University of Toronto
Department of Electrical and Computer Engineering
ECE557F Systems Control
Problem Set #7

1. (a) The one-dimensional exosystem is $\dot{w} = 0$. The plant with exosystem is

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ \dot{w} &= 0 \\ e &= r - y = w - x_1,\end{aligned}$$

where the reference signal is $r(t) = w(t)$. The pair (A, B) is clearly controllable. The problem requests that the poles of the closed-loop system $(A + BK)$ be placed at $\{-1, -2\}$. This gives $K = [-4 \quad -4]$.
The regulator equations are:

$$\begin{aligned}0 &= \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \Pi + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Gamma \\ &\begin{bmatrix} 1 & 0 \end{bmatrix} \Pi = 1\,.\end{aligned}$$

The solution is

$$\Pi = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \Gamma = -2.$$

The full information output regulator is

$$u = \Gamma w + K(x - \Pi w) = -2w + \begin{bmatrix} -4 & -4 \end{bmatrix} \left(x - \begin{bmatrix} 1 \\ 0 \end{bmatrix} w \right) = -4x_1 - 4x_2 + 2w\,.$$

(b) The observability matrix of the augmented system above

$$Q_0 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix}$$

is full rank, so the system is observable. Placing the poles of $(\overline{A} - G\overline{C})$ at $\{-10, -20, -30\}$ is the same as placing the poles of $(\hat{A}^\top + \hat{C}^\top(-G^\top))$ at $\{-10, -20, -30\}$, so we look at the pair

$$\overline{A}^\top = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \overline{C}^\top = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

We get $G = \begin{bmatrix} -3061 \\ -1163 \\ -3000 \end{bmatrix}$. The robust regulator is

$$\begin{aligned}\dot{\xi} &= \begin{bmatrix} -3061 & 1 & 3061 \\ -1165 & -3 & 1165 \\ -3000 & 0 & 3000 \end{bmatrix} \xi + \begin{bmatrix} -3061 \\ -1163 \\ -3000 \end{bmatrix} e \\ u &= \begin{bmatrix} -4 & -4 & 2 \end{bmatrix} \xi\end{aligned}$$

(c) The transfer function of the controller from e to u is

$$C(s) = -\frac{10896s^2 + 22896s + 12000}{s(s^2 + 64 + 1348)}.$$

Note that it has a pole at zero, as one would expect.

2. We can easily verify that (A, B) is stabilizable. Therefore we can first design the asymptotic part of the controller. We want $\sigma(A + BK) = \{ -1, -2 \}$. Using pole placement we obtain

$$K = [-2 \ -2] .$$

Then we obtain

$$A + BK = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} .$$

Now we design the feedforward or exact matching part of the controller. That is, we must find a pair (Π, Γ) satisfying the FBI equations

$$\Pi S = A\Pi + B\Gamma \tag{1}$$

$$C\Pi = C_r . \tag{2}$$

To that end, we must find a state space model for the exosystem. We take $p(t) = t^2$ and differentiate three times to obtain the ODE $p^{(3)} = 0$. Now we put this ODE in state space form. Letting $w_1 = p$, $w_2 = \dot{p}$, and $w_3 = \ddot{p}$, the exosystem model is

$$\dot{w} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} w , \qquad y_r = [\ 2\ 0\ 0]w := C_r w ,$$

where the initial condition of the exosystem is $w(0) = [\ 0\ 0\ 1\ ]^T$. Then for $\Pi : \mathbb{R}^3 \to \mathbb{R}^2$ and $\Gamma : \mathbb{R}^3 \to \mathbb{R}$, (1) gives

$$\begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \\ \pi_4 & \pi_5 & \pi_6 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \\ \pi_4 & \pi_5 & \pi_6 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [\ \gamma_1 \quad \gamma_2 \quad \gamma_3\ ] .$$

We obtain the following solutions:

$$\begin{aligned} \pi_1 &= \pi_5 = 2 \\ \pi_2 &= \pi_3 = \pi_4 = \pi_6 = 0 \\ \gamma_1 &= 0 \\ \gamma_2 &= 2 \\ \gamma_3 &= 2 . \end{aligned}$$

The tracking controller is then of the form:

$$\begin{aligned} u &= \Gamma w + K(x - \Pi w) \\ &= -2x_1 - 2x_2 + 4w_1 + 6w_2 + 2w_3 . \end{aligned}$$

3. We must design a tracking controller so that x_2 tracks a constant value of 20m/s. The velocity dynamics are

$$\dot{x}_2 = -x_2 + u\,, \qquad y = x_2\,.$$

The exosystem is

$$\dot{w} = 0\,, \qquad w(0) = 20\,, \qquad y_r = w\,.$$

The regulator equations are

$$\begin{aligned} \Pi S &= A\Pi + B\Gamma \\ C\Pi &= C_r\,. \end{aligned}$$

This yields the constraints

$$-\Pi + \Gamma = 0\,, \qquad \Pi = 1\,.$$

Thus, $\Gamma = 1$. We can design K by pole placement. Since this is a scalar system, we obtain immediately that $K = -4$. The final tracking controller is

$$\begin{aligned} u &= \Gamma w + K(x_2 - \Pi w) \\ &= 5w - 4x_2\,. \end{aligned}$$

4. (a) The reference trajectory $r(t)$ satisifies $\ddot{r}(t) = 0$. The desired distance d satisfies $\dot{d} = 0$. Therefore, the exosystem is

$$\begin{aligned} \dot{w}_1 &= w_2 \\ \dot{w}_2 &= 0 \\ \dot{w}_3 &= 0. \end{aligned}$$

If we set $r + d = -(w_1 + w_3)$, we get an output regulation problem for the system

$$\begin{aligned} \dot{x} &= u \\ \dot{w}_1 &= w_2 \\ \dot{w}_2 &= 0 \\ \dot{w}_3 &= 0 \end{aligned}\,, \qquad e = x + w_1 + w_3.$$

(b) The regulator equations take the form

$$\Pi \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \Gamma$$

$$\Pi + \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} = 0.$$

Their solution is

$$\Pi = \begin{bmatrix} -1 & 0 & -1 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0 & -1 & 0 \end{bmatrix}.$$

Being a pure integrator, the plant is obviously controllable. In order to make the tracking error e converge to zero with rate $\exp(-5t)$, we choose $K = -5$. The full information output regulator is

$$u = \Gamma w + K(x - \Pi w) = \begin{bmatrix}0 & -1 & 0\end{bmatrix} w - 5\left(x - \begin{bmatrix}-1 & 0 & -1\end{bmatrix} w\right)$$

$$= \begin{bmatrix}-5 & -5 & -1 & -5\end{bmatrix} \begin{bmatrix} x \\ w_1 \\ w_2 \\ w_3. \end{bmatrix}$$

(c) While the car model is observable, the augmented system

$$\dot{\overline{x}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overline{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$
$$e = \begin{bmatrix}1 & 1 & 0 & 1\end{bmatrix} \overline{x}$$

is undetectable because $\overline{A}$ has all evalues at 0, and the PBH test at this evalue gives

$$\operatorname{rank} \begin{bmatrix} \overline{A} \\ \overline{C} \end{bmatrix} = 2 < 4.$$

(d) The observability matrix for the augmented system is

$$\overline{Q}_o = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \ker(\overline{Q}_o) = \operatorname{sp} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Define

$$P = \begin{bmatrix} -2 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

Note that the first two columns span the kernel of $\overline{Q}_o$. The third column has the form $\begin{bmatrix} I_n \\ 0_{q\times n} \end{bmatrix}$, where $n = 1, q = 3$, and the fourth column is chosen to complete the basis. Using $z = P^{-1}\overline{x}$ we get

$$\dot{z} = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & -1/2 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] z + \left[\begin{array}{c} 0 \\ 0 \\ \hline 1 \\ 0 \end{array}\right] u$$
$$e = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \end{array}\right] z.$$

After pruning the unobservable subsystem, we are left with the second-order system

$$\begin{aligned} \dot{z}_3 &= u + z_4 \quad && \text{new plant} \\ \dot{z}_4 &= 0 && \text{new exosystem} \\ e &= z_3. \end{aligned}$$

The regulator equations for this reduced system are guaranteed to be solvable. The reg. equations are

$$\begin{aligned}\Pi \cdot 0 &= 1 \cdot \Gamma + 1 \\ 1 \cdot \Pi &= 0,\end{aligned}$$

where Π and Γ are scalars. The solution is $\Pi = 0$, $\Gamma = -1$, so a full information output regulator for the pruned system is given by $u = -z_4 - 2z_3$, where we have placed the pole of the closed-loop system at -2. Note that we could have directly defined a full information output regulator by observing that e is equal to the state of the "new plant," and so the controller should simply stabilize z_3. The way to do that is to cancel out the term $+z_4$ in the equation of the "new plant," and assign the pole somewhere in OLHP. The controller above does precisely that. We now design the observer gain G to place the poles of the matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - G \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

at -10. Doing so, we obtain $G = \begin{bmatrix} 20 \\ 100 \end{bmatrix}$. In conclusion, the error feedback output regulator is

$$\begin{aligned}\dot{\xi} &= \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} -2 & -1 \end{bmatrix} - \begin{bmatrix} 20 \\ 100 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right) \xi + \begin{bmatrix} 20 \\ 100 \end{bmatrix} e = \begin{bmatrix} -22 & 0 \\ -100 & 0 \end{bmatrix} \xi + \begin{bmatrix} 20 \\ 100 \end{bmatrix} e \\ u &= \begin{bmatrix} -2 & -1 \end{bmatrix} \xi\end{aligned}$$

(e) The transfer function of the controller is

$$U(s)/E(s) = -\frac{140s + 200}{s(s + 22)}.$$

Note this transfer function has just one pole at zero. This is consistent with the fact, from classical feedback control theory, that since the plant has already one pole at zero, in order to make it asymptotically track a reference signal of the type $c_0 t + c_1$ the controller should have one pole at zero.

5. (a) We would like the state x to asymptotically track the signal $(r_1(t), r_2(t)) = (\cos t, -\sin t)$. Notice that $\dot{r}_1(t) = r_2(t)$, and that $\ddot{r}_1(t) = -r_1(t)$. Let $w_1 = -r_1$, $w_2 = -r_2$. We obtain the exosystem

$$\begin{aligned}\dot{w}_1 &= w_2 \\ \dot{w}_2 &= -w_1.\end{aligned}$$

Note that this equation has the canonical form of an LTI system with eigenvalues at $\pm i$. We can now formulate the output regulation problem as

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ \dot{w} &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} w \\ e &= \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} x_1 + w_1 \\ x_2 + w_2 \end{bmatrix} = I_2 x + I_2 w.\end{aligned}$$

(I_2 denotes the 2×2 identity matrix). The regulator equations are

$$\begin{aligned} \Pi \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Pi + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Gamma \\ I_2 \Pi + I_2 &= 0. \end{aligned}$$

Their solution is easily computed as

$$\Pi = -I_2, \quad \Gamma = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Since the regulation equations are solvable and the plant is controllable, the problem is solvable. A full information output regulator solving it is

$$u = \begin{bmatrix} 1 & 0 \end{bmatrix} w + K(x + w),$$

where $K = \begin{bmatrix} -1 & -2 \end{bmatrix}$ places the poles of the closed-loop system at -1.

(b) We need to add one equation to the exosystem as follows

$$\dot{w}_3 = 0,$$

and the plant becomes

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} w,$$

with output

$$e = I_2 x + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} w.$$

The regulator equations in this case are

$$\begin{aligned} \Pi \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Pi + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Gamma + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \\ I_2 \Pi + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} &= 0, \end{aligned}$$

with $\Pi \in \Re^{2\times 3}$ and $\Gamma \in \Re^{1\times 3}$. The second equation gives

$$\Pi = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

Replacing Π into the first equation, we see that there is no matrix Γ solving it, so the problem is unsolvable!!