Chapter 8

Output Regulation with Disturbances

In this chapter we extend the output regulation methodology of the previous chapter to the case when the system is affected by an unmeasurable disturbance signal whose frequency content is known. It will be seen that the regulator equations again play a central role in solving this problem.

Disturbance signals are sometimes modelled as unmeasurable white noise inputs with certain statistical properties. See Section 6.4 on Kalman filter design. On the other hand, much may be know about a disturbance signal, even it is not directly measurable. A more realistic assumption is that the frequency content of the disturbance signal is known. There are many applications of this situation. For example:

- Landing a helicopter on a rolling ship.
- Rejecting engine vibration in autopilot actuators.
- Velocity tracking for UTIAS rovers subject to uneven weight distribution in the wheels due to the batteries.
- Control of a lift bridge in the presence of wind disturbances and a constant disturbance due to uncentered center of mass of the bridge span.

8.1 Problem Statement

Consider the linear system

$$\begin{aligned} \dot{x} &= Ax + Bu + Dw \\ y &= Cx + C_w w\,. \end{aligned} \tag{8.1}$$

The term Dw represents an unmeasurable disturbance signal which is generated by an exosystem whose dynamics are known. The output y represents the measurements of both the system and exosystem states. Let $y_d(t) \neq 0$ denote the desired signal to track asymptotically. We assume that both the disturbance and y_d are generated by an exosystem given by

$$\begin{aligned} \dot{w} &= Sw \\ y_d &= E_d w \end{aligned}$$

where $w \in \mathbb{R}^q$. The tracking error is

$$e = Ex - E_d w\,.$$

The control objective is to design a dynamic feedback of the form

$$\begin{aligned} \dot{\xi} &= G\xi + Hy && (8.2)\\ u &= F\xi && (8.3) \end{aligned}$$

such that

(AS) If $w(t) = 0$, then the closed-loop system

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} A & BF \\ HC & G \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix},$$

is asymptotically stable.

(R) For all initial conditions $(x(0), \xi(0), w(0))$, the closed-loop system satisfies $e(t) \to 0$ as $t \to \infty$.

A controller that satisfies the above objectives is called a *regulator*. In this section we assume neither x nor w are fully measurable. In addition, we require:

- (A, B) is stabilizable.
- (C_c, A_c) is detectable, where

$$A_c = \begin{bmatrix} A & D \\ 0 & S \end{bmatrix}, \qquad C_c = \begin{bmatrix} C & C_w \end{bmatrix}. \tag{8.4}$$

The detectability assumption says that both x and w can be recovered from the measurement y. The strategy for designing the regulator (8.2)-(8.3) will be to build an observer for x and w based on y.

8.2 Disturbance Decoupling

Before proceeding we discuss a simpler problem called *disturbance decoupling*. Suppose that we do not know the frequency content of the disturbance signal, so we do not have an exosystem model for it. We want to design a feedback control $u = Fx$ such that the disturbance does not appear in the output of the system. Consider the closed-loop system

$$\dot{x} = (A + BF)x + Dw\,, \qquad y = Cx\,.$$

The solution is

$$y(t) = Ce^{(A+BF)t}x(0) + \int_0^t Ce^{(A+BF)(t-\tau)}Dw(\tau)d\tau\,.$$

The requirement that y is unaffected by w implies that for all signals w,

$$C\int_0^t e^{(A+BF)(t-\tau)}Dw(\tau)d\tau = 0\,. \tag{8.5}$$

Let $\tilde{Q}_c$ be the controllability matrix of $(A+BF, D)$. Then condition (8.5) is equivalent to $C\mathcal{R}(\tilde{Q}_c) = 0$, or

$$\mathcal{R}(\tilde{Q}_c) \subset \mathcal{N}(C)\,. \tag{8.6}$$

Thus, the controller must be designed to guarantee the geometric condition (8.6) holds. See *W.M. Wonham, Linear Multivariable Control: A Geometric Approach* for the details.

8.3 Disturbance Rejection

Now we return to our output regulation problem with disturbance rejection. The main result is the following.

Theorem 8.3.1. *Suppose that* $eig(S) \in \mathbb{C}^+$, (A,B) *is stabilizable, and* (C_c, A_c) *is detectable. A regulator of the form* (8.2)-(8.3) *exists if and only if there exist maps* $\Pi : \mathbb{R}^q \to \mathbb{R}^n$ *and* $\overline{F}_2 : \mathbb{R}^q \to \mathbb{R}^m$ *satisfying*

$$\Pi S = A\Pi + B\overline{F}_2 + D \tag{8.7}$$

$$E\Pi = E_d\,. \tag{8.8}$$

The main idea is to construct an observer for the composite state (x, w). To this end, define $\eta = (x, w) \in \mathbb{R}^{n+q}$. The composite system is

$$\dot{\eta} = A_c\eta + B_c u\,, \qquad y = C_c\eta$$

where

$$B_c = \begin{bmatrix} B \\ 0 \end{bmatrix}.$$

An observer for the composite system is

$$\dot{\hat{\eta}} = A_c\hat{\eta} + B_c u + L(y - \hat{y})\,, \qquad \hat{y} = C_c\hat{\eta}\,. \tag{8.9}$$

The estimator error $\delta\eta = \eta - \hat{\eta}$ has dynamics $\dot{\delta\eta} = (A_c - LC_c)\delta\eta$. Since (C_c, A_c) is detectable we can choose L such that $A_c - LC_c$ is Hurwitz. We will construct a feedback

$$u = F\hat{\eta} = F_1\hat{x} + F_2\hat{w} \stackrel{\triangle}{=} \overline{F}_2\hat{w} + F_1(\hat{x} - \Pi\hat{w}). \tag{8.10}$$

Since (A, B) is stabilizable we can choose F_1 such that $\overline{A} \stackrel{\triangle}{=} A + BF_1$ is Hurwitz.

Proof. ($\Longleftarrow$) Suppose $(\Pi, \overline{F}_2)$ is a solution of (8.7)-(8.8). Consider the controller described by (8.9)-(8.10). We will show this controller is a regulator with $\xi = \hat{\eta}$. First check the the asymptotic stability requirement. Suppose $w(t) = 0$. Then the dynamics of the closed-loop system are given by

$$\begin{bmatrix} \dot{x} \\ \dot{\delta\eta} \end{bmatrix} = \begin{bmatrix} \overline{A} & -BF \\ 0 & (A_c - LC_c) \end{bmatrix} \begin{bmatrix} x \\ \delta\eta \end{bmatrix}.$$

Since the system matrix is Hurwitz we have $x(t) \to 0$ as desired.

Next we consider the regulation requirement. Define $z = x - \Pi w$. We have

$$\begin{aligned} \dot{z} &= Ax + B(F_1\hat{x} + F_2\hat{w}) + Dw - \Pi S w \\ &= \overline{A}z + [\overline{A}\Pi + BF_2 + D - \Pi S]w - BF_1\delta x - BF_2\delta w \\ &= \overline{A}z - BF_1\delta x - BF_2\delta w\,. \end{aligned}$$

Combining with the dynamics of $\delta\eta$ we have the composite dynamics

$$\begin{bmatrix} \dot{z} \\ \dot{\delta\eta} \end{bmatrix} = \begin{bmatrix} \overline{A} & -BF \\ 0 & (A_c - LC_c) \end{bmatrix} \begin{bmatrix} z \\ \delta\eta \end{bmatrix}.$$

Hence $z(t) \to 0$. Next we have

$$e(t) = Ex - E_d w = Ez + (E\Pi - E_d)w = Ez\,.$$

Hence, $e(t) \to 0$ as desired.

($\Longrightarrow$) Suppose we have a regulator of the form (8.2)-(8.3). When $w(t) = 0$, the closed-loop dynamics are

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} A & BF \\ HC & G \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix}.$$

By assumption, the closed-loop dynamics are asymptotically stable. Now consider the Sylvester equation

$$\begin{bmatrix} \Pi \\ \Sigma \end{bmatrix} S = \begin{bmatrix} A & BF \\ HC & G \end{bmatrix} \begin{bmatrix} \Pi \\ \Sigma \end{bmatrix} + \begin{bmatrix} D \\ HC_w \end{bmatrix}. \tag{8.11}$$

Because the eigenvalues of S and of the closed-loop system matrix are disjoint, by Sylvester's theorem there exists a unique solution for (Π, Σ). In particular, setting $\overline{F}_2 = F\Sigma$, we obtain (8.7). Next, let $z_1 = x - \Pi w$ and $z_2 = \xi - \Sigma w$. Then using (8.11) we obtain

$$\dot{z} = \begin{bmatrix} A & BF \\ HC & G \end{bmatrix} z$$

so $z(t) \to 0$. Finally, we get $e(t) = Ez(t) + (E\Pi - E_d)w(t)$. By assumption $e(t) \to 0$. Also, $z(t) \to 0$. Since $eig(S) \subset \mathbb{C}^+$, for all $w(0)$, $w(t) \to 0$. This implies $E\Pi = E_d$, as desired. $\square$