University of Toronto
Department of Electrical and Computer Engineering
ECE557F Systems Control
Problem Set #6

1. The system clearly has (A, B) stabilizable and $(\sqrt{Q}, A)$ detectable. The unique solution of the algebraic Riccati equation (ARE) is given by

$$\begin{aligned} &\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \\ - &\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0 \end{aligned}$$

This gives the equations

$$\begin{aligned} p_2^2 &= 1 \\ p_1 - p_2 - p_2 p_3 &= 0 \\ 2p_2 - 2p_3 - p_3^2 + 1 &= 0 \end{aligned}$$

The unique positive semidefinite solution is given by

$$p_2 = 1$$

Then we have

$$p_3^2 + 2p_3 - 3 = 0$$

giving

$$p_3 = 1$$

Finally,

$$p_1 = 2$$

The optimal control law is given by

$$u = -\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} x = -\begin{bmatrix} 1 & 1 \end{bmatrix} x$$

The closed loop system matrix is given by

$$A_c = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

The characteristic polynomial is given by $s^2 + 2s + 1$ so that the closed loop poles are both at -1.

2. Again it is easy to check stabilizability and detectability. The ARE is given by

$$\begin{bmatrix} 0 & -10 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}+\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -10 & -2 \end{bmatrix}$$
$$-\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix}\begin{bmatrix} 0 & 2 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

This gives the equations

$$\begin{aligned} -20p_2 - 4p_2^2 &= 0 \\ p_1 - 2p_2 - 10p_3 - 4p_2p_3 &= 0 \\ 2p_2 - 4p_3 - 4p_3^2 + 1 &= 0 \end{aligned}$$

We see that

$$p_2 = 0$$

Then

$$4p_3^2 + 4p_3 - 1 = 0$$

giving

$$p_3 = \frac{-4 \pm \sqrt{32}}{8} = \frac{\sqrt{2}-1}{2}$$
$$p_1 = 10p_3 = 5(\sqrt{2}-1)$$

The optimal feedback law is

$$\begin{aligned} u &= -\begin{bmatrix} 0 & 2 \end{bmatrix}\begin{bmatrix} 5(\sqrt{2}-1) & 0 \\ 0 & \frac{\sqrt{2}-1}{2} \end{bmatrix} x \\ &= -\begin{bmatrix} 0 & (\sqrt{2}-1) \end{bmatrix} x \end{aligned}$$

The closed loop system matrix is given by

$$A_c = \begin{bmatrix} 0 & 1 \\ -10 & -2\sqrt{2} \end{bmatrix}$$

The poles are the roots of $s^2 + 2\sqrt{2}s + 10$, which are at $-\sqrt{2} \pm 2\sqrt{2}i$.

3. This is a standard LQR problem, with $Q = I_{2\times 2}$ and $R = \epsilon > 0$. The pair (A, B) is controllable, hence stabilizable, which guarantees this problem to be solvable. The optimal control is $K^* = -R^{-1}B^TP$, where P is the positive definite solution of (ARE):

$$A^TP + PA - PBR^{-1}B^TP + Q = 0\,,$$

i.e.

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}P + P\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \frac{1}{\epsilon}P\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}P + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0\,.$$

Solving the above equation, we obtain

$$P = \begin{bmatrix} \sqrt{1+2\sqrt{\epsilon}} & \sqrt{\epsilon} \\ \sqrt{\epsilon} & \sqrt{(1+2\sqrt{\epsilon})\epsilon} \end{bmatrix}.$$

Therefore, the associated control law is

$$\begin{aligned} u^* &= K^* x \\ K^* &= \begin{bmatrix} -\frac{1}{\sqrt{\epsilon}} & -\sqrt{\frac{1+2\sqrt{\epsilon}}{\epsilon}} \end{bmatrix}. \end{aligned}$$

The magnitude of u at $t = 0$, $\|u(0)\|$, is

$$\begin{aligned} u(0) &= \|K^* x(0)\| \\ &= \frac{1}{\sqrt{\epsilon}} + \sqrt{\frac{1+2\sqrt{\epsilon}}{\epsilon}}. \end{aligned}$$

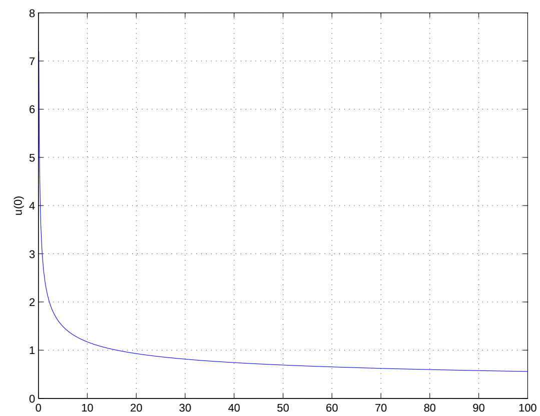

Figure 1: u(0)

In Figure 1, we observe that as $\epsilon \longrightarrow 0$, the control effort at the initial time goes to ∞, as expected.

4. The state-space model of the system is

$$\begin{aligned} \dot{x} &= \lambda x + u \\ y &= x\,. \end{aligned}$$

Stabilizability and detectability are easily verified so the problem is solvable. We solve

$$A^T P + PA - PBR^{-1}B^T P + C^T QC = 0$$

to obtain $2\lambda P - \frac{P^2}{\epsilon} + 1 = 0$. Solving for P and keeping in mind that $P > 0$, we get

$$P = \epsilon\lambda + \sqrt{\lambda^2\epsilon^2 + \epsilon}\,.$$

and

$$u^* = \left(-\lambda - \sqrt{\lambda^2 + \frac{1}{\epsilon}}\right) x\,.$$

Next we consider the closed-loop poles of the system

$$eig(A + BK^*) = -\sqrt{\lambda^2 + \frac{1}{\epsilon}}\,.$$

We see that regardless of whether $\lambda < 0$ or $\lambda > 0$, if $\epsilon \to 0$ then the closed-loop poles approach $-\infty$, whereas if $\epsilon \to \infty$ then the poles approach $-|\lambda|$.

5. We show that if any of the pair is not detectable, so are the others. Suppose $(\sqrt{Q}, A)$ is not detectable. There exists a λ with $Re\,\lambda \geq 0$, such that

$$Rank \begin{bmatrix} \sqrt{Q} \\ A - \lambda I \end{bmatrix} < n$$

Thus there exists a v such that

$$\begin{bmatrix} \sqrt{Q} \\ A - \lambda I \end{bmatrix} v = 0$$

But $\sqrt{Q}v = 0$ if and only if $Qv = 0$ if and only $Cv = 0$. This shows that $(\sqrt{Q}, A)$ is not detectable if and only if (Q, A) is not detectable if and only if (C, A) is not detectable.