University of Toronto
Department of Electrical and Computer Engineering
ECE557F Systems Control
Problem Set #5

1. (a) By direct computation,

$$\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

which is easily verified to be nonsingular. Hence (C, A) is observable.

(b) For $K = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, we find

$$A + BK = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

We then have

$$\begin{bmatrix} C \\ C(A+BK) \\ C(A+BK)^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -2 \\ 1 & 4 & 4 \end{bmatrix}$$

which is easily verified to be singular. Hence $(C, A+BK)$ is not observable. To construct an example of a system becoming observable after state feedback, just reverse the steps in part (b). Set

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

Then (C, A) is not observable. Let $K = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}$. $(C, A+BK)$ is then observable.

2. The system equations are given by

$$\dot{x} = \begin{bmatrix} 0 & \omega & 0 \\ -\omega & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} x \qquad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

To construct a minimal observer, we identify

$$A_{11} = 0 \qquad A_{12} = \begin{bmatrix} \omega & 0 \end{bmatrix} \qquad A_{21} = \begin{bmatrix} -\omega \\ 0 \end{bmatrix} \qquad A_{22} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Original system is observable if and only if $[A_{12},\ A_{22}]$ is observable, i.e. $\begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix}$ is nonsingular, which is true if $\omega \neq 0$. Decompose the system equations into

$$\begin{aligned} \dot{x}_1 &= \omega x_2 \\ \dot{x}_2 &= -\omega x_1 + x_3 \\ \dot{x}_3 &= x_3 \end{aligned}$$

Thus the minimal observer is of the form

$$\begin{bmatrix} \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} + \begin{bmatrix} -\omega \\ 0 \end{bmatrix} y + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} (\dot{y} - \omega \hat{x}_2)$$

For the observer to have poles at -1 and -2 we require the eigenvalues of

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} [\ \omega \quad 0\ ] = \begin{bmatrix} -l_1\omega & 1 \\ -l_2\omega & 1 \end{bmatrix}$$

to be at -1 and -2

$$\begin{aligned} \det \left\{ sI - \begin{bmatrix} -l_1\omega & 1 \\ -l_2\omega & 1 \end{bmatrix} \right\} &= \det \begin{bmatrix} s + l_1\omega & -1 \\ l_2\omega & s-1 \end{bmatrix} \\ &= s^2 + (l_1\omega - 1)s + (l_2 - l_1)\omega \\ &= s^2 + 3s + 2 \qquad \text{for pole placement} \\ \therefore \qquad l_1 = \frac{4}{\omega},\ l_2 &= \frac{6}{\omega} \\ \begin{bmatrix} \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ -6 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} &+ \begin{bmatrix} -\omega \\ 0 \end{bmatrix} y + \begin{bmatrix} \frac{4}{\omega} \\ \frac{6}{\omega} \end{bmatrix} \dot{y} \end{aligned}$$

Letting $\begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} - \begin{bmatrix} \frac{4}{\omega} \\ \frac{6}{\omega} \end{bmatrix} y = z$, we obtain

$$\begin{aligned} \dot{z} &= \begin{bmatrix} -4 & 1 \\ -6 & 1 \end{bmatrix} \left(z + \begin{bmatrix} \frac{4}{\omega} \\ \frac{6}{\omega} \end{bmatrix} y \right) + \begin{bmatrix} -\omega \\ 0 \end{bmatrix} y \\ &= \begin{bmatrix} -4 & 1 \\ -6 & 1 \end{bmatrix} z + \begin{bmatrix} -\frac{10}{\omega} - \omega \\ -\frac{18}{\omega} \end{bmatrix} y \end{aligned}$$

The minimal order observer is given by

$$\begin{aligned} \hat{x}_1 &= y \\ \begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} &= z + \begin{bmatrix} \frac{4}{\omega} \\ \frac{6}{\omega} \end{bmatrix} y \end{aligned}$$

3. (i) State space realization of $\dfrac{1}{s^2(s+1)}$: Choose controllable canonical form to give

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & \vdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \vdots & 0 & 1 \\ 0 & \vdots & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ \cdots \\ 0 \\ 1 \end{bmatrix} u \\ y &= [\ 1 \quad 0 \quad 0\ ] x \end{aligned}$$

(ii) Pole placement by state feedback: Since we desire all transients to decay at least as fast as e^{-2t}, choose desired closed loop polynomial to be $r(s) = (s+2)^3 = s^3 + 6s^2 + 12s + 8$. Since system is in controllable canonical form the desired feedback law is

$$u = -[\ 8 \quad 12 \quad 5\ ]x$$

(iii) Minimal order observer design: we need to estimate x_2 and x_3.

$$A_{22} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \qquad A_{12} = [\ 1 \quad 0\ ]$$

The pair (A_{12}, A_{22}) is observable. Again, we choose the observer poles to be at -2 so that the desired $r(s) = s^2 + 4s + 4$

$$A_{22} - LA_{12} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} [1 \quad 0] = \begin{bmatrix} -l_1 & 1 \\ -l_2 & -1 \end{bmatrix}$$

with $\det(sI - A_{22} + LA_{12}) = s^2 + (l_1 + 1)s + (l_1 + l_2)$

Hence $\quad l_1 = 3,\ l_2 = 1$

Also

$$(A_{22} - LA_{12})L + (A_{21} - LA_{11}) = \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ -4 \end{bmatrix}$$

$$(B_2 - LB_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Thus the minimal order observer equations are given by

$$\begin{aligned} \dot{z} &= \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix} z + \begin{bmatrix} -8 \\ -4 \end{bmatrix} y + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ \begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} &= z + \begin{bmatrix} 3 \\ 1 \end{bmatrix} y \qquad \hat{x}_1 = y \end{aligned}$$

(iv) Compensator design:

$$\begin{aligned} u &= k^T \hat{x} = -8y - [12 \quad 5] \begin{bmatrix} z_1 + 3y \\ z_2 + y \end{bmatrix} \\ &= -49y - 12z_1 - 5z_2 \end{aligned}$$

To find the transfer function of the compensator, we substitute the expression for u into the equation for z to give

$$\begin{aligned} \dot{z} &= \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix} z + \begin{bmatrix} -8 \\ -4 \end{bmatrix} y + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (-49y - 12z_1 - 5z_2) \\ &= \begin{bmatrix} -3 & 1 \\ -13 & -6 \end{bmatrix} z + \begin{bmatrix} -8 \\ -53 \end{bmatrix} y \end{aligned}$$

Therefore

$$\hat{z}(s) = \begin{bmatrix} s+3 & -1 \\ 13 & s+6 \end{bmatrix}^{-1} \begin{bmatrix} -8 \\ -53 \end{bmatrix} \hat{y}(s)$$

Hence

$$\begin{aligned} \hat{u}(s) &= -49y + \frac{[12 \quad 5] \begin{bmatrix} s+6 & 1 \\ -13 & s+3 \end{bmatrix} \begin{bmatrix} 8 \\ 53 \end{bmatrix}}{s^2+9s+31} \\ &= -\frac{49s^2+80s+32}{s^2+9s+31} \hat{y}(s) \end{aligned}$$

It is straightforward to verify that the closed loop poles are the roots of $s^5 + 10s^4 + 40s^3 + 80s^2 + 80s + 32 = (s+2)^5$.