University of Toronto
Department of Electrical and Computer Engineering
ECE557F Systems Control
Problem Set #2
Selected Solutions

1. (a) $A = \begin{bmatrix} 4 & 1 & -1 \\ 3 & 2 & -3 \\ 1 & 3 & 0 \end{bmatrix}$ can be reduced by column operations to

$$\begin{bmatrix} 4 & 1 & -1 \\ 3 & 2 & -3 \\ 1 & 3 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 4 & -1 \\ 2 & 3 & -3 \\ 3 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & -5 & -1 \\ 3 & -11 & 3 \end{bmatrix}$$
$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -3 & 1 \end{bmatrix}$$

Then

(i) $\text{Rank}(A) = 3$, $\text{Im } A = R^3$

(ii) A basis for $\text{Im } A = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

(iii) kernel $A = \{0\}$.

(b) $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 1 & 2 \\ 3 & 4 & 5 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & \star \\ 2 & -1 & -2 & -7 & \star \\ 3 & -2 & -4 & -12 & \star \end{bmatrix}$

From columns 1, 2, and 4, we see that

(i) Rank $A = 3$.

(ii) Basis for $\text{Im } A = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

To find Ker A, we do elementary row operations

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 1 & 2 \\ 3 & 4 & 5 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 7 & 8 \\ 0 & 2 & 4 & 12 & 15 \end{bmatrix}$$
$$\longrightarrow \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 7 & 8 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 6 & 0 \\ 0 & 1 & 2 & 9 & 0 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix}$$
$$\longrightarrow \begin{bmatrix} 1 & \frac{4}{3} & \frac{5}{3} & 0 & 0 \\ 0 & 1 & 2 & 9 & 0 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix}$$

Thus

$$Ax = 0 \;\Rightarrow\; \begin{aligned} 2x_4 + x_5 &= 0 \\ x_2 + 2x_3 + 9x_4 &= 0 \\ x_1 + \frac{4}{3}x_2 + \frac{5}{3}x_3 &= 0 \end{aligned}$$

$$\begin{aligned}
x_5 &= -2x_4 \\
x_2 &= -2x_3 - 9x_4 \\
x_1 &= -\frac{4}{3}x_2 - \frac{5}{3}x_3 = -\frac{4}{3}(-2x_3 - 9x_4) - \frac{5}{3}x_3 \\
&= x_3 + 12x_4 \\
x &= \begin{bmatrix} x_3 + 12x_4 \\ -2x_3 - 9x_4 \\ x_3 \\ x_4 \\ -2x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 12 \\ -9 \\ 0 \\ 1 \\ -2 \end{bmatrix} x_4
\end{aligned}$$

Hence $\text{Ker } A = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 12 \\ -9 \\ 0 \\ 1 \\ -2 \end{bmatrix} \right\}$

2. Let $A, U \in \mathbb{R}^n$, $U \neq I$, and U nonsingular.

   (a) $\mathcal{N}(A) = \mathcal{N}(UA)$. T or F? **Ans:** True. Proof: Suppose $x \in \mathcal{N}(A)$. Then $Ax = 0$. So $UAx = u0 = 0$, or $x \in \mathcal{N}(A)$. Conversely, suppose $x \in \mathcal{N}(UA)$. Then $UAx = 0$. Multiply on left and right by U^{-1}, to obtain $Ax = 0$, or $x \in \mathcal{N}(A)$.

   (b) $\mathcal{N}(A) = \mathcal{N}(AU)$. T or F? **Ans:** False. Try

   $$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

   Then if $v = [2 \;\; 1]^T$, $v \in \mathcal{N}(AU)$ but $v \notin \mathcal{N}(A)$.

   (c) $\mathcal{N}(A^2) \subseteq \mathcal{N}(A)$. T or F? **Ans:** False. Take

   $$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

3. Let x_i be real numbers. Is $\{\ (x_1, x_2, x_3) : 2x_1 + 3x_2 + 6x_3 - 5 = 0\}$ a subspace of $\mathbb{R}^3$? **No:** Zero is not a solution of this constraint so the properties of a subspace are not satisfied.

4. You are given $A : \mathbb{R}^2 \to \mathbb{R}^2$, $A \neq 0$ and $\mathcal{R}(A) \subseteq \mathcal{N}(A)$. Can you find A? Explain. **Ans:** No, you cannot find A. The only thing you know is that $A^2x = 0$ for all x, so $A^2 = 0$, i.e. A is nilpotent.

5. You are given the n eigenvalues of the matrix $A \in \mathbb{R}^{n \times n}$. Can you determine $rank(A)$? If yes, given an expression for $rank(A)$. If no, can you give bounds on $rank(A)$? **Ans:** No, for example

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

both have eigenvalues $\{0, 0, 0\}$ but $rank(A_1) = 2$ and $rank(A_2) = 1$. Let m be the number of zero eigenvalues. Then $rank(A) \geq n - m$. If there is exactly one Jordan block associated with the zero eigenvalue, then $rank(A) = n - 1$. Hence, if $m = 0$, $rank(A) = n$ and if $m > 0$ then $n - m \leq rank(A) \leq n - 1$.

6. Suppose you have a ray starting at $x(0) = x_0$ and not including the origin. Can it be the trajectory of a linear system $\dot{x} = Ax$, $x \in \mathbb{R}^2$? What about in higher dimensions? Explain. **Ans:** Yes, one can have trajectories on a line not through the origin, in any dimension. But this will require at least one zero eigenvalue of A.

7. Two matrices $A, B \in \mathbb{R}^{n \times n}$ each have distinct eigenvalues. Also they share the same set of eigenvectors. Is $AB = BA$? **Ans:** Yes, since $A = P^{-1}\Lambda_A P$ and $B = P^{-1}\Lambda_B P$ where P is the matrix whose columns are the eigenvectors of A (or B). Hence,

$$AB = P^{-1}\Lambda_A P P^{-1}\Lambda_B P = P^{-1}\Lambda_A \Lambda_B P = P^{-1}\Lambda_B \Lambda_A P = P^{-1}\Lambda_B P P^{-1}\Lambda_A P = BA\,.$$

8. We know that $rank(AB) = m - dim(\mathcal{N}(AB))$ so the key idea to finding a necessary and sufficient condition for AB to be rank m is that $\mathcal{N}(AB)$ be the trivial subspace $\{0\}$. This can be guaranteed if no vector in $\mathcal{R}(B)$ lies in $\mathcal{N}(A)$. So we propose the condition

$$\mathcal{R}(B) \cap \mathcal{N}(A) = \{0\}\,.$$

Proof: (Necessity) Suppose $rank(AB) = m$. Suppose by way of contradiction that there exists $v \neq 0$ and $v \in \mathcal{R}(B) \cap \mathcal{N}(A)$. Then $ABv = 0$, i.e. $dim(\mathcal{N}(AB)) \geq 1$, so $rank(AB) < m$.

(Sufficiency) Suppose $\mathcal{R}(B) \cap \mathcal{N}(A) = \{0\}$. Now suppose by way of contradiction that $rank(AB) < m$. This means there exists $v \neq 0$ such that $ABv = 0$. This can happen either if

(a) $Bv = 0$ implying $v \in \mathcal{N}(B)$. But this is impossible since $dim(\mathcal{N}(B)) = m - rank(B) = 0$, or

(b) $Bv \neq 0$ implying $Bv \in \mathcal{N}(A)$, which contradicts the assumption $\mathcal{R}(B) \cap \mathcal{N}(A) = \{0\}$.

9. (a) Using the form of F, we can write down the following equations for the components of $\phi(t)$:

$$\begin{aligned} \dot{\phi}_0(t) &= -\alpha_0 \phi_{n-1}(t) \\ \dot{\phi}_k(t) &= \phi_{k-1}(t) - \alpha_k \phi_{n-1}(t) \quad \text{for } k = 1, \ldots, n-1 \end{aligned}$$

Putting these equations together, we can write

$$
\begin{aligned}
\dot{G}(t) &= \sum_{i=0}^{n-1} \dot{\phi}_i(t)A^i \\
&= -\alpha_0\phi_{n-1}(t)I + \sum_{k=1}^{n-1}(\phi_{k-1}(t) - \alpha_k\phi_{n-1}(t))A^k \\
&= -(\alpha_0 I + \sum_{k=1}^{n-1}\alpha_k A^k)\phi_{n-1}(t) + \sum_{k=1}^{n-1}(\phi_{k-1}(t)A^k \\
&= A^n\phi_{n-1}(t) + \sum_{k=1}^{n-1}(\phi_{k-1}(t)A^k \quad \text{using Cayley-Hamilton} \\
&= A\sum_{j=0}^{n-1}(\phi_j(t)A^j \\
&= AG(t)
\end{aligned}
$$

Finally

$$G(0) = A^0 = I$$

so that G(t) satisfies the same differential equation and initial condition as e^{At}. This proves $G(t) = e^{At}$.

(b) The characteristic polynomial of A is $s^2 + 3s + 2$. Hence the matrix F is given by

$$F = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}$$

We can compute e^{Ft} by using Laplace transforms.

$$
\begin{aligned}
\begin{bmatrix} s & 2 \\ -1 & s+3 \end{bmatrix}^{-1} &= \frac{\begin{bmatrix} s+3 & -2 \\ 1 & s \end{bmatrix}}{(s+1)(s+2)} \\
&= \begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{-2}{s+1} + \frac{2}{s+2} \\ \frac{1}{s+1} - \frac{1}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix} \\
&= \begin{bmatrix} 2e^{-t} - e^{-2t} & -2e^{-t} + 2e^{-2t} \\ e^{-t} - e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}
\end{aligned}
$$

Hence

$$\phi(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

so that

$$e^{At} = (2e^{-t} - e^{-2t})I + (e^{-t} - e^{-2t})A$$

After simplification, we get

$$e^{At} = \begin{bmatrix} e^{-t} & e^{-t} - e^{-2t} \\ 0 & e^{-2t} \end{bmatrix}$$

By direct evaluation using Laplace transforms, we have

$$\begin{aligned}(sI - A)^{-1} &= \begin{bmatrix} s+1 & -1 \\ 0 & s+2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix}\end{aligned}$$

Hence

$$e^{At} = \begin{bmatrix} e^{-t} & e^{-t} - e^{-2t} \\ 0 & e^{-2t} \end{bmatrix}$$

the same result as before.