^{\lambda t}\theta$. For zero initial conditions, the solution $x(t)$ is given by $e^{\lambda t}\xi$. Substituting into (7.1), we get the following equation

$$\lambda\xi = A\xi + B\theta\,. \tag{7.7}$$

Suppose this input results in no output. Then we must also have

$$C\xi = 0\,. \tag{7.8}$$

In the single-input single-output case (i.e., θ is a scalar), this gives the condition

$$C(\lambda I - A)^{-1}B = 0\,. \tag{7.9}$$

Such a λ is therefore a zero of the transfer function

$$G(s) = C(sI - A)^{-1}B$$

. For the multivariable case, we can combine equations (7.7) and (7.8) into

$$\begin{bmatrix} (A - \lambda I) & B \\ C & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \theta \end{bmatrix} = 0\,. \tag{7.10}$$

(7.10) can be solved for nonzero $\begin{bmatrix} \xi \\ \theta \end{bmatrix}$ if and only if

$$\begin{bmatrix} (A - \lambda I) & B \\ C & 0 \end{bmatrix} \quad \text{is not full rank}\,. \tag{7.11}$$

By analogy with the single-input single-output case, we call such a λ a *transmission zero* of the system.

From this discussion, we see that condition (7.6) corresponds to having no transmission zero at the origin. We can now state the conditions for solvability of the tracking problem:

1. (A, B) stabilizable
2. The system (7.1) has no transmission zeros at 0 .

If, in addition, (A, B) is in fact controllable, then the rate of convergence to 0 of the tracking error can be pre-assigned.

Example 7.2.1.

Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 24 & -10 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

The characteristic polynomial of the plant is given by

$$det(sI - A) = s^3 + 10s^2 - 24s = s(s^2 + 10s - 24) = s(s + 12)(s - 2)$$

The transfer function is given by

$$G(s) = \frac{1}{s(s + 12)(s - 2)}$$

so that there are no transmission zeros at 0. To solve for the steady state values Π and $\overline{F}_2$, we set

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 24 & -10 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Pi \\ \overline{F}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ y_d \end{bmatrix}.$$

Successively from the equation for each row, we see that $\Pi_2 = 0$, $\Pi_3 = 0$, $\overline{F}_2 = 0$, and $\Pi_1 = y_d$. Note that (A, B) is controllable. Suppose we choose the closed loop poles to be at -1, $-1 \pm i$. This corresponds to the desired characteristic polynomial

$$r(s) = (s^2 + 2s + 2)(s + 1) = s^3 + 3s^2 + 4s + 2\,.$$

Since (A, B) is in controllable canonical form, we immediately obtain

$$F_1 = \begin{bmatrix} -2 & -28 & 7 \end{bmatrix}.$$

The asymptotic tracking controller is

$$\begin{aligned} u &= \begin{bmatrix} -2 & -28 & 7 \end{bmatrix} \begin{bmatrix} x_1 - y_d \\ x_2 \\ x_3 \end{bmatrix} && (7.12) \\ &= 2(x_1 - y_d) - 28x_2 + 7x_3\,. && (7.13) \end{aligned}$$

To determine the transfer function from the reference input $y_d(t)$ to the output $y(t)$, first note that

$$\begin{aligned} A + BF_1 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 24 & -10 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & -28 & 7 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix}. && (7.14) \end{aligned}$$

Writing

$$u = F_1(x - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} y_d)$$

we can write the closed loop system as

$$
\begin{aligned}
\dot{x} &= (A+BF_1)x - BF_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} y_d \\
&= (A+BF_1)x + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} y_d
\end{aligned} \tag{7.15}
$$

Noting that (7.15) is again in controllable canonical form, we can immediately write down the transfer function from y_d to y as

$$
\begin{aligned}
y(s) &= C(sI - A - BF_1)^{-1} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} y_d(s) \\
&= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \frac{\begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix}}{s^3 + 3s^2 + 4s + 2} 2y_d(s) \\
&= \frac{2}{s^3 + 3s^2 + 4s + 2} y_d(s) \, .
\end{aligned} \tag{7.16}
$$

Since $y_d(s) = \frac{y_d}{s}$ and the closed loop system is stable, the steady state value of y can be determined from the final value theorem of Laplace transforms

$$
\begin{aligned}
\lim_{t \to \infty} y(t) &= \lim_{s \to 0} \frac{2}{s^3 + 3s^2 + 4s + 2} y_d \\
&= y_d
\end{aligned}
$$

so that asymptotic tracking is indeed achieved. In this example, no additional feedforward control is needed since there is a pole at the origin for the open loop plant. From classical control theory, we know that for such (type-1) systems, asymptotic step tracking is guaranteed using unity feedback as long as the closed loop system is stable. The state space formulation gives exactly this structure.

Example 7.2.2.

As another example, consider the linear system (7.1), but with

$$
A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} .
$$

The open loop characteristic polynomial is given by

$$
det(sI - A) = s^3 - 2s^2 - s + 2 = (s-1)(s+1)(s-2)
$$

Again to solve for the steady state values of Π and $\overline{F}_2$, put

$$
\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Pi \\ \overline{F}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} y_d
$$

First, second, and 4th rows give respectively $\Pi_2 = 0$, $\Pi_3 = 0$, $\Pi_1 = y_d$, while the 3rd row gives $\overline{F}_2 = 2y_d$.

Suppose we would like to place the closed loop poles at $-2,\ -1 \pm i$, so that the desired characteristic polynomial is

$$r(s) = (s^2 + 2s + 2)(s + 2) = s^3 + 4s^2 + 6s + 4\,.$$

This results in

$$F_1 = \left[\begin{array}{ccc} -2 & -7 & -6 \end{array}\right].$$

The control law is given by

$$\begin{aligned} u &= \overline{F}_2 w + F_1(x - \Pi w) \\ &= 4y_d - 2x_1 - 7x_2 - 6x_3\,. \end{aligned}$$

The transfer function from y_d to y is easily evaluated to be

$$y(s) = \frac{4}{s^3 + 4s^2 + 6s + 4} y_d(s)\,.$$

7.3 Output Regulation with Partial State Information

The extension to observer-based output regulation is straightforward using the separation principle. Whether one uses the full-order or reduced-order observer, the observer estimation error $x(t) - \hat{x}(t)$ satisfies a homogeneous stable equation. By replacing the control law (7.4) with

$$u = F_1 \hat{x} + F_2 w \tag{7.17}$$

we are guaranteed a solution of the tracking problem with a controller which is based on output feeback combined with a feedforward term.

To illustrate the procedure, we re-visit Example 7.2.2, using a reduced-order observer to estimate x_2 and x_3 and employing the feedback law (7.17). The decomposed system equations are given by

$$\begin{aligned} \begin{bmatrix} \dot{x_2} \\ \dot{x_3} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ -2 \end{bmatrix} y \\ \dot{x}_1 &= \left[\begin{array}{cc} 1 & 0 \end{array}\right] \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}. \end{aligned}$$

Hence the reduced-order observer is given by

$$\begin{bmatrix} \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ -2 \end{bmatrix} y + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \left(\dot{y} - \left[\begin{array}{cc} 1 & 0 \end{array}\right] \begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \end{bmatrix}\right). \tag{7.18}$$

The system matrix for the observer is given by

$$H = \begin{bmatrix} -l_1 & 1 \\ 1 - l_2 & 2 \end{bmatrix},$$

where l_1 and l_2 are to be chosen to place the poles of the observer. Its characteristic polynomial is given by

$$det(sI - H) = s^2 + (l_1 - 2)s + (l_2 - 1 - 2l_1)$$

Let us choose the observer poles to be at $-4, -4$. The desired observer characteristic polynomial is given by

$$r_o(s) = s^2 + 8s + 16$$

On matching coefficients, we see that $l_1 = 10$ and $l_2 = 37$. Thus the reduced-order observer is given by

$$\begin{bmatrix} \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \end{bmatrix} = \begin{bmatrix} -10 & 1 \\ -36 & 2 \end{bmatrix} \begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ -2 \end{bmatrix} y + \begin{bmatrix} 10 \\ 37 \end{bmatrix} \dot{y} \tag{7.19}$$

We shall use (7.19) for derivation of the controller and so we shall not perform the transformation to remove $\dot{y}$. The control law is given by

$$\begin{aligned} u &= \overline{F}_2 w + F_1(\hat{x} - \Pi w) \\ &= 4y_d - 2\hat{x}_1 - 7\hat{x}_2 - 6\hat{x}_3 \\ &= 4y_d - 2y - 7\hat{x}_2 - 6\hat{x}_3\,. \end{aligned} \tag{7.20}$$

Substituting the control law into (7.19), we find

$$\begin{aligned} \begin{bmatrix} \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \end{bmatrix} &= \begin{bmatrix} -10 & 1 \\ -36 & 2 \end{bmatrix} \begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (4y_d - 2y - \begin{bmatrix} 7 \\ 6 \end{bmatrix} \begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \end{bmatrix}) + \begin{bmatrix} 0 \\ -2 \end{bmatrix} y + \begin{bmatrix} 10 \\ 37 \end{bmatrix} \dot{y} \\ &= \begin{bmatrix} -10 & 1 \\ -43 & -4 \end{bmatrix} \begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} 4y_d + \begin{bmatrix} 0 \\ -4 \end{bmatrix} y + \begin{bmatrix} 10 \\ 37 \end{bmatrix} \dot{y}\,. \end{aligned} \tag{7.21}$$

Using (7.21), we can determine the transfer functions from y and y_d to $\begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \end{bmatrix}$ as

$$\begin{bmatrix} \hat{x}_2(s) \\ \hat{x}_3(s) \end{bmatrix} = \begin{bmatrix} s+10 & -1 \\ 43 & s+4 \end{bmatrix}^{-1} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} 4y_d + \left(\begin{bmatrix} 0 \\ -4 \end{bmatrix} + \begin{bmatrix} 10s \\ 37s \end{bmatrix} \right) y \right).$$

Finally, substituting into (7.20), we obtain

$$\begin{aligned} u &= -\begin{bmatrix} 7 \\ 6 \end{bmatrix} \begin{bmatrix} s+10 & -1 \\ 43 & s+4 \end{bmatrix}^{-1} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} 4y_d + \left(\begin{bmatrix} 0 \\ -4 \end{bmatrix} + \begin{bmatrix} 10s \\ 37s \end{bmatrix} \right) y \right) + 4y_d - 2y \\ &= \frac{s^2+8s+16}{s^2+14s+83} 4y_d - \frac{2s^2+4s-102}{s^2+14s+83} y - \frac{292s^2+179s}{s^2+14s+83} y \\ &= -\frac{294s^2+183s-102}{s^2+14s+83} y + \frac{s^2+8s+16}{s^2+14s+83} 4y_d\,. \end{aligned} \tag{7.22}$$

If we express (7.22) in the form

$$u(s) = -F(s)y(s) + C(s)y_d(s)$$

the closed loop transfer function from y_d to y is given by

$$y(s) = \frac{G(s)}{1+G(s)F(s)} C(s) y_d(s)\,.$$

Substituting, we finally get

$$\begin{aligned} y(s) &= \frac{4(s^2+8s+16)}{s^5+12s^4+54s^3+116s^2+128s+64} y_d(s) \\ &= \frac{4(s^2+8s+16)}{(s+4)^2(s+2)(s^2+2s+2)} y_d(s)\,. \end{aligned} \tag{7.23}$$

In the final transfer function (7.23), the observer poles are in fact cancelled, leaving

$$y(s) = \frac{4}{(s+2)(s^2+2s+2)} y_d(s)\,. \tag{7.24}$$