ECE557F
Systems Control

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Contents

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Chapter 1

Linear Differential Equations

In this chapter, we discuss the solution of the linear differential equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) \\ x(0) &= x_0\,, \end{aligned} \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$. Very often for simplicity we suppress the time dependence in the notation. We give a systematic method for solving (1.1) and discuss properties of the solution. Finally, we discuss linear systems with inputs and outputs.

1.1 Existence and Uniqueness of Solutions

It is known from the theory of ordinary differential equations that under certain regularity assumptions, a (nonlinear) differential equation of the form

$$\dot{x}(t) = f(x,t), \qquad t \in [t_0, t_1] \tag{1.2}$$

$$x(t_0) = x_0 \tag{1.3}$$

has a unique solution passing through x_0 at $t = t_0$. The regularity conditions are

(i) $f(\cdot,\cdot)$ is a continuous function from $\mathbb{R}^n \times [t_0, t_1]$ to $\mathbb{R}^n$,

(ii) f satisfies a global Lipschitz condition

$$||f(x_1,t) \; - \; f(x_2,t)|| \; \leq \; k(t)||x_1 - x_2|| \qquad \forall \; x_1, x_2 \in \mathbb{R}^n$$

where $k(t)$ is continuous on $[t_0, t_1]$.

One way to obtain the solution to (1.3) is by a *Picard iteration.* That is, we consider the iterations

$$\begin{aligned} x_0(t) &= x_0 \\ x_{n+1}(t) &= x_0 + \textstyle\int_{t_0}^t f(x_n(s), s)ds\,. \end{aligned} \tag{1.4}$$

Then

$$\lim_{n\to\infty} \; \sup_{t\in[t_0,t_1]} \; ||x_n(t) - x(t)|| \; = \; 0$$

where $x(t)$ is the unique solution of (1.3). Let us apply the above results to (1.1). The R.H.S. of (1.1) clearly satisfies the regularity conditions (i) and (ii). Hence there exists a unique solution to (1.1). The Picard iteration gives

$$\begin{aligned} x_0(t) &= x_0 \\ x_{n+1}(t) &= x_0 + \textstyle\int_0^t Ax_n(s)ds\,. \end{aligned} \tag{1.5}$$

From (1.5), we see that $x(t)$ is given by an infinite series of the form

$$\begin{aligned} x(t) &= x_0 + \int_0^t Ads x_0 + \int_0^t \int_0^{s_1} A^2 ds_2 ds_1\, x_0 \\ &\quad +\ldots+ \int_0^t \int_0^{s_1} \ldots \int_0^{s_{n-1}} A^n ds_n ds_{n-1} \ldots ds_1 x_0 + \ldots \end{aligned}$$

The integrals can be easily evaluated, with the nth term of the series given by

$$\frac{A^n t^n}{n!} x_0\,.$$

In analogy with the scalar exponential function, the matrix exponential of a square matrix M is defined to be

$$e^M := \sum_{n=0}^{\infty} \frac{M^n}{n!}$$

where M^0 is defined to be I, the identity matrix. This series can be shown to converge. Observe that e^M satisfies two properties:

1. e^M is invertible and $(e^M)^{-1} = e^{-M}$.
2. $e^{M+N} = e^M e^N \quad \Leftrightarrow \quad M$ and N commute, i.e., $MN = NM$.

The matrix function $t \mapsto e^{At} : \mathbb{R} \to \mathbb{R}^{n \times n}$ is now defined, and the Picard iteration suggests that the solution of (1.1) is

$$x(t) = e^{At} x_0\,. \tag{1.6}$$

1.2 State Transition Matrix

The matrix function $t \mapsto e^{At}$ is called the *state transition matrix* of (1.1). In this section we prove several useful properties of the state transition matrix and then consider three methods to compute it: when A is diagonal or diagonalizable, when A is not diagonalizable, and a Laplace transform based approach.

The following two properties of e^{At} can be considered its defining properties:

$$\begin{aligned} \frac{d}{dt} e^{At} &= Ae^{At} \\ e^{At}|_{t=0} &= I\,. \end{aligned} \tag{1.7}$$

To see that these hold, note that they form a linear differential equation satisfying the regularity conditions for existence and uniqueness. Since the infinite series defining e^{At} is uniformly convergent, its derivative can be determined by differentiating term by term. It is easily verified that the result satisfies (1.7). Uniqueness of solutions then shows that (1.7) defines e^{At}.

Lemma 1.2.1. *The state transition matrix satisfies*

$$\begin{aligned} e^{A(t+s)} &= e^{At} e^{As} \qquad \forall\, t, s \qquad \textit{semi-group property} \\ (e^{At})^{-1} &= e^{-At}\,. \end{aligned}$$

Proof. First, let $F(t) = e^{At}e^{As}$. Then

$$\begin{aligned} F(0) &= e^{As} \\ \frac{d}{dt}F(t) &= AF(t) \end{aligned}$$

so that $F(t)$ satisfies the same differential equation and initial condition as $e^{A(t+s)}$. By uniqueness, they must be equal, which verifies the first property. Second, we can verify from the definition that $e^{-At}e^{At} = I$, which proves the second property. □

1.2.1 Computing e^{At}: Diagonalizable Case

To study the structure of e^{At}, let us first study the special case when

$$A = \begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & \\ & & \ddots & 0 \\ 0 & & 0 & \lambda_n \end{bmatrix}.$$

The λ_i's are not necessarily distinct and they may be complex, provided we interpret (1.1) as a differential equation in $\mathbb{C}^n$. In this case (1.1) is completely decoupled into n differential equations

$$\dot{x}_i(t) = \lambda_i x_i(t) \qquad i = 1, ..., n \tag{1.8}$$

so that

$$x_i(t) = e^{\lambda_i t} x_{0i} \tag{1.9}$$

where x_{0i} is the ith component of x_0. It follows that

$$x(t) = \begin{bmatrix} e^{\lambda_1 t} & 0 & & 0 \\ 0 & e^{\lambda_2 t} & & \\ & & \ddots & 0 \\ 0 & & 0 & e^{\lambda_n t} \end{bmatrix} x_0 \tag{1.10}$$

and the matrix exponential e^{At} can simply be read off from the R.H.S. of (1.10):

$$e^{At} = \begin{bmatrix} e^{\lambda_1 t} & 0 & & 0 \\ 0 & e^{\lambda_2 t} & & \\ & & \ddots & 0 \\ 0 & & 0 & e^{\lambda_n t} \end{bmatrix} \tag{1.11}$$

To extend the above results to more general situations, assume that A is *diagonalizable*, i.e., there exists a nonsingular matrix T (in general complex) such that $T^{-1}AT$ is a diagonal matrix. That is,

$$T^{-1}AT := \Lambda = \begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & \\ & & \ddots & 0 \\ 0 & & 0 & \lambda_n \end{bmatrix}. \tag{1.12}$$

Then

$$e^{At} = e^{T\Lambda T^{-1}t} \tag{1.13}$$

Lemma 1.2.2. *For any $n \times n$ matrix B*

$$e^{TBT^{-1}t} = Te^{Bt}T^{-1}. \tag{1.14}$$

Proof. By definition

$$e^{TBT^{-1}t} = \sum_{k=0}^{\infty} \frac{(TBT^{-1})^k t^k}{k!}.$$

The kth term of this series is

$$\frac{t^k}{k!}(TBT^{-1})^k = \frac{t^k}{k!}(TBT^{-1})(TBT^{-1})\underset{\cdots}{\overset{k\ times}{\overline{\qquad\quad}}}(TBT^{-1}).$$

Other than the T on the left most side and the T^{-1} on the right most side, all the other T's and T^{-1}'s multiply out to the identity matrix I. Hence

$$\frac{(TBT^{-1})^k t^k}{k!} = \frac{TB^kT^{-1}t^k}{k!}$$

which is precisely the kth term of the infinite series for $Te^{Bt}T^{-1}$. □

From (1.11), (1.12) and (1.14), we find that whenever A is diagonalizable,

$$e^{At} = T\begin{bmatrix} e^{\lambda_1 t} & 0 & & 0 \\ 0 & e^{\lambda_2 t} & & \\ & & \ddots & 0 \\ 0 & & 0 & e^{\lambda_n t} \end{bmatrix}T^{-1} \tag{1.15}$$

From linear algebra, we know that either of the following two conditions will guarantee diagonalizability:

(i) A has distinct eigenvalues.

(ii) A is a symmetric matrix.

In these two cases, (1.15) completely characterizes the structure of e^{At}. The diagonalizing matrix T can be taken to be the matrix whose columns are the independent eigenvectors of A.

Example 1.2.1.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

Due to the form of A, we see right away that the eigenvalues are 1, 2 and -1 so that

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

The eigenvector corresponding to the eigenvalue 2 is given by $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}'$, while that of -1 is $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}'$. To find the eigenvector for the eigenvalue 1, we have

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix},$$

so that $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ is an eigenvector. The diagonalizing matrix T is then

$$
\begin{aligned}
T &= \begin{bmatrix} 2 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\
T^{-1} &= \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 1 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$$
e^{At} = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 1 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} = \begin{bmatrix} e^t & 0 & 0 \\ -e^t + e^{2t} & e^{2t} & 0 \\ \frac{1}{2}(e^t - e^{-t}) & 0 & e^{-t} \end{bmatrix}.
$$

1.2.2 Computing e^{At} using Jordan form

If A is not diagonalizable, then the above procedure cannot be carried out. A is, in general, not diagonalizable if it does not have distinct eigenvalues. Let us consider the following $l \times l$ matrix A which has the eigenvalue λ with multiplicity l:

$$
A = \begin{bmatrix} \lambda & 1 & 0 & & 0 \\ & & & \ddots & \\ & & & & 0 \\ & & & & 1 \\ 0 & & 0 & & \lambda \end{bmatrix} \tag{1.16}
$$

Write $A = \lambda I + N$ where

$$
N = \begin{bmatrix} 0 & 1 & 0 & & 0 \\ & & & \ddots & \\ & & & & 0 \\ & & & & 1 \\ 0 & & & & 0 \end{bmatrix} \tag{1.17}
$$

Direct calculation shows that

$$
[N^k]_{j,j+k} = \begin{cases} 1 & j = 1, 2, ..., l \\ 0 & \text{all other entries}, \end{cases}
$$

so that $N^l = 0$. N is thus a nilpotent matrix. Then the matrix exponential e^{Nt} is easily evaluated to be

$$
e^{Nt} = \sum_{j=0}^{l-1} \frac{N^j t^j}{j!} = \begin{bmatrix} 1 & t & \cdots & \frac{t^{l-1}}{(l-1)!} \\ 0 & & & \vdots \\ & & & t \\ 0 & 0 & & 1 \end{bmatrix}. \tag{1.18}
$$

To compute e^{At} when A is of the form (1.16), recall the following property of matrix exponentials, which we now prove.

Lemma 1.2.3. *If A and B commute, i.e. $AB = BA$, then $e^{(A+B)t} = e^{At}e^{Bt}$.*

Proof. Considering $e^{At}e^{Bt}$, we have

$$\frac{d}{dt}\left(e^{At}e^{Bt}\right) = Ae^{At}e^{Bt} + e^{At}Be^{Bt}. \tag{1.19}$$

If A and B commute, $e^{At}B = Be^{At}$ so that the R.H.S. of (1.19) becomes $(A+B)e^{At}e^{Bt}$. Furthermore, at $t = 0$, $e^{At}e^{Bt} = I$. Hence $e^{At}e^{Bt}$ satisfies the same differential equation as well as initial condition as $e^{(A+B)t}$. By uniqueness, they must be equal. □

If A is of the form (1.16), then since λI commutes with N, we can write

$$\begin{aligned} e^{At} &= e^{(\lambda I)t}e^{Nt} = (e^{\lambda t}I)(e^{Nt}) \\ &= e^{\lambda t}\begin{bmatrix} 1 & t & \cdots & \frac{t^{l-1}}{(l-1)!} \\ 0 & & & \vdots \\ & & & t \\ 0 & 0 & & 1 \end{bmatrix} \end{aligned} \tag{1.20}$$

using (1.18).

We are now in a position to show the general structure of the matrix exponential. From linear algebra, we know that there always exists a matrix T such that

$$T^{-1}AT = J = \begin{bmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_k \end{bmatrix}$$

i.e., the *Jordan canonical form* of A with

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ & \lambda_i & 1 & & 0 \\ & & & & \vdots \\ & & & & 1 \\ 0 & & & & \lambda_i \end{bmatrix},$$

an $n_i \times n_i$ matrix with $\sum_{i=1}^{k} n_i = n$. But

$$e^{Jt} = \begin{bmatrix} e^{J_1t} & & & 0 \\ & e^{J_2t} & & \\ & & \ddots & \\ 0 & & & e^{J_kt} \end{bmatrix} \tag{1.21}$$

and each e^{J_it} is of the form

$$e^{J_it} = e^{\lambda_it}\begin{bmatrix} 1 & t & \cdots & \frac{t^{n_i-1}}{(n_i-1)!} \\ & & \ddots & \\ & & \ddots & t \\ 0 & & & 1 \end{bmatrix} \tag{1.22}$$

Finally,

$$e^{At} = Te^{Jt}T^{-1} \tag{1.23}$$

Equations (1.21), (1.22) and (1.23) give the complete form for the matrix exponential e^{At}.

The above procedure in principle enables us to evaluate e^{At}, the only problem being to find the matrix T which transforms A either to diagonal or Jordan canonical form. In general, this can be a very tedious task. The above formula is most useful as a means to study the qualitative dependence of e^{At} on t, and will be particularly important in our discussion of stability. Finally, we remark that the above results hold regardless of whether the λ_i's are real or complex. In the latter case, we take the underlying vector space to be complex. The results then go through without modification.

1.2.3 Computing e^{At} using Laplace Transforms

Another method to evaluate e^{At} analytically is to use Laplace transforms. If we let $G(t) = e^{At}$, then the Laplace transform of $G(t)$, denoted by $\hat{G}(s)$, satisfies

$$s\hat{G}(s) \;=\; A\hat{G}(s) \;+\; I$$

or

$$\hat{G}(s) \;=\; (sI - A)^{-1} \tag{1.24}$$

Applying the inversion integral, we find

$$e^{At} \;=\; \frac{1}{2\pi i} \lim_{\beta\to\infty} \int_{\sigma-i\beta}^{\sigma+i\beta} (sI - A)^{-1} e^{st} ds \tag{1.25}$$

Each term in the inversion integral may be evaluated by residue calculus.

$(sI - A)^{-1}$ may be evaluated in a recursive way: Let $\det(sI - A) \;=\; s^n + p_1 s^{n-1} + ... + p_n \;=\; p(s)$. Write

$$(sI - A)^{-1} \;=\; \frac{B(s)}{p(s)} \;=\; \frac{s^{n-1}B_1 + s^{n-2}B_2 + ... + B_n}{p(s)}$$

In your problem set, you are asked to use the identity $(sI - A)B(s) = p(s)I$ to derive recursive equations for the B_k matrices.

The above procedure is particularly easy to use if A has distinct eigenvalues $\lambda_1, ..., \lambda_n$, for then

$$(sI - A)^{-1} \;=\; \frac{B(s)}{\sum_{i=1}^{n} (s - \lambda_i)} \;=\; \sum_{i=1}^{n} \frac{R_i}{(s - \lambda_i)}$$

by partial fractions expansion with

$$R_i \;=\; \lim_{s\to\lambda_i} (s - \lambda_i) \frac{B(s)}{p(s)} \;=\; \frac{B(\lambda_i)}{\sum_{k=1,k\neq i}^{n} (\lambda_i - \lambda_k)}$$

Hence,

$$e^{At} \;=\; \sum_{i=1}^{n} R_i e^{\lambda_i t}\,. \tag{1.26}$$

Example 1.2.2.

To illustrate this procedure for evaluating e^{At}, again consider

$$A \quad = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$$p(s) = \det \begin{bmatrix} s-1 & 0 & 0 \\ -1 & s-2 & 0 \\ -1 & 0 & s+1 \end{bmatrix} = (s-1)(s-2)(s+1)$$
$$= s^3 - 2s^2 - s + 2$$

The matrix polynomial $B(s)$ can then be determined using the recursive procedure as

$$B(s) = \begin{bmatrix} s^2-s-2 & s+1 & s-2 \\ 0 & s^2-1 & 0 \\ 0 & 0 & s^2-3s+2 \end{bmatrix}$$
$$(sI-A)^{-1} = \frac{1}{(s-1)(-2)} \begin{bmatrix} -2 & 0 & 0 \\ 2 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + \frac{1}{(s-2)(3)} \begin{bmatrix} 0 & 3 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$+\frac{1}{(s+1)(6)} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -3 & 0 & 6 \end{bmatrix}$$
$$e^{At} = \begin{bmatrix} e^t & 0 & 0 \\ -e^t & 0 & 0 \\ \frac{1}{2}e^t & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & e^{2t} & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -\frac{1}{2}e^{-t} \\ 0 & 0 & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$$
$$= \begin{bmatrix} e^t & -e^t + e^{2t} & \frac{1}{2}(e^t - e^{-t}) \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$$

the same result as before.

Example 1.2.3.

As a second example, let $A = \begin{bmatrix} \sigma & -\omega \\ \omega & \sigma \end{bmatrix}$. Then

$$A = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix} + \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \quad .$$

But

$$e^{\begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} t} = \mathbf{L}^{-1} \left\{ \begin{bmatrix} s & \omega \\ -\omega & s \end{bmatrix}^{-1} \right\}$$

where $\mathbf{L}^{-1}$ is the inverse Laplace transform operator

$$= \mathbf{L}^{-1} \begin{bmatrix} \frac{s}{s^2+\omega^2} & -\frac{\omega}{s^2+\omega^2} \\ \frac{\omega}{s^2+\omega^2} & \frac{s}{s^2+\omega^2} \end{bmatrix} = \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix}$$
$$e^{At} = e^{(\sigma I)t} \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix} = \begin{bmatrix} e^{\sigma t}\cos\omega t & e^{\sigma t}\sin\omega t \\ -e^{\sigma t}\sin\omega t & e^{\sigma t}\cos\omega t \end{bmatrix}$$

Of course, the above procedures of evaluating e^{At} analytically would be virtually impossible to carry out when the dimension n is say ≥ 5. In general, we must resort to numerical techniques. Numerically stable and efficient methods of evaluating the matrix exponential can be found in the research literature.

1.3 Modal Decomposition

We can give a dynamical interpretation to the above results. Suppose A has distinct eigenvalues $\lambda_1, ..., \lambda_n$ so that it has a set of linearly independent eigenvectors $v_1, ..., v_n$. In this case A can be diagonalized, $\Lambda = T^{-1}AT$, where the diagonalizing matrix is $T = [v_1 \dots v_n]$. We have seen that $e^{At} = Te^{\Lambda t}T^{-1}$, so that

$$e^{At}[v_1 \dots v_n] = [v_1 \dots v_n]e^{\Lambda t} .$$

or

$$e^{At}v_i = e^{\lambda_i t}v_i .$$

Now if we set $x_0 = v_j$, then

$$\begin{aligned} x(t) &= e^{At}x_0 = e^{At}v_j \\ &= e^{\lambda_j t}v_j \end{aligned}$$

so that the solution is just the stretching or shrinking of the eigenvector v_j. In general, since $\{v_i\}$ form a basis for $\mathbb{R}^n$, we can write

$$x_0 = \sum_{j=1}^{n} \xi_j v_j = T\xi \tag{1.27}$$

for some $\xi = \begin{bmatrix}\xi_1 & \dots & \xi_n\end{bmatrix}'$ so that

$$\begin{aligned} x(t) &= e^{At}\sum_{j=1}^{n} \xi_j v_j \\ &= \sum_{j=1}^{n} \xi_j e^{\lambda_j t} v_j . \end{aligned} \tag{1.28}$$

That is, $x(t)$ is expressible as a (time-varying) linear combination of the eigenvectors of A. More directly, we have

$$\begin{aligned} e^{At}x_0 &= Te^{\Lambda t}T^{-1}x_0 = Te^{\Lambda t}\xi \\ &= T\begin{bmatrix} \xi_1 e^{\lambda_1 t} & 0 & & 0 \\ 0 & \xi_2 e^{\lambda_2 t} & & \\ & & \ddots & 0 \\ 0 & & 0 & \xi_n e^{\lambda_n t} \end{bmatrix} \\ &= \sum_j \xi_j e^{\lambda_j t} v_j , \end{aligned}$$

the same result as in (1.28). The representation (1.28) of the solution $x(t)$ in terms of the eigenvectors of A is called the *modal decomposition* of $x(t)$.

1.4 Phase Portraits

Consider again the linear time-invariant system (1.1) and suppose that the system is second order, so that $x = [x_1\ x_2]'$. A *phase portrait* of (1.1) is a plot in the $x_1 - x_2$ plane of trajectories of (1.1) for a set of initial conditions spread over the $x_1 - x_2$ plane. The picture gives the *qualitative* behavior of the system in the sense that one cannot deduce the exact form of the solutions (often distinguished as the *quantitative* behavior), but can deduce features such as stability, existence of equilibria, existence of

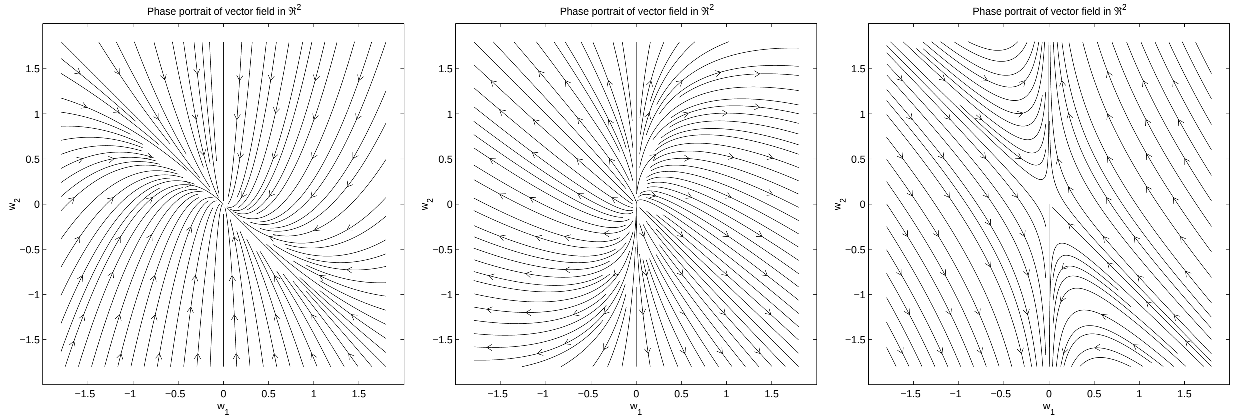


Figure 1.1: Phase portraits for a stable node, unstable node, and saddle point.

periodic trajectories, and so forth. In the case when (1.1) has a single equilibrium point at the origin, phase portraits can be used to visually classify the types of equilibria at the origin.

Recall that in general the solution of (1.1) is

$$x(t) = Te^{Jt}T^{-1}x_0$$

where T is a nonsingular matrix and J is the Jordan form of A. If we define $z = T^{-1}x$ then we can sketch the phase portrait in the $z_1 - z_2$ plane and perform a linear transformation back to the $x_1 - x_2$ plane to obtain the system phase portrait. Alternatively one can work directly in x coordinates. There are several cases depending on the eigenvalues of A.

1.4.1 Case 1: Real Eigenvalues

In this case there is a single equilibrium point at the origin and A can be diagonalized so the Jordan form is

$$J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},$$

where $\lambda_1 \neq \lambda_2 \neq 0$. We know from our discussion of modal decomposition that if a trajectory starts along the eigenvector v_1 (v_2) then it remains on it and grows or shrinks as $e^{\lambda_1 t}$ ($e^{\lambda_2 t}$). If a trajectory starts at an arbitrary initial condition its behavior is governed by a linear combination of trajectories starting at the components of the initial condition in the directions of the two eigenvectors. Figure 1.1, generated by the Matlab command `streamslice` shows the possible cases. If $\lambda_1 < 0$ and $\lambda_2 < 0$ the equilibrium point is called a *stable node*. If $\lambda_1\lambda_2 < 0$ the equilibrium point is called a *saddle point*. If $\lambda_1 > 0$ and $\lambda_2 > 0$ the equilibrium point is called an *unstable node*.

1.4.2 Case 2: Complex Conjugate Eigenvalues

In this case the real Jordan form is

$$J = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix},$$

where the complex conjugate eigenvalues are $\lambda_{1,2} = \alpha \pm j\beta$. If one transforms z to polar coordinates (r, θ) the differential equation obtained is

$$\dot{r} = \alpha r\,, \qquad \dot{\theta} = \beta\,.$$

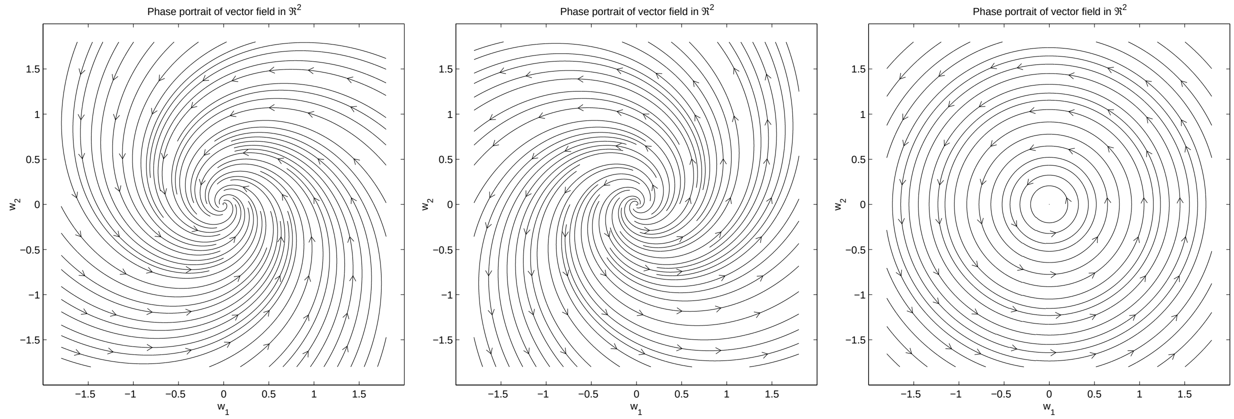


Figure 1.2: Phase portraits for a stable focus, unstable focus, and center.

The solution of these uncoupled equations is a logarithmic spiral in the $z_1 - z_2$ plane, which can be transformed to the $x_1 - x_2$ plane. Figure 1.2 shows the possible cases. If $\alpha < 0$ then the equilibrium point is called a *stable focus*. If $\alpha = 0$ the equilibrium point is called a *center* and all the trajectories form *periodic orbits*. If $\alpha > 0$ the equilibrium point is called an *unstable focus*.

1.4.3 Case 3: Nonzero Real Repeated Eigenvalues

In this case the eigenvalues of A are $\lambda_1 = \lambda_2 = \lambda \neq 0$ and the Jordan form is

$$J = \begin{bmatrix} \lambda & k \\ 0 & \lambda \end{bmatrix},$$

where k is either 0 or 1. If $k = 0$ then we have the same situation as the first case: the equilibrium is either a stable or unstable node. However, every vector starting at the origin is an eigenvector. If $k = 1$ there is only one eigenvector associated with the two eigenvalues. If $\lambda < 0$ ($\lambda > 0$) the equilibrium point is, as before, called a *stable node* (*unstable node*).

1.4.4 Case 4: Zero Eigenvalues

If there are zero eigenvalues then there is not a unique equilibrium point at the origin but a set of equilibrium points called the *equilibrium set*. It is an exercise for you to sketch the qualitative types of phase portraits that occur in this case.

The table below summarizes the types of equilibrium points for second-order linear systems. Cases with zero eigenvalues are omitted.

λ_1, λ_2 real, $\lambda_1 < 0$, $\lambda_2 < 0$	Stable node
λ_1, λ_2 real, $\lambda_1 > 0$, $\lambda_2 > 0$	Unstable node
λ_1, λ_2 real, $\lambda_1 \lambda_2 < 0$	Saddle point
λ_1, λ_2 complex conjugate, $Re\lambda_1 > 0$	Unstable focus
λ_1, λ_2 complex conjugate, $Re\lambda_1 < 0$	Stable focus
λ_1, λ_2 complex conjugate, $Re\lambda_1 = 0$	Center

1.5 Linear System with Inputs

The solution of the homogeneous equation (1.1) can be easily generalized to differential equations with inputs. Consider the equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ x(0) &= x_0 \end{aligned} \tag{1.29}$$

where u is a piecewise continuous $\mathbb{R}^m$-valued function. Define a function $z(t) = e^{-At}x(t)$. Then $z(t)$ satisfies

$$\begin{aligned} \dot{z}(t) &= -e^{-At}Ax(t) + e^{-At}Ax(t) + e^{-At}Bu(t) \\ &= e^{-At}Bu(t) \end{aligned}$$

Since the above equation does not depend on $z(t)$ on the right hand side, it can be directly integrated to give

$$z(t) = z(0) + \int_0^t e^{-As}Bu(s)ds = x(0) + \int_0^t e^{-As}Bu(s)ds\,.$$

Hence

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}Bu(s)ds\,.$$

This is the *variation of constants formula* for solving (1.29). Uniqueness follows from uniqueness of the homogeneous equation.

Now consider linear systems with inputs and outputs

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ x(0) &= x_0 && (1.30) \\ y(t) &= Cx(t) + Du(t)\,. && (1.31) \end{aligned}$$

From the above results, the solution is obtained immediately

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-s)}Bu(s)ds + Du(t)\,. \tag{1.32}$$

In the case $x_0 = 0$ and $D = 0$, we find

$$y(t) = \int_0^t Ce^{A(t-s)}Bu(s)ds$$

which is of the form $y(t) = \int_0^t H(t-s)u(s)ds$, a convolution integral. $H(t) = Ce^{At}B$ is called the impulse response of the system. If we allow generalized functions, we can incorporate a nonzero D term as

$$H(t) = Ce^{At}B + D\delta(t) \tag{1.33}$$

where $\delta(t)$ is the Dirac δ-function. Noting that the Laplace transform of e^{At} is $(sI - A)^{-1}$, the transfer function of the linear system from u to y, which is the Laplace trasnform of the impulse response, is given by

$$H(s) = C(sI - A)^{-1}B + D\,. \tag{1.34}$$

1.6 Stability

Let us consider the unforced state equation first: $\dot{x} = Ax, \quad x(0) = x_0$.

Theorem 1.6.1. *The vector $x(t) \to 0$ as $t \to \infty$ for every $x_0 \Leftrightarrow$ all the eigenvalues of A lie in the open left half-plane.*

Proof.

$$\hat{x}(s) = (sI - A)^{-1} x_0$$

Thus

$$x(t) \to 0 \quad \forall x_0$$

$$\begin{aligned} &\Leftrightarrow \text{all poles of } (sI - A)^{-1} \text{ lie in } \{s : \text{Re}s < 0\} \\ &\Leftrightarrow \text{all eigenvalues of } A \text{ lie in } \{\lambda : \text{Re}\lambda < 0\} \end{aligned}$$

□

Now let us look at the full system model:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \ . \end{aligned}$$

The transfer matrix from u to y is

$$H(s) = C(sI - A)^{-1}B + D \ .$$

We can write this as

$$H(s) = \frac{1}{\det(sI - A)} C \cdot \text{adj}(sI - A) \cdot B + D \ .$$

Notice that the elements of the matrix $\text{adj}(sI - A)$ are all polynomials in s; consequently, they have no poles. Notice also that $\det(sI - A)$ is the characteristic polynomial of A. We can therefore conclude from the preceding equation that

$$\{\text{eigenvalues of } A\} \supset \{\text{poles of } H(s)\} \ .$$

Hence, if all the eigenvalues of A are in the open left half-plane, then $H(s)$ is a stable transfer matrix. The converse is not necessarily true.

Example 1.6.1.

Consider

$$A = \begin{bmatrix} 0 & 1 & 1 \\ -2 & -2 & 0 \\ 2 & 1 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$C = [2 \quad 2 \quad 1], \qquad D = [0 \quad 0]$$

$$\{\text{eigenvalues of } A\} = \{0, -1, -2\}$$

$$
\begin{aligned}
H(s) &= \frac{1}{\det(sI-A)} C \cdot \operatorname{adj}(sI-A) \cdot B \\
&= \frac{1}{s^2+3s^2+2s} \cdot [2 \;\; 2 \;\; 1] \cdot \begin{bmatrix} s^2+3s+2 & s+2 & s+2 \\ -2s-2 & s^2+s-2 & -2 \\ 2s+2 & s+2 & s^2+2s+2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ -1 & -1 \end{bmatrix} \\
&= \frac{1}{s^2+3s^2+2s} [2s^2+4s+2 \;\; 2s^2+5s+2 \;\; s^2+4s+2] \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ -1 & -1 \end{bmatrix} \\
&= \frac{1}{s^3+3s^2+2s} [s^2+2s \;\; s^2+s] \\
&= \begin{bmatrix} \dfrac{s+2}{s^2+3s+2} & \dfrac{s+1}{s^2+3s+2} \end{bmatrix}
\end{aligned}
$$

Thus {poles of $H(s)$} = {-1,-2}. Hence the eigenvalue of A at $\lambda = 0$ does not appear as a pole of $H(s)$. Actually, this shouldn't be surprising – a more obvious example is

$$
A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
$$
$$
C = [1 \;\; 0], \qquad D = 0
$$
$$
H(s) = \frac{1}{s-1} \qquad .
$$

Chapter 2

Linear Algebra

This chapter gives an introduction to those parts of linear algebra that are needed to understand two key concepts of linear system theory: controllability and observability. It is assumed that the reader has some background in linear algebra.

2.1 Vector Spaces

A *linear space* (or *vector space*) $\mathcal{X}$ over the field $\mathbb{R}$ of reals is a set of elements (called *vectors*) with two operations: addition of vectors and scalar multiplication. (Check P. Halmos, *Finite Dimensional Vector Spaces* for a precise definition). In this course, we will mostly be working with $\mathcal{X} = \mathbb{R}^n$. For vectors $x_1, \dots, x_n$ in $\mathcal{X}$, Span $\{x_1, \dots, x_n\}$ denotes the linear span of the vectors, i.e.,

$$\left\{ \sum_{i=1}^{n} c_i \, x_i \ : \ c_i \in \mathbb{R} \right\}.$$

We say $\mathcal{X}$ is *finite-dimensional* if there exist vectors $x_1, \dots, x_n$ such that $\mathcal{X} = \text{Span}\,\{x_1, \dots, x_n\}$. The least such n is the *dimension* of $\mathcal{X}$, denoted $\dim(\mathcal{X})$. A set of vectors $\{x_1, \dots, x_n\}$ is *linearly independent* if

$$(\forall c_i) \ \ c_1 x_1 + \cdots + c_n x_n = 0 \quad \Longrightarrow \quad c_1 = \cdots = c_n = 0.$$

A set $\{x_1, \dots, x_n\}$ is a *basis* for $\mathcal{X}$ if

$$\mathcal{X} = \text{Span}\,\{x_1, \dots, x_n\} \text{ and } \{x_1, \dots, x_n\} \text{ is lin. indep.}$$

Then every x in $\mathcal{X}$ can be written

$$x = c_1 x_1 + \cdots + c_n x_n$$

where the coefficients are unique.

Definition 2.1.1. A subset $\mathcal{V}$ of $\mathcal{X}$ is a *subspace*, and we write $\mathcal{V} \subset \mathcal{X}$, if $\mathcal{V}$ is closed under addition, i.e.,

$$x, y \in \mathcal{V} \Longrightarrow x + y \in \mathcal{V},$$

and closed under scalar multiplication, i.e.,

$$x \in \mathcal{V}, \ c \in \mathbb{R} \Longrightarrow c\, x \ \in \ \mathcal{V}.$$

For example, in $\mathbb{R}^2$ the subspaces are $\{0\}$, straight lines through 0, $\mathbb{R}^2$ itself. In $\mathcal{X}$, the subspace $\{0\}$ is sometimes denoted 0 (the zero subspace). The empty set is not a subspace.

Let $\mathcal{V}, \mathcal{W}$ be subspaces of $\mathcal{X}$. Then $\mathcal{V}+\mathcal{W}$ denotes the set

$$\{v+w : v \in \mathcal{V}, w \in \mathcal{W}\},$$

and it is a subspace of $\mathcal{X}$. The set union $\mathcal{V} \cup \mathcal{W}$ is not a subspace in general (when is it?). The intersection $\mathcal{V} \cap \mathcal{W}$ is however a subspace. As an example:

$$\mathcal{X} = \mathbb{R}^3, \quad \mathcal{V} \text{ a line through 0}, \quad \mathcal{W} \text{ a plane through 0.}$$

Then $\mathcal{V}+\mathcal{W} = \mathbb{R}^3$ if $\mathcal{V}$ does not lie in $\mathcal{W}$. If $\mathcal{V} \subset \mathcal{W}$, then of course $\mathcal{V}+\mathcal{W} = \mathcal{W}$.

As an exercise show that

$$\dim(\mathcal{V}+\mathcal{W}) = \dim(\mathcal{V}) + \dim(\mathcal{W}) - \dim(\mathcal{V} \cap \mathcal{W}).$$

Two subspaces $\mathcal{V}, \mathcal{W}$ are *independent* if $\mathcal{V} \cap \mathcal{W} = 0$. This is not the same as being orthogonal. For example two lines through the origin in the plane are independent iff they are not colinear (i.e., the angle between them is not 0), while they are orthogonal iff the angle is 90°. Three subspaces $\mathcal{U}, \mathcal{V}, \mathcal{W}$ are *independent* if $\mathcal{U}, \mathcal{V}+\mathcal{W}$ are independent, $\mathcal{V}, \mathcal{U}+\mathcal{W}$ are independent, and $\mathcal{W}, \mathcal{U}+\mathcal{V}$ are independent. This is not the same as being pairwise independent. As an example, let $\mathcal{U}, \mathcal{V}, \mathcal{W}$ be 1-dimensional subspaces of $\mathbb{R}^3$, i.e., three lines through 0. When are they independent? Pairwise independent?

More generally, subspaces $\mathcal{V}_1, \ldots, \mathcal{V}_k$ are *independent* if

$$\mathcal{V}_i \cap \left(\sum_{j \neq i} \mathcal{V}_j \right) = 0 \text{ for every } i.$$

The following three conditions are equivalent:

1. $\mathcal{V}_1, \ldots, \mathcal{V}_k$ are independent.
2. $(\forall v \in \mathcal{V}_1 + \cdots + \mathcal{V}_k)\ (\exists \text{ unique } v_i \in \mathcal{V}_i)\ \ v = v_1 + \cdots + v_k$.
3. $\dim(\mathcal{V}_1 + \cdots + \mathcal{V}_k) = \dim(\mathcal{V}_1) + \cdots + \dim(\mathcal{V}_k)$.

If $\mathcal{V}, \mathcal{W}$ are independent subspaces, we write their sum as $\mathcal{V} \oplus \mathcal{W}$. This is called an *internal direct sum*. Likewise for more than two.

Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two vector spaces, not necessarily subspaces of a larger space. The Cartesian product $\mathcal{X}_1 \times \mathcal{X}_2$ is the set of ordered pairs (x_1, x_2). It's a vector space under componentwise addition and scalar multiplication. This space is denoted $\mathcal{X}_1 \oplus \mathcal{X}_2$ too, the *external direct sum*.

Let $\mathcal{X}$ be a vector space and $\mathcal{V} \subset \mathcal{X}$. It is a fact that every subspace has an independent complement, i.e.,

$$\mathcal{V} \subset \mathcal{X} \implies (\exists \mathcal{W} \subset \mathcal{X})\ \mathcal{X} = \mathcal{V} \oplus \mathcal{W}.$$

In fact, $\mathcal{V}$ has many complements, but there is a vector space that uniquely captures the notion of "$\mathcal{X}$ minus $\mathcal{V}$." It's called the *quotient space*, denoted $\mathcal{X}/\mathcal{V}$. We will not go into the details of the construction here, but we will at times make use of the notation (see the Representation theorem below).

Finally, the *orthogonal complement* $\mathcal{V}^\perp$ (called "$\mathcal{V}$ perp") of a subspace $\mathcal{V} \subset \mathcal{X}$ is given by

$$\mathcal{V}^\perp = \{x \in \mathcal{X} \mid x \cdot v = 0\,, v \in \mathcal{V}\}\,.$$

It is the set of all vectors orthogonal to every vector in V.

2.2 Linear Transformations

Let $\mathcal{X}, \mathcal{Y}$ be two vector spaces. A function $A : \mathcal{X} \to \mathcal{Y}$ is a *linear transformation* (LT) iff

$$\begin{aligned} A(x_1 + x_2) &= Ax_1 + Ax_2, \quad x_1, x_2 \in \mathcal{X} \\ A(ax) &= aA\,x, \quad a \in \mathbb{R},\ x \in \mathcal{X}. \end{aligned}$$

An LT is uniquely determined by its action on a basis. That is, if $A : \mathcal{X} \to \mathcal{Y}$ is an LT and if $\{e_1, \ldots, e_n\}$ is a basis for $\mathcal{X}$, then if we know the vectors Ae_i, we can compute Ax for every $x \in \mathcal{X}$, by linearity.

Example 2.2.1.

Let $\mathcal{X}$ be a vector space of dimension n and let $\{e_1, \ldots, e_n\}$ be a basis. Every vector x in $\mathcal{X}$ has a unique expansion

$$x = a_1 e_1 + \cdots + a_n e_n, \quad a_i \in \mathbb{R}.$$

The function

$$x \longmapsto \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

defines an LT $Q : \mathcal{X} \to \mathbb{R}^n$. It maps x to its *vector of coordinates* with respect to the basis.

For example, let $\mathcal{X} = \mathbb{R}^2$. Take the natural basis

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

In this case

$$Q : x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \longmapsto \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

i.e., Q is the identity LT. If the basis instead is

$$e_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

then for any x in $\mathcal{X}$

$$\begin{aligned} x &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= a_1 e_1 + a_2 e_2 \\ &= a_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \end{aligned}$$

so

$$Q : x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \longmapsto \quad \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Every LT on finite-dimensional vector spaces has a *matrix representation*. Let A be an LT $\mathcal{X} \to \mathcal{Y}$, with

$$\dim(\mathcal{X}) = n, \text{ basis } \{e_1, \ldots, e_n\}; \quad \dim(\mathcal{Y}) = p, \text{ basis } \{f_1, \ldots, f_p\}.$$

The following is a recipe for constructing the matrix A:

1. Take the i^{th} basis vector e_i of $\mathcal{X}$.
2. Apply the LT A to get Ae_i.
3. Express the vector Ae_i in the basis for $\mathcal{Y}$ and enter this coordinate vector as column i of A.

Example 2.2.2.

Let $A : \mathbb{R}^{2\times 2} \to \mathbb{R}$ map a 2×2 matrix B to trace B. Let's find its matrix representation. We need a basis for $\mathbb{R}^{2\times 2}$; let's take, say,

$$E_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \ E_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \ E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

You can check these are linearly independent. And we need a basis for $\mathbb{R}$; let's take $f = 3$. To find the first column of A, apply the recipe with $i = 1$:

1. $E_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$
2. $AE_1 = \text{trace } E_1 = 1$
3. $1 = \frac{1}{3}f$, so 1^{st} col of A is $\frac{1}{3}$.

Proceeding column by column, we get

$$A = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}.$$

An LT induces two special subspaces.

Definition 2.2.1. Let $A : \mathcal{X} \to \mathcal{Y}$ be an LT. The *kernel* or *nullspace* of A is the subspace of $\mathcal{X}$

$$\mathcal{N}(A) = \text{Ker } A := \{x : Ax = 0\}.$$

The *image* or *range space* of A is the subspace of $\mathcal{Y}$

$$\mathcal{R}(A) = Im(A) := \{y : (\exists\, x \in \mathcal{X}) y = Ax\}.$$

More generally, if $\mathcal{V} \subset \mathcal{X}$, the *image* of $\mathcal{V}$ under A is

$$A\mathcal{V} := \{y : (\exists\, x \in \mathcal{V}) y = Ax\}.$$

Thus

$$\mathcal{R}(A) = A\mathcal{X}.$$

Example 2.2.3.

The following example shows how to compute $\mathcal{N}(A)$. Suppose we have

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix},$$

and we want to find all $x \in \mathbb{R}^3$ such that $Ax = 0$ or

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ x_1 + x_3 &= 0\,. \end{aligned}$$

Equivalently

$$\begin{aligned} x_1 &= -x_3 \\ x_2 &= x_3\,. \end{aligned}$$

Hence

$$\mathcal{N}(A) = \text{span of } \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Example 2.2.4.

The following example illustrates the determination of $\mathcal{R}(A)$ and its use in solution of linear equations. Consider the linear system of equations

$$Ax = b\,,$$

with

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \qquad b = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

Then $x = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ are all solutions.

The range of A is the column span of A or

$$\mathcal{R}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

We get an under-determined set of equations

$$\begin{aligned} x_2 + x_3 - x_4 &= 1 \\ x_1 + 2x_2 + 3x_3 - x_4 &= 4 \end{aligned}$$

which has an infinite number of solutions. However, if $b = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$, then there is no solution.

Recall that an LT A is *one-to-one* if $v_1 \neq v_2$ implies $Av_1 \neq Av_2$. An LT A is *onto* if for every $y \in \mathcal{Y}$ there exists an $x \in \mathcal{X}$ such that $Ax = y$. Clearly A is onto if $\mathcal{R}(A) = \mathcal{Y}$. Can we find a characterization of whether A is one-to-one in terms of $\mathcal{N}(A)$ or $\mathcal{R}(A)$?

Lemma 2.2.1. *Let A be a linear transformation from $\mathcal{X}$ to $\mathcal{Y}$. Then A is one-to-one iff $\mathcal{N}(A) = 0$.*

Lemma 2.2.2. *Let A be a linear transformation from $\mathcal{X}$ to $\mathcal{Y}$ where $dim(\mathcal{X}) = n$. Then*

$$dim\mathcal{R}(A) + dim\mathcal{N}(A) = n\,.$$

Whether a matrix A is one-to-one or onto (or both) can be easily checked as follows:

$$A \text{ is onto} \iff A \text{ has full row rank;}$$

$$A \text{ is one-to-one} \iff A \text{ has full column rank.}$$

The rank of a matrix is the dimension of $\mathcal{R}(A)$.

2.3 Invariant Subspaces

Example 2.3.1.

Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be the LT that maps $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ to $\begin{bmatrix} x_1 + x_2 \\ 2x_1 + 2x_2 \end{bmatrix}$. Thus, with respect to the natural bases, the matrix representation is

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

Clearly, $\mathcal{N}(A)$ is the 1-dimensional subspace spanned by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Also,

$$x \in \mathcal{N}(A) \Rightarrow Ax = 0 \in \mathcal{N}(A),$$

or equivalently,

$$A\mathcal{N}(A) \subset \mathcal{N}(A).$$

Definition 2.3.1. If $A : \mathcal{X} \to \mathcal{X}$ is an LT, a subspace $\mathcal{V} \subset \mathcal{X}$ is A-*invariant* if $A\mathcal{V} \subset \mathcal{V}$.

For example, the zero subspace, $\mathcal{X}$ itself, $\mathcal{N}(A)$, and Im A are A-invariant. $\mathcal{N}(A)$ is the eigenspace for the zero eigenvalue, assuming $\lambda = 0$ *is* an eigenvalue (as in the example above). More generally, suppose λ is an eigenvalue of A, and assume $\lambda \in \mathbb{R}$. Then $Ax = \lambda x$ for some $x \neq 0$ so $\mathcal{V} = \text{span}\,\{x\}$ is A-invariant. So is the *eigenspace*

$$\{x : Ax = \lambda x\} = \{x : (A - \lambda I)\, x = 0\} = \mathcal{N}(A - \lambda I).$$

Theorem 2.3.1 (Representation Theorem)**.** *Suppose $\mathcal{V} \subset \mathcal{X}$ is an A-invariant subspace for LT $A : \mathcal{X} \to \mathcal{X}$. The A has a matrix representation*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}.$$

The LT $A_{11} : \mathcal{V} \to \mathcal{V}$ is called the *restriction* of A to $\mathcal{V}$. Likewise $A_{22} : \mathcal{X}/\mathcal{V} \to \mathcal{X}/\mathcal{V}$ is the restriction of A to $\mathcal{X}/\mathcal{V}$.

Proof. Let $\{e_1, \dots, e_k\}$ be a basis for $\mathcal{V}$ and let $\{e_1, \dots, e_k, \dots, e_n\}$ be a basis for $\mathcal{X}$. We know that for any $x \in \mathcal{V}$, $Ax \in \mathcal{V}$. In particular, if we pick $x = e_i$, $i = 1, \dots, k$ then since $Ae_i \in \mathcal{V}$,

$$Ae_i = a_{1i}e_1 + \ldots + a_{ki}e_k$$

for suitable constants a_{ij}, We know that the coefficients a_{ij}, $j = 1, \dots, k$ form the ith column of A. Hence, A takes the form

$$A = \begin{bmatrix} A_{11} & * \\ 0 & * \end{bmatrix}.$$

□

Example 2.3.2.

Let $\mathcal{X} = \mathbb{R}^3$, let $\mathcal{V}$ be the (x_1, x_2)-plane, and let $A : \mathcal{X} \to \mathcal{X}$ be the LT that rotates a vector 90° about the x_3-axis using the right-hand rule. Thus $\mathcal{V}$ is A-invariant.

Let us take the bases

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ for } \mathcal{V}$$

$$e_1,\ e_2,\ e_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ for } \mathcal{X}.$$

The matrix representation of A with respect to the latter basis is

$$A = \left[\begin{array}{cc|c} 0 & -1 & -2 \\ 1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right].$$

So, in particular, the induced matrix representation of A_{22} is 1.

2.4 Appendix

This Appendix provides some background material used in the main result on controllability in Section 3.1. In particular, we present the Cayley-Hamilton theorem and adjoints of linear operators. It is intended primarily for readers who want to have a more in depth understanding of the mathematical foundations of certain arguments appearing in the proofs of Chapter 3. This material may be skipped on a first reading.

2.4.1 Cayley-Hamilton Theorem

Theorem 2.4.1. *A square matrix satisfies its own characteristic equation.*

Proof. Let $\Delta(s) = \det(sI - A) = s^n + d_1 s^{n-1} + \cdots + d_n = 0$ be the characteristic equation of a matrix $A : X \to X$. Also let

$$Adj(sI - A) = B_0 s^{n-1} + B_1 s^{n-2} + \cdots + B_{n-2} s + B_{n-1}.$$

where B_i are constant matrices in $\mathbb{R}^{n\times n}$. We have that

$$(sI - A)Adj(sI - A) = \det(sI - A) \cdot I\,. \tag{2.1}$$

Substituting the expression for $Adj(sI - A)$ on the l.h.s. we obtain

$$\begin{aligned}&(sI - A)\left[B_0 s^{n-1} + B_1 s^{n-2} + \cdots + B_{n-2} s + B_{n-1}\right]\\ &\quad = B_0 s^n + (B_1 - AB_0)s^{n-1} + \cdots + (B_{n-1} - AB_{n-2})s - AB_{n-1}.\end{aligned}$$

When we compare coefficients with the r.h.s. of Equation (2.1) we find that

$$\begin{aligned} B_0 &= I \\ (B_1 - AB_0) &= d_1 I \\ (B_2 - AB_1) &= d_2 I \\ &\vdots \\ (B_{n-1} - AB_{n-2}) &= d_{n-1} I \\ -AB_{n-1} &= d_n I\,. \end{aligned}$$

When these coefficients are substituted in $\Delta(s)$ with $s = A$ we find

$$\begin{aligned}\Delta(A) = A^n B_0 + A^{n-1}(B_1 - AB_0) + A^{n-2}(B_2 - AB_1) +&\\ \cdots + A(B_{n-1} - AB_{n-2}) - AB_{n-1} &= 0\end{aligned}$$

□

2.4.2 Adjoints of Linear Maps

$\mathbb{R}^n$ is a vector space with an *inner product*

$$\langle x, y\rangle_{\mathbb{R}^n} := x'y\,,$$

where x' is the transpose of x, and $x, y \in \mathbb{R}^n$. The associated *norm* is

$$\|x\|_{\mathbb{R}^n} := \langle x, x\rangle^{\frac{1}{2}}\,.$$

More generally, a vector space V equipped with an inner product is called an *inner product space*. (An inner product space is called a *Hilbert space* if it is complete, i.e., if every Cauchy sequence converges.)

Let V be an inner product space with an inner product $\langle \cdot\,,\cdot\rangle$. Recall

$$\big|\langle x, y\rangle\big| \le \|x\|\,\|y\|\,, \qquad \forall x, y \in V\,. \qquad \text{(Schwartz inequality)}$$

Definition 2.4.1. (Adjoint) Let V and W be inner product spaces with inner products $\langle \cdot\,,\cdot\rangle_V$ and $\langle \cdot\,,\cdot\rangle_W$, respectively and $L : V \to W$ a continuous linear map (also called a linear transformation). The adjoint of L is defined as the linear map $L^* : W \to V$ which satisfies:

$$\langle w, Lv\rangle_W = \langle L^* w, v\rangle_V\,, \qquad \forall v \in V\,, \quad \forall w \in W\,.$$

One can show that the adjoint of a linear map, as defined above, always exists and is unique. We denote by $\|\cdot\|_V$ and $\|\cdot\|_W$ the associated norms on V and W, respectively.

Example 2.4.1. *Let $V = \mathbb{R}^n$, $W = \mathbb{R}^m$, endowed with the standard Cartesian basis and inner product, and consider a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ having a matrix representation A. The transpose A' of A is the matrix representation of the adjoint of the map.*

Example 2.4.2. *Let $\mathcal{L}^2([0,t_1])$ denote the space of square integrable, vector-valued functions defined on $[0,t_1]$, i.e., the set of functions $u(\cdot) : [0,t_1] \to \mathbb{R}^k$ with $\int_0^{t_1} \|u(t)\|^2\,dt < \infty$. The inner product $\langle \cdot\,,\cdot\rangle_2$ in $\mathcal{L}^2([0,t_1])$ is defined by*

$$\langle u_1, u_2\rangle_2 := \int_0^{t_1} u_1'(t) u_2(t)\,dt\,,$$

where $u_1'(t)$ denotes the transpose of $u_1(t) \in \mathbb{R}^k$. Let $F(t)$ be an $n \times k$ real matrix defined for $t \in [0,t_1]$ and satisfying $\int_0^{t_1} \|F(t)\|^2\,dt < \infty$. Define the linear map $L : \mathcal{L}^2([0,t_1]) \to \mathbb{R}^n$ by:

$$L(u) = \int_0^{t_1} F(t)u(t)\,dt\,.$$

One can find the adjoint $L^* : \mathbb{R}^n \to \mathcal{L}^2([0,t_1])$ of the map L as follows. We have

$$\begin{aligned}
\langle Lu, x\rangle_{\mathbb{R}^n} &= \left[\int_0^{t_1} F(t)u(t)dt\right]' x \\
&= \int_0^{t_1} u'(t)F'(t)x dt \\
&= \int_0^{t_1} u'(t)(L^*x)(t) dt \\
&= \langle u, L^*x\rangle_2\,.
\end{aligned}$$

Hence

$$\big(L^*x\big)(t) = F'(t)x\,, \qquad x \in \mathbb{R}^n\,, \quad t \in [0,t_1]\,,$$

where, again, $F'(t)$ is the transpose of $F(t)$. Note that LL^* is a map from $\mathbb{R}^n$ to $\mathbb{R}^n$ and so it has a matrix representation given by:

$$LL^* = \int_0^{t_1} F(t)F'(t)\,dt\,.$$

Definition 2.4.2. (Orthogonality) Let V be an inner product space with inner product $\langle \cdot\,,\cdot\rangle$. We say that two vectors $x, y \in V$ are *orthogonal*, and we use the notation $x \perp y$, if $\langle x, y\rangle = 0$. Let M be a subspace of V. The *orthogonal complement* of M in V is denoted by $M^\perp$ and defined by:

$$M^\perp := \big\{x \in V \;\big|\; \langle x, y\rangle = 0\,,\ \forall y \in M\big\}$$

It is clear that $M^\perp$ is a subspace of V. We say that two subspaces M and N of V are orthogonal, denoted by $M \perp N$ if whenever $x \in M$ and $y \in N$, then $x \perp y$. We will use the notation $V = M \overset{\perp}{\oplus} N$ and say that V is the *orthogonal direct sum* of M and N to denote that $V = M \oplus N$ and $M \perp N$.

We are going to make use of the following fact which we state here without proof.

Theorem 2.4.2. *If M is a finite dimensional subspace of an inner product space V then $V = M \oplus M^\perp$ and, in addition, $M = \left(M^\perp\right)^\perp$.*

Lemma 2.4.1. *Let L be as in Definition 2.1 with V, W inner product spaces. Then*

(a) $\mathcal{N}(L^*) = \mathcal{R}(L)^\perp$ *and* $\mathcal{N}(L) = \mathcal{R}(L^*)^\perp$.

(b) $\mathcal{N}(LL^*) = \mathcal{N}(L^*)$ *and* $\mathcal{N}(L^*L) = \mathcal{N}(L)$,

where $\mathcal{N}$ and $\mathcal{R}$ denote the nullspace and the range respectively.

Proof. The second identities in (a) and (b) can be derived from the first by interchanging the roles of L and L^*. For (a) note that

$$w \in \mathcal{N}(L^*) \iff \langle L^*w, v\rangle_V = 0\,, \quad \forall v \in V \tag{2.2}$$
$$\iff \langle w, Lv\rangle_W = 0\,, \quad \forall v \in V \tag{2.3}$$
$$\iff w \in \mathcal{R}(L)^\perp\,. \tag{2.4}$$

To prove (b) it is enough to show $\mathcal{N}(LL^*) \subset \mathcal{N}(L^*)$, since the inclusion $\mathcal{N}(L^*) \subset \mathcal{N}(LL^*)$ is obvious. We have,

$$w \in \mathcal{N}(LL^*) \Longrightarrow \langle w, LL^*w\rangle_W = 0 \tag{2.5}$$
$$\Longrightarrow \langle L^*w, L^*w\rangle_V = \|L^*w\|_V^2 = 0 \tag{2.6}$$
$$\Longrightarrow w \in \mathcal{N}(L^*)\,, \tag{2.7}$$

and the proof is complete. □

Theorem 2.4.3. *Let V and W be inner product spaces such that either V or W is finite dimensional. Let $L : V \to W$ be a continuous linear map. Then*

(a) $W = \mathcal{R}(L) \overset{\perp}{\oplus} \mathcal{N}(L^*)$ *and* $V = \mathcal{R}(L^*) \overset{\perp}{\oplus} \mathcal{N}(L)$

(b) $\mathcal{R}(L) = \mathcal{R}(LL^*)$ *and* $\mathcal{R}(L^*) = \mathcal{R}(L^*L)$

(c) If $w_0 \in \mathcal{R}(L)$, then $v_0 = L^*\eta_0$, where η_0 satisfies $LL^*\eta_0 = w_0$ is the solution of minimum norm of the equation $Lv = w_0$, i.e., if v is any other solution, then $\|v_0\|_V \le \|v\|_V$.

(d) If $w_0 \in W$, then any $v_0 \in V$ satisfying $L^*Lv_0 = L^*w_0$, solves the minimization problem $\inf\limits_{v\in V} \|Lv - w_0\|_W$.

Proof. Since $\mathcal{R}(L)$ is a finite dimensional subspace it follows from Theorem 2.1 and Lemma 2.2 that

$$\mathcal{N}(L^*)^\perp = \left(\mathcal{R}(L)^\perp\right)^\perp = \mathcal{R}(L)$$

and hence

$$W = \mathcal{R}(L) \oplus \mathcal{R}(L)^\perp = \mathcal{R}(L) \overset{\perp}{\oplus} \mathcal{N}(L^*)\,.$$

Also, since $(LL^*)^* = LL^*$,

$$\mathcal{R}(L) = \mathcal{N}(L^*)^\perp = \mathcal{N}(LL^*)^\perp = \mathcal{R}(LL^*)\,.$$

The proofs of the second parts of (a) and (b) are analogous.

For part (c), note that since $\mathcal{R}(L) = \mathcal{R}(LL^*)$ and $w_0 \in \mathcal{R}(L)$ there exists $\eta_0 \in W$ such that $LL^*\eta_0 = w_0$. If v is any solution of $Lv = w_0$, then $L(v - v_0) = 0$ or equivalently $(v - v_0) \in \mathcal{N}(L)$; also, $v_0 \in \mathcal{R}(L^*)$. Hence, by (a) of Lemma 1, $(v - v_0) \perp v_0$, implying that

$$\|v\|_V^2 = \|v - v_0 + v_0\|_V^2 = \|v - v_0\|_V^2 + \|v_0\|_V^2\,.$$

It follows that $\|v_0\|_V \le \|v\|_V$.

For part (d), first note that since $\mathcal{R}(L^*) = \mathcal{R}(L^*L)$ the existence of some v_0 satisfying $L^*Lv_0 = L^*w_0$ is guaranteed. Note that $(Lv_0 - w_0) \in \mathcal{N}(L^*)$ Let $v \in V$ be arbitrary. Since $(Lv - Lv_0) \in \mathcal{R}(L)$, it follows by Lemma 1 that $(Lv - Lv_0) \perp (Lv_0 - w_0)$; therefore,

$$\|Lv - w_0\|_W^2 = \|Lv - Lv_0 + Lv_0 - w_0\|_W^2 = \|Lv - Lv_0\|_W^2 + \|Lv_0 - w_0\|_W^2\,.$$

Thus, $\|Lv - w_0\|_W \ge \|Lv_0 - w_0\|_W$. □

Chapter 3

Controllability

All the material in this chapter pertains to the differential equation

$$\dot{x} = Ax + Bu, \, , \qquad x(0) = 0 \, , \tag{3.1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$, so $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. We study the concept of controllability and two important properties of controllability: invariance under change of basis and invariance under state feedback. We also discuss controllable canonical form for single input systems, which is very useful for the pole assignment problem discussed in the next chapter.

3.1 Reachable States

Consider the following problem. For fixed t_1 and a given vector v in $\mathbb{R}^n$, does there exist a control input u, defined on $[0, t_1]$, such that the solution of (3.1) satisfies $x(t_1) = v$? We shall refer to this problem as the *reachability problem*.

Let $\mathcal{U}$ be the space of piecewise continuous functions with finite energy in every finite time interval. Define the linear operator $L_c : \mathcal{U} \rightarrow \mathbb{R}^n$ by

$$L_c u = \int_0^{t_1} e^{A(t_1 - s)} B u(s) ds \, .$$

The reachability problem is equivalent to the solvability of the linear equation

$$L_c u = v \, .$$

To bring out the analogy a bit further, recall that the linear equation

$$A\xi = b$$

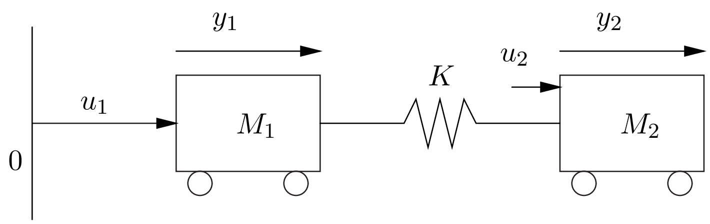
has a solution if and only if $b \in \mathcal{R}(A)$ where $\mathcal{R}(A)$ is the range of the matrix A. Similarly, the reachability problem is solvable if and only if $v \in \mathcal{R}(L_c)$. Then we say that the state v is reachable at time t_1. The set of reachable states is given by $\mathcal{R}(L_c)$. Every state is reachable iff L_c is onto. If every state is reachable then we say that the system (A, B) is *controllable*.

Define the *controllability matrix* Q_c, which has dimension $n \times nm$, as

$$Q_c = [\; B \;\; AB \;\; A^2B \;\; \ldots \;\; A^{n-1}B \;] \, .$$

We will see shortly that the rank of this matrix determines whether a system is controllable.

Example 3.1.1.



The system equations are

$$\begin{aligned} M_1\ddot{y}_1 &= -K(y_1 - y_2) + u_1 \\ M_2\ddot{y}_2 &= -K(y_2 - y_1) + u_2\,. \end{aligned}$$

We define the state vector

$$x = \begin{bmatrix} y_1 \\ \dot{y}_1 \\ y_2 \\ \dot{y}_2 \end{bmatrix}.$$

Then the state equations are

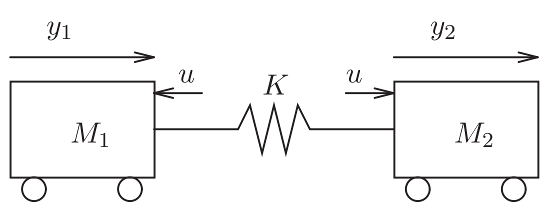
$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-K}{M_1} & 0 & \frac{K}{M_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{K}{M_2} & 0 & \frac{-K}{M_2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{1}{M_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{M_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

The controllability matrix is

$$Q_c \;=\; \begin{bmatrix} 0 & 0 & \frac{1}{M_1} & 0 & 0 & 0 & \frac{-K}{M_1^2} & \frac{K}{M_1M_2} \\ \frac{1}{M_1} & 0 & 0 & 0 & \frac{-K}{M_1^2} & \frac{K}{M_1M_2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{M_2} & 0 & 0 & \frac{K}{M_1M_2} & \frac{-K}{M_2^2} \\ 0 & \frac{1}{M_2} & 0 & 0 & \frac{K}{M_1M_2} & \frac{-K}{M_1^2} & 0 & 0 \end{bmatrix}$$

which has 4 linearly independent columns so that it is full rank.

Example 3.1.2.



The system equations are

$$\begin{aligned} M_1\ddot{y}_1 &= -u - K(y_1 - y_2) \\ M_2\ddot{y}_2 &= u - K(y_2 - y_1) \end{aligned}$$

and defining the state vector

$$x = \begin{bmatrix} y_1 \\ \dot{y}_1 \\ y_2 \\ \dot{y}_2 \end{bmatrix},$$

we obtain the state equations

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-K}{M_1} & 0 & \frac{K}{M_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{K}{M_2} & 0 & \frac{-K}{M_2} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -\frac{1}{M_1} \\ 0 \\ \frac{1}{M_2} \end{bmatrix} u \,.$$

The controllability matrix is

$$Q_c = \begin{bmatrix} 0 & -\frac{1}{M_1} & 0 & \frac{K}{M_1^2} + \frac{K}{M_1 M_2} \\ -\frac{1}{M_1} & 0 & \frac{K}{M_1^2} + \frac{K}{M_1 M_2} & 0 \\ 0 & \frac{1}{M_2} & 0 & -\frac{K}{M_1 M_2} - \frac{K}{M_2^2} \\ \frac{1}{M_2} & 0 & -\frac{K}{M_1 M_2} - \frac{K}{M_2^2} & 0 \end{bmatrix} .$$

Note that for this Q_c

$$\begin{aligned} \text{the 3rd column} \quad &= \quad \begin{bmatrix} 0 \\ \frac{1}{M_1}\left(\frac{K}{M_1} + \frac{K}{M_2}\right) \\ 0 \\ \frac{1}{M_2} \end{bmatrix} \\ &= \quad -\left(\frac{K}{M_1} + \frac{K}{M_2}\right) \begin{bmatrix} \text{1st} \\ \text{column} \end{bmatrix} \end{aligned}$$

so that only two columns in Q_c are linearly independent.

Associated with Q_c, we define the *controllable subspace* which is $\mathcal{R}(Q_c)$. Note that for the two cart-two force system $\mathcal{R}(Q_c) = \mathbb{R}^4$ while for the two cart-one force system $\mathcal{R}(Q_c) = \mathbb{R}^2$.

The infinite dimensional vector space $\mathcal{U}$ has an inner product

$$\langle u, w \rangle_{\mathcal{U}} := \int_0^{t_1} u(\tau)^T w(\tau) d\tau \,.$$

Referring to the Appendix, the adjoint operator $L_c^* : \mathbb{R}^n \to \mathcal{U}$ of L_c is

$$(L_c^* v)(\tau) = B^T e^{(t_1 - \tau)A^T} v \,.$$

We define, for each $t > 0$, the *controllability gramian*

$$\begin{aligned} W_c(t) := (L_c L_c^*)(t) \quad &= \quad \int_0^t e^{A(t-\tau)} B B^T e^{A^T(t-\tau)} d\tau \\ &= \quad \int_0^t e^{A\tau} B B^T e^{A^T \tau} d\tau \,. \end{aligned}$$

We will shortly make use of the fact that $\mathcal{R}(L_c) = \mathcal{R}(L_c L_c^*)$; the proof is in the Appendix. This result is useful because $L_c L_c^* : \mathbb{R}^n \to \mathbb{R}^n$ is a matrix.

Example 3.1.3.

Suppose we have

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$\begin{aligned}
(sI-A)^{-1} &= \begin{bmatrix} s+1 & -1 \\ 0 & s+2 \end{bmatrix}^{-1} = \frac{\begin{bmatrix} s+2 & 1 \\ 0 & s+1 \end{bmatrix}}{(s+1)(s+2)} \\
&= \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} - \frac{1}{s+2} \\ 0 & \frac{1}{s+2} \end{bmatrix} \\
e^{At} &= \mathcal{L}^{-1}(sI-A)^{-1} = \begin{bmatrix} e^{-t} & e^{-t} - e^{-2t} \\ 0 & e^{-2t} \end{bmatrix} \\
W_c(t) &= \int_0^t \begin{bmatrix} e^{-\tau} & e^{-\tau} - e^{-2\tau} \\ 0 & e^{-2\tau} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} [0 \ \ 1] \begin{bmatrix} e^{-\tau} & 0 \\ e^{-\tau} - e^{-2\tau} & e^{-2\tau} \end{bmatrix} d\tau \\
&= \int_0^t \begin{bmatrix} e^{-\tau} & e^{-\tau} - e^{-2\tau} \\ 0 & e^{-2\tau} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ e^{-\tau} - e^{-2\tau} & e^{-2\tau} \end{bmatrix} d\tau \\
&= \int_0^t \begin{bmatrix} e^{-2\tau} - 2e^{-3\tau} + e^{-4\tau} & e^{-3\tau} - e^{-4\tau} \\ e^{-3\tau} - e^{-4\tau} & e^{-4\tau} \end{bmatrix} d\tau \\
&= \begin{bmatrix} \frac{1}{2}(1-e^{-2t}) - \frac{2}{3}(1-e^{-3t}) + \frac{1}{4}(1-e^{-4t}) & \frac{1}{3}(1-e^{-3t}) - \frac{1}{4}(1-e^{-4t}) \\ \frac{1}{3}(1-e^{-3t}) - \frac{1}{4}(1-e^{-4t}) & \frac{1}{4}(1-e^{-4t}) \end{bmatrix}.
\end{aligned}$$

Note that W_c is a symmetric $n \times n$ matrix.

We want to find a characterization of $\mathcal{R}(L_c)$ that allows us to determine in a computationally direct way whether a linear system is controllable. The main result of this section is the following.

Theorem 3.1.1. $\mathcal{R}(L_c) = \mathcal{R}(Q_c)$.

Proof. First we will show that $\mathcal{R}(L_c) \subset \mathcal{R}(Q_c)$. The Cayley-Hamilton theorem (refer to the Appendix for a proof) says that a matrix satisfies its own characteristic polynomial; that is, if $p(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0$ is the characteristic polynomial of the matrix A, then $p(A) = A^n + a_{n-1}A^{n-1} + ... + a_0 I = 0$. A consequence is that A^n is a linear combination of $\{A^j, \ \ j = 0, ..., n-1\}$ and hence A^k, $k \geq n$ is also. Since

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

the Cayley-Hamilton theorem allows us to conclude that

$$e^{At} = \varphi_0(t)I + \varphi_1(t)A + ... + \varphi_{n-1}(t)A^{n-1}$$

for certain functions $\{\varphi_i(t)\}$. Let $x \in \mathcal{R}(L_c)$. Then there exists a control u such that

$$\begin{aligned}
x &= \int_0^t e^{A(t-\tau)} B u(\tau) d\tau \\
&= \int_0^t [\varphi_0(t-\tau)I + ... + \varphi_{n-1}(t-\tau)A^{n-1}] B u(\tau) d\tau.
\end{aligned}$$

Let

$$v_j = \int_0^t \varphi_j(t-\tau)u(\tau)d\tau\,.$$

Then

$$x = [B \;\; AB \;\dots\; A^{n-1}B] \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{bmatrix} \in \mathcal{R}(Q_c)\,.$$

Second, we show that $\mathcal{R}(Q_c) \subset \mathcal{R}(L_c)$. Using facts about adjoints and orthogonal subspaces found in the Appendix, we have that

$$\begin{aligned} \mathcal{R}(L_c) &= \mathcal{R}(L_c L_c^*) = \mathcal{N}(L_c L_c^*)^{\perp} \\ \mathcal{R}(Q_c) &= \mathcal{N}({Q_c}^T)^{\perp}\,. \end{aligned}$$

From these facts, showing that $\mathcal{R}(Q_c) \subset \mathcal{R}(L_c)$ is equivalent to showing that

$$\mathcal{N}(L_c L_c^*) \subset \mathcal{N}(Q_c^T)\,.$$

To this end, let $x \in \mathcal{N}(L_c L_c^*)$. Then we have

$$\begin{aligned} 0 &= x^T W_c x \\ &= \int_0^t x^T e^{A\tau} B B^T e^{A^T\tau} x d\tau \\ &= \int_0^t \|B^T e^{A^T\tau} x\|^2 d\tau\,. \end{aligned}$$

This yields

$$B^T e^{A^T\tau} x = 0\,, \qquad 0 \le \tau \le t\,.$$

Setting $\tau = 0$ gives

$$B^T x = 0$$

For $k = 1, \dots, n-1$, take the kth derivative of $B^T e^{A^T\tau}$ with respect to τ and evaluate the result at $\tau = 0$. This gives successively

$$\begin{aligned} B^T A^T x &= 0 \\ B^T (A^T)^2 x &= 0 \\ &\vdots \\ B^T (A^T)^{n-1} x &= 0 \end{aligned}$$

so that $Q_c^T x = 0$. That is $x \in \mathcal{N}(Q_c^T)$, as desired. □

We summarize our results as follows. We say a linear system $\dot{x} = Ax + Bu$ or the pair (A, B) is controllable if any one (hence all) of the following conditions holds:

$$\begin{array}{lrcl} (i) & \mathcal{R}(L_c) &=& \mathbb{R}^n \quad \text{for some (hence all)} \quad t > 0. \\ (ii) & \mathcal{R}(Q_c) &=& \mathbb{R}^n. \\ (iii) & rank \;\; [B\; AB\; \cdots\; A^{n-1}B] &=& n. \end{array}$$

Note that if the system is controllable, then the state can be transferred from any state x_0 (not just 0) at $\tau = 0$ to any other state x_1 at time $\tau = t_1$. This is because if we want to transfer the state from x_0 to x_1, we can simply use the control which transfers the state from 0 to $z = x_1 - e^{At_1}x_0$. In fact, a control input that achieves the transfer is given by

$$u(\tau) = B^T e^{A^T(t_1-\tau)} W_c^{-1}(t_1)(x_1 - e^{At_1}x_0)\,.$$

You can verify for yourself that this control achieves the transfer.

3.2 Alternate Proof of Controllability

Controllability is a deep property of a control system. The proof in the previous section, while the most common one in the textbooks, has the shortcomings that it requires background on linear operators and adjoints and that it does not give much intuition about controllability. Also, it is not easily extended to the nonlinear setting. In this section, we give another proof that a system is controllable if and only if all states are reachable from the origin, in order to provide an alternate view and further insight.

Theorem 3.2.1. *$\mathcal{R}(L_c) = \mathbb{R}^n$ if and only if $rank(Q_c) = n$.*

Proof. (Necessity) We show that if $\mathcal{R}(L_c) = \mathbb{R}^n$ then $rank(Q_c) = n$. Suppose $rank(Q_c) < n$. Consider the expression

$$e^{-At_1}x_1 - x_0 = \int_0^{t_1} e^{-A\tau}Bu(\tau)d\tau\,.$$

By assumption there exists a non-zero vector $v \in \mathbb{R}^n$ such that $v^T Q_c = 0$. This implies

$$v^T B = 0\,, \quad v^T AB = 0\,, \quad \dots \qquad , v^T A^{n-1}B = 0\,.$$

By the Cayley-Hamilton theorem (see the Appendix), we obtain $v^T A^k B = 0$, for all $k = 0, 1, \dots$. It follows that $v^T e^{-A\tau}B = 0$ since $e^{-A\tau}B = B - AB\tau + A^2B\frac{\tau^2}{2!} + \cdots$. Therefore

$$v^T\left(e^{-At_1}x_1 - x_0\right) = \int_0^{t_1} v^T e^{-A\tau}Bu(\tau)d\tau = 0\,.$$

This means there is a constraint on x_0 and x_1. But x_0 and x_1 must be arbitrary because $\mathcal{R}(L_c) = \mathbb{R}^n$. Thus, we arrive at a contradiction.

(Sufficiency) We show that if $rank(Q_c) = n$ then $\mathcal{R}(L_c) = \mathbb{R}^n$. Suppose not. That is, suppose the map L_c is not onto. Equivalently, the linear map

$$e^{-At_1}L_c = \int_0^{t_1} e^{-A\tau}Bu(\tau)d\tau$$

is not onto. This means there is a non-zero vector $v \in \mathbb{R}^n$ such that

$$v^T \int_0^{t_1} e^{-A\tau}Bu(\tau)d\tau = 0\,.$$

Choose a control of the form $u(\tau) = (0, \dots, 0, u_i^s(\tau), 0, \dots, 0)$ where the ith component is

$$u_i^s(\tau) = \begin{cases} 1 & 0 \le \tau \le 2 \\ 0 & \tau > s \end{cases}$$

and $s \in \mathbb{R}$ is a parameter. Then we have

$$v^T \int_0^s e^{-A\tau}b_i d\tau = 0\,, \qquad i = 1, \dots, m$$

where b_i is the ith column of B. This expression holds for all $s \in \mathbb{R}$. This means

$$v^T e^{-As}b_i = 0\,, \qquad s \in \mathbb{R}, \qquad i = 1, \dots, m\,.$$

Differentiating this expression repeatedly with respect to s and setting $s = 0$ we obtain

$$v^T A^k b_i = 0\,, \qquad i = 1, \dots, m$$

for all $k = 0, 1, \dots$ Equivalently, $v^T Q_c = 0$, which contradicts $rank(Q_c) = n$. □

3.3 Invariance under Change of Basis

Recall that if x is a state vector, so is $T^{-1}x$ for any nonsingular matrix T. In fact, if we let $z = T^{-1}x$,

$$\begin{aligned} \dot{z} &= T^{-1}\dot{x} \\ &= T^{-1}ATz + T^{-1}Bu \end{aligned}$$

so that with z as the state vector, the system matrices change from (A, B) to $(T^{-1}AT, T^{-1}B)$. We refer to this as a change of basis because if we let the columns of T form a new basis for $\mathbb{R}^n$, z is the representation of x in this new basis.

Theorem 3.3.1. *(A, B) is controllable if and only if $(T^{-1}AT, T^{-1}B)$ is controllable for every nonsingular T.*

Proof. Consider the controllability matrix $\tilde{Q}_c$ for the pair $(T^{-1}AT, T^{-1}B)$:

$$\begin{aligned} \tilde{Q}_c &= [T^{-1}B \;\; T^{-1}ATT^{-1}B \cdots] \\ &= T^{-1}[B \;\; AB \cdots] \\ &= T^{-1}Q_c\,. \end{aligned}$$

Since $rank(T^{-1}Q_c) = rank(Q_c)$ the result is proved. □

If the pair (A, B) is not controllable, there is a particular basis in which the controllable and uncontrollable parts are displayed transparently. We illustrate the choice of basis and the computation involved with the following example.

Example 3.3.1.

For the two cart-one force system, if we take $M_1 = K = 1$, $M_2 = \frac{1}{2}$, we obtain the system matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \end{bmatrix}.$$

We know that this system is not controllable, and that the first two columns of the controllability matrix span $\mathcal{R}(Q_c)$. It is easily verified that we can take the following two vectors as a basis for $\mathcal{R}(Q_c)$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix}.$$

We complete this to a basis in $\mathbb{R}^4$ by augmenting with, say, the vectors

$$v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Define the matrix

$$T = \begin{bmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix}.$$

With respect to this basis, the state x is transformed to $z = T^{-1}x$. Since

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

the equation governing z is given by

$$\dot{z} = \tilde{A}z + \tilde{B}u$$

where

$$\begin{aligned} \tilde{A} &= T^{-1}AT \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} \end{aligned} \tag{3.1}$$

where $\tilde{A}_{11}$ is the upper left 2x2 block, $\tilde{A}_{12}$ the upper right 2x2 block, and $\tilde{A}_{22}$ the lower right 2x2 block.

$$\begin{aligned} \tilde{B} &= T^{-1}B \\ &= \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix}. \end{aligned} \tag{3.2}$$

Note that the pair $(\tilde{A}_{11}, \tilde{B}_1)$ is controllable because $rank(Q_c) = rank([\tilde{B}_1 \ \tilde{A}_{11}\tilde{B}_1 \ \tilde{A}_{11}^2\tilde{B}_1])$, while z_3 and z_4 corresponding to the $\tilde{A}_{22}$ block is uncontrollable, since they are decoupled from z_1 and z_2 and are unaffected by u.

In general, whenever Q_c is not full rank, we can find a basis so that in the new basis, A and B take the form given in (3.1) and (3.2), respectively, with $(\tilde{A}_{11}, \tilde{B}_1)$ controllable. The procedure is:

1. Find a basis for $\mathcal{R}(Q_c)$. Denote the vectors in this basis by $\{v_1, v_2, \cdots, v_k\}$.

2. Complete the basis to form a basis for $\mathbb{R}^n$. Define the matrix T to have as its columns the basis vectors $\{v_1, v_2, \cdots, v_n\}$.

3. Compute $\tilde{A} = T^{-1}AT$ and $\tilde{B} = T^{-1}B$. $\tilde{A}$ will take the form (3.1) and $\tilde{B}$ will take the form (3.2).

3.4 Invariance under State Feedback

A control law of the form

$$u(t) = Kx(t) + v(t)$$

with $v(t)$ a new input, is referred to as a *state feedback*. The closed-loop system equation is given by

$$\dot{x} = (A + BK)x(t) + Bv(t)$$

It is an important property that controllability is unaffected by state feedback.

Theorem 3.4.1. *(A, B) is controllable if and only if $(A + BK, B)$ is controllable for all K.*

A proof can be obtained using the PBH test. For details, refer to the problem sets.

3.5 Controllable Canonical Form

For single input systems, there is a special form of system matrices for which controllability always holds. This special form is referred to as the *controllable canonical form*. Using a lower case b to indicate explicitly that the input matrix is a column vector for a single input system, the controllable canonical form is given by

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & \cdots & 0 & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & & -\alpha_{n-1} \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

It is easy to verify that the controllability matrix for this pair (A, b) always has rank n, regardless of the values of the coefficients α_j; hence the name controllable canonical form. An A matrix taking the above form is referred to as a *companion form* matrix. It is straightforward to show that the characteristic polynomial of the companion form matrix is given by

$$\det(sI - A) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_0 .$$

It will be seen in the next chapter that when using pole assignment in single-input systems, the controllable canonical form is particularly convenient for control design. To prepare for that discussion, we have that

Theorem 3.5.1. *If (A, b) is controllable there exists a similarity transformation T such that $(T^{-1}AT, T^{-1}b)$ is in controllable canonical form.*

Proof. Consider the matrix

$$
\begin{aligned}
T &= [A^{n-1}b \ \ A^{n-2}b \ \ \cdots \ \ b]
\begin{bmatrix}
1 & 0 & \dots & & 0 \\
\alpha_{n-1} & & & & \\
\alpha_{n-2} & & & & \vdots \\
\vdots & & & & \\
\alpha_2 & & & & 0 \\
\alpha_1 & \alpha_2 & \dots & \alpha_{n-1} & 1
\end{bmatrix} \\
&= [b \ \cdots \ A^{n-1}b]
\begin{bmatrix}
\alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & 1 \\
\alpha_2 & \cdots & \alpha_{n-1} & 1 & 0 \\
\vdots & \iddots & 1 & 0 & 0 \\
\alpha_{n-1} & \iddots & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 & 0
\end{bmatrix} \\
&= \begin{bmatrix}
| & & | \\
v_1 & \cdots & v_n \\
| & & |
\end{bmatrix}.
\end{aligned}
$$

The second matrix (called a Toeplitz matrix) forming T is non-singular. The first matrix forming T is Q_c. Thus, controllability ensures T^{-1} exists, so that its columns $v_1, ..., v_n$ form a basis of $\mathbb{R}^n$.

Note that

$$
\begin{aligned}
v_1 &= A^{n-1}b + \alpha_{n-1}A^{n-2}b + ... + \alpha_1 b \\
v_2 &= A^{n-2}b + \alpha_{n-1}A^{n-3}b + \ldots + \alpha_2 b \\
&\vdots \\
v_{n-1} &= Ab + \alpha_{n-1}b \\
v_n &= b
\end{aligned}
$$

and that

$$
\begin{aligned}
Av_1 &= A^n b + ... + \alpha_1 Ab + \alpha_0 b - \alpha_0 b \\
&= -\alpha_0 b \qquad \text{by the Cayley-Hamilton Theorem} \\
&= -\alpha_0 v_n \\
Av_2 &= v_1 - \alpha_1 v_n \\
&\vdots \\
Av_n &= v_{n-1} - \alpha_{n-1} v_n
\end{aligned}
$$

Thus the matrix representation of A with respect to the basis $\{v_1 ... v_n\}$ looks like

$$
\tilde{A} = \begin{bmatrix}
0 & 1 & \cdots & 0 & 0 \\
\vdots & & & \ddots & \\
0 & 0 & \cdots & 0 & 1 \\
-\alpha_0 & -\alpha_1 & \cdots & & -\alpha_{n-1}
\end{bmatrix}
$$

Similarly, the vector b looks like

$$
\begin{bmatrix}
0 \\ 0 \\ \vdots \\ 0 \\ 1
\end{bmatrix} = \tilde{b}
$$

But $\tilde{A}$ and $\tilde{b}$ are then related to the original matrices through

$$\begin{aligned}\tilde{A} &= T^{-1}AT \\ \tilde{b} &= T^{-1}b\end{aligned}$$

so that they are related by a similarity transformation. Thus the new system $z(t) = T^{-1}x(t)$ will satisfy an equation of the form

$$\dot{z} = \tilde{A}z + \tilde{b}u$$

with $(\tilde{A}, \tilde{b})$ in controllable canonical form. □

3.6 PBH Test

There is a very useful test for controllability, referred to as the PBH test.

Theorem 3.6.1 (PBH)**.** *(A, B) is controllable if and only if rank$[A - \lambda I \quad B] = n$ for all eigenvalues λ of A.*

This theorem can be proved using the change of basis described above for uncontrollable systems. Details are provided in the problem sets. It is important to note that $Rank[A - \lambda I \quad B] = n$ for all eigenvalues λ of A if and only if $Rank[A - \lambda I \quad B] = n$ for all complex numbers λ. This is because for λ not an eigenvalue of A, $Rank(A - \lambda I) = n$.

Chapter 4

Pole Assignment for Linear Systems

In the previous chapters we examined the open loop response of a linear system and structural properties such as stability, and particularly controllability; later we will also study observability. In this chapter we study the design of control laws for regulating a linear system to the origin, called the *stabilization* problem. We confine ourselves to LTI systems of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ x(0) &= x_0 \end{aligned} \tag{4.1}$$

and we assume that the full state vector $x(t)$ is available for measurement. A natural control law for (4.1) is to use *linear state feedback* of the form

$$u = Kx(t)\,. \tag{4.2}$$

The closed-loop system under (4.2) is then

$$\dot{x} = (A + BK)x(t)\,, \tag{4.3}$$

and the closed-loop response is completely determined by $(A + BK)$, while the stability of the closed-loop system as well as the rate of regulation of x to zero is determined by the eigenvalues of $(A + BK)$, which are called the *poles* of the closed-loop system. In particular, the system (4.3) is asymptotically stable if and only if all eigenvalues of (A+BK) lie in $Re\ s < 0$. The immediate question that arises is: can we use state feedback to arbitrarily assign the eigenvalues of $A + BK$ for a given (A, B) pair? This problem of finding K to achieve a prescribed set of eigenvalues is called the *pole assignment problem*.

To facilitate the subsequent discussion, let us first formulate the problem in a more precise form. The eigenvalues of $A + BK$ are just the roots of the characteristic polynomial of $A + BK$, which we denote by $p_K(s) = \det(sI - A - BK)$. $p_K(s)$ is a monic polynomial of degree n with real coefficients. Specifying the poles of the closed-loop system to be $\lambda_1, ..., \lambda_n$ (where a complex λ_i is included if and only if its complex conjugate λ_i^* is also included) is equivalent to specifying the nth degree monic polynomial with real coefficients $r(s) = (s - \lambda_1)(s - \lambda_2)...(s - \lambda_n)$. The pole assignment problem can be formulated as follows.

Pole Assignment Problem *Given an nth degree monic polynomial with real coefficients $r(s)$, find a matrix K such that $p_K(s) = \det(sI - A - BK) = r(s)$.*

In this chapter we give necessary and sufficient conditions for the solvability of the pole assignment problem and give a constructive procedure for finding K.

4.1 Single-Input Systems

First consider the pole assignment problem for single-input systems of the form $\dot{x} = Ax + bu$. The control law is of the form

$$u = k^T x \tag{4.4}$$

for some column vector k, with the closed-loop system given by

$$\dot{x} = (A + bk^T)x\,. \tag{4.5}$$

The solution of this problem rests on two observations:

(i) The eigenvalues of $A + bk^T$ are invariant under similarity transformation, i.e., $\det(sI - A) = \det(sI - T^{-1}AT)$. Let $z(t) = T^{-1}x(t)$, T nonsingular. Then

$$\dot{z} = T^{-1}ATz + T^{-1}bu\,. \tag{4.6}$$

Suppose we solve the pole assignment problem for (4.6), i.e., for a given monic polynomial $r(s)$ whose roots appear in complex conjugate pairs, there exists a vector k_1 such that

$$\det(sI - T^{-1}AT - T^{-1}bk_1^T) = r(s)\,. \tag{4.7}$$

Then since

$$\det(sI - T^{-1}AT - T^{-1}bk_1^T) = \det(sI - A - bk_1^T T^{-1}) \tag{4.8}$$

we find that the pole assignment problem for (4.4) is solved by taking

$$k^T = k_1^T T^{-1}\,. \tag{4.9}$$

(ii) We know how to solve the pole assignment problem if (A, b) is in controllable canonical form,

$$A = A_c = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & \cdots & 0 & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & & -\alpha_{n-1} \end{bmatrix}, \qquad b = b_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

For,

$$A_c + b_c l^T = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ & & & & \\ & & & \ldots & 1 \\ l_0 - \alpha_0 & \ldots & & & l_{n-1} - \alpha_{n-1} \end{bmatrix}, \qquad l = \begin{bmatrix} l_0 \\ l_1 \\ \vdots \\ l_{n-1} \end{bmatrix}.$$

Thus

$$\det(sI - A_c - b_c l^T) = s^n + (\alpha_{n-1} - l_{n-1})s^{n-1} + \ldots + (\alpha_1 - l_1)s + (\alpha_0 - l_0)$$

which can be made into any monic nth degree $r(s)$ with real coefficients, $r(s) = s^n + r_{n-1}s^{n-1} + \ldots + r_0$ by simply choosing

$$l = \begin{bmatrix} \alpha_0 - r_0 \\ \alpha_1 - r_1 \\ \vdots \\ \alpha_{n-1} - r_{n-1} \end{bmatrix}. \tag{4.10}$$

We can now put (i) and (ii) together. We know from the previous chapter that if (A, b) is controllable, there exists a nonsingular transformation T such that $(T^{-1}AT, T^{-1}b) = (A_c, b_c)$ is in controllable canonical form. Hence we have the following result.

Proposition 4.1.1. *Assume (A, b) is controllable. Let the desired closed-loop characteristic polynomial be given by $r(s)$, a monic polynomial of nth degree with real coefficients. Then the following vector k solves the pole assignment problem:*

$$k^T = [(\alpha_0 - r_0) \quad (\alpha_1 - r_1)...(\alpha_{n-1} - r_{n-1})]T^{-1} \tag{4.11}$$

where

$$T = [A^{n-1}b \quad A^{n-2}b...b] \begin{bmatrix} 1 & 0 & & ... & 0 \\ \alpha_{n-1} & 1 & 0 & ... & 0 \\ \vdots & & & & \\ & & & & 0 \\ \alpha_1 & & ... & \alpha_{n-1} & 1 \end{bmatrix} \tag{4.12}$$

and α_i's are the coefficients of the characteristic polynomial of A, (i.e. $det(sI - A) = s^n + \alpha_{n-1}s^{n-1} + ... + \alpha_0$).

4.2 Multivariable Systems

The pole assignment problem for a single-input controllable system is relatively straightforward, but the pole assignment problem for multivariable systems is considerably harder. Interestingly, its solution relies on the solution of the single-input case. The generalization rests on the following result, which takes advantage of the invariance of controllability under state feedback.

Theorem 4.2.1. *Let (A, B) be controllable, and let $b_1, ..., b_m$ be the columns of the B matrix. For each i such that $b_i \neq 0$ there exists a $m \times n$ matrix K_i such that $(A + BK_i, b_i)$ is controllable.*

Proof. Without loss of generality, let $i = 1$. By controllability, the matrix

$$\tilde{Q}_c = [b_1 \quad Ab_1...A^{n-1}b_1 \quad b_2...A^{n-1}b_2...b_m...A^{n-1}b_m]$$

has rank n ($\tilde{Q}_c$ is obtained from Q_c by reordering its columns which does not affect the rank). We now look for the first n linearly independent columns in the matrix Q_c, giving rise to a matrix U of the form:

$$U = [b_1 \quad Ab_1...A^{\nu_1 - 1}b_1 \quad b_2 \quad Ab_2...A^{\nu_2 - 1}b_2...b_m \quad Ab_m...A^{\nu_m - 1}b_m]$$

in which some of the b_i's may be missing (the corresponding $\nu_i = 0$). By controllability

$$\sum_{i=1}^{m} \nu_i = n$$

We now associate the following matrix S with the above U matrix:

$$\begin{array}{lllllllllllll} U & = & [b_1 & ... & A^{\nu_1-1}b_1 & b_2... & A^{\nu_2-1}b_2 & \cdots & A^{\nu_{m-1}-1}b_{m-1} & b_m & ...A^{\nu_m-1}b_m &] \\ & & & \updownarrow & \updownarrow \ \nu_1 - \text{th column} & \updownarrow & \updownarrow \ (\nu_1+\nu_2)\text{th column} & \updownarrow & \updownarrow \ (\nu_1+\nu_2+\nu_{m-1})\text{th column} & \updownarrow & & \\ S & = & [0 & ...0 & e_2 & 0...0 & e_3 \ ... & 0 & e_m & 0 & ...0 &] \end{array}$$

where e_i is a $m \times 1$ vector with the only nonzero element being 1 in the ith position.

Let $K_1 = SU^{-1}$ or $K_1 U = S$. By the nonsingularity of U, such a K_1 is well-defined. We claim $(A+BK_1, b_1)$ is controllable. For,

$$\begin{array}{ccccc} K_1 b_1 = 0 & K_1 b_2 = 0 & & K_1 b_m = 0 \\ K_1 A b_1 = 0 & & & \\ \vdots & \vdots & \cdots & \vdots \\ K_1 A^{\nu_1 - 1} b_1 = e_2 & K_1 A^{\nu_2 - 1} b_2 = e_3 & & K_1 A^{\nu_m - 1} b_m = 0 \end{array}$$

Thus

$$\begin{array}{rcll} b_1 & = & b_1 & \\ (A+BK_1)b_1 & = & Ab_1 & \\ (A+BK_1)^{\nu_1} b_1 & = & (A+BK_1)A^{\nu_1 - 1} b_1 & = A^{\nu_1} b_1 + Be_2 = A^{\nu_1} b_1 + b_2 \\ & & & = b_2 + \text{lin. comb. of prev. columns} \\ (A+BK_1)^{\nu_1 + 1} b_1 & = & (A+BK_1)(A^{\nu_1} b_1 + b_2) & = Ab_2 + \text{lin. comb. of prev. columns} \\ (A+BK_1)^{\nu_1 + \ldots + \nu_m - 1} b_1 & = & A^{\nu_m - 1} b_m + \text{lin. comb. of} & \text{prev. columns} \end{array}$$

Hence all the above vectors are linearly independent by definition of the ν_i's, so that $[b_1 \ \ (A+BK_1)b_1 ... (A+BK_1)^{n-1} b_1]$ is nonsingular. We conclude that

$$\dot{x} = (A+BK_1)x + b_1 v \tag{4.13}$$

is controllable. □

The pole assignment problem for the linear multivariable case has now been reduced to the single-input case. The procedure is the following. Let $\dot{x} = Ax + Bu$ be controllable with $b_1 \neq 0$. Construct K_1 such that (4.13) is controllable. Find k_1 such that with $v = k_1^T x$, (4.13) has the pre-assigned poles. The closed-loop system is then

$$\begin{aligned} \dot{x} &= (A + BK_1 + b_1 k_1^T)x \\ &= [A + B(K_1 + e_1 k_1^T)]x \end{aligned} \tag{4.15}$$

(4.15) may thus be obtained by letting

$$u = Kx = (K_1 + e_1 k_1^T)x\,. \tag{4.16}$$

4.3 Pole Assignability Implies Controllability

We have proved that controllability is sufficient for pole assignability. We now prove the converse so that controllability is also necessary for pole assignability. Let λ_i, $i = 1, ..., n$ be a set of n distinct real numbers, none of which is an eigenvalue of A. By the assumption of pole assignability, there exists a K such that

$$(A+BK)x_i = \lambda_i x_i$$

where x_i are the eigenvectors associated with λ_i. Thus

$$(\lambda_i I - A)^{-1} BK x_i = x_i \qquad i = 1, ..., n$$

But $(\lambda_i I - A)^{-1} = \sum_{j=0}^{n-1} \rho_j(\lambda_i) A^j$ for some suitable functions ρ_j, $j = 0, ..., n-1$. Hence

$$\sum_{j=0}^{n-1} \rho_j(\lambda_i) A^j BK x_i = x_i \qquad i = 1, ..., n$$

The left hand side are elements in the range space of $[B \ \ AB ... A^{n-1}B]$. By linear independence of x_i, we conclude $\mathcal{R}[B \ \ AB ... A^{n-1}B] = n$, i.e. (A, B) is controllable. Combining the above results, we have

Theorem 4.3.1 (Pole Assigment)**.** *The closed-loop poles of (4.1) can be arbitrarily assigned if and only if (A, B) is controllable.*

4.4 Stabilizability

The results of the previous sections show that controllability is equivalent to pole assignability. In applications only a portion of the system may be controllable. We would like to see how this affects the ability to assign poles of the closed-loop system. Recall from before that there exists a nonsingular matrix T such that by letting $z = T^{-1}x$, we can transform the system into

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix} u \tag{4.17}$$

We see that the z_2 component is decoupled from z_1 and we have no control whatsoever over z_2. Hence the eigenvalues of $\tilde{A}_{22}$ will be unchanged regardless of what feedback law we choose. On the other hand, since $(\tilde{A}_{11}, \tilde{B}_1)$ is controllable, by suitable choice of feedback

$$u = \tilde{K}z = \tilde{K}_1 z_1 + \tilde{K}_2 z_2$$

we can make $\tilde{A}_{11} + \tilde{B}_1\tilde{K}_1$ in the closed-loop system

$$\dot{z} = \begin{bmatrix} \tilde{A}_{11} + \tilde{B}_1\tilde{K}_1 & \tilde{A}_{12} + \tilde{B}_1\tilde{K}_2 \\ 0 & \tilde{A}_{22} \end{bmatrix} z$$

have any desired set of eigenvalues. Thus we see that we can, by feedback, modify at will q poles of the closed-loop system, corresponding to those in $\tilde{A}_{11}$, but $n - q$ poles, corresponding to those in $\tilde{A}_{22}$, will remain fixed. The eigenvalues which can be modified by feedback are called *controllable eigenvalues.*

These considerations suggest the following definition, which describes a weaker property than pole assignability.

Definition 4.4.1. The system (4.1) is said to be *stabilizable* if there exists a matrix K such that for $u = Kx$, the closed-loop system (4.3) is stable.

It is evident from the decomposition (4.17) that a system is stabilizable if and only if $\tilde{A}_{22}$ is (asymptotically) stable. Informally, we have that: *system (4.1) is stabilizable if and only if all the unstable modes are controllable.* Finally, a nice test for stabilizability based on the PBH test is discussed in the problem sets.

Chapter 5

Observability, Observers, and Feedback Compensators

In the previous chapter we studied the design of state feedback laws using pole assignment. Such control laws require the state to be measured. For many systems, we may only get partial information about the state through the measured output. In this chapter, we shall study the property of observability and show that whenever the system is observable, we can estimate the state accurately using an observer. The state estimate can then be used to design feedback compensators.

5.1 Observability

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0 \tag{5.1}$$
$$y(t) = Cx(t) \tag{5.2}$$

where $y \in \mathbb{R}^p$ is the output.

We assume that the only signals that are measured are the inputs $u(t)$ and outputs $y(t)$. Since the state plays such an important role in control design, we want to know whether we can determine the state $x(t)$ from the input-output measurements on $[0\ t]$. This motivates the following definition.

Definition 5.1.1. The system (5.1), (5.2) is said to be *observable* at time t if the state $x(t)$ can be determined from $u(s), y(s)\,, \quad 0 \leq s \leq t$.

We can include a term $Du(t)$ on the right hand side of (5.2). However, since we can consider $y(t) - Du(t)$ as a new known measurement, there is no loss of generality in assuming that $D = 0$. Since $x(t)$ can be determined once the initial state x_0 and the inputs are known, the state determination problem is equivalent to finding the initial state from input-output measurements. The system will not be observable if the initial state cannot be determined from input-output measurements.

Definition 5.1.2. Two initial state vectors ξ and η are said to be *indistinguishable* at t if for the same input $u(s)$, $0 \leq s \leq t$, the outputs corresponding to ξ and η are the same.

The system (5.1), (5.2) is observable if and only if there are no initial state vectors which are indistinguishable from each other. Now fix t. We define the *observability Gramian* W_o to be

$$W_o = \int_0^t e^{A^T\tau} C^T C e^{A\tau} d\tau\,. \tag{5.3}$$

Theorem 5.1.1 (Observability)**.** *The system (5.1), (5.2) is observable at time t if and only if the observability Gramian W_o is nonsingular.*

Proof. First assume that W_o is nonsingular. The output is given by

$$y(t) \quad = \quad Ce^{At}x_0 + \int_0^t Ce^{A(t-s)}Bu(s)ds\,.$$

Let

$$\tilde{y}(t) = y(t) - \int_0^t Ce^{A(t-s)}Bu(s)ds\,.$$

Then

$$Ce^{A\tau}x_0 = \tilde{y}(\tau)\,, \qquad 0 \le \tau \le t\,.$$

Multiplying both sides of (5.1) by $e^{A^T\tau}C^T$ and integrate from 0 to t gives

$$W_o x_0 = \int_0^t e^{A^T\tau}C^T\tilde{y}(\tau)d\tau\,. \tag{5.4}$$

Since W_o is nonsingular, we can invert (5.4) to obtain x_0:

$$x_0 = W_o^{-1}\int_0^t e^{A^T\tau}C^T\tilde{y}(\tau)d\tau\,.$$

Now suppose W_o is singular. There exists a nonzero vector v such that $W_o v = 0$. This in turn implies that $v^T W_o v = 0$, from which we find

$$Ce^{A\tau}v = 0\,, \qquad 0 \le \tau \le t\,.$$

This means that for the input $u = 0$, the vectors v and 0 are indistinguishable initial conditions, both giving rise to an output $y = 0$. Hence the system is not observable. □

Since observability of the system depends only on the pair (C, A), we shall also say (C, A) is observable. On comparing the observability Gramian for the pair (C, A) and the controllability Gramian for the pair (A, B), we see that they are very similar in form. In particular, if we make the correspondence

$$\begin{array}{ccc} A^T & \longleftrightarrow & A \\ C^T & \longleftrightarrow & B \end{array}$$

then we have changed the observability Gramian into the controllability Gramian. Using this correspondence, the following result is immediate.

Theorem 5.1.2 (Duality Theorem)**.** *(C, A) is observable if and only if (A^T, C^T) is controllable.*

Let us define the observability matrix Q_o as

$$Q_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

From the duality theorem, we can immediately deduce the following algebraic criterion for observability: *(C, A) is observable if and only if $Rank(Q_o) = n$.* Equivalently, (C, A) is observable if and only if $\mathcal{N}(Q_o) = \{0\}$.

5.2 Full State Observers

In the previous chapter, we saw how stabilizing control laws can be designed using pole placement whenever the state is available. For many systems, the state is not available for measurement. We know that for the system (5.1), (5.2), the initial state, and hence the state trajectory, can in principle be determined if (C, A) is observable. However, the procedure involves integration and inversion of a matrix which is ill-conditioned. An alternative, more robust and practical approach to estimate the state is desired.

A natural approach for designing a state estimator is suggested by the following idea. If we build a duplicate of system (5.1), in general, as the initial conditions would not match, the outputs of the two systems would be different, but the error might be used as a feedback signal to improve the state estimation. Specifically, we seek an estimator of the form

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(y - \hat{y})\,, \qquad \hat{x}(0) = \hat{x}_0 \\ \hat{y} &= C\hat{x} \end{aligned} \tag{5.5}$$

where $\hat{x}_0$ can be arbitrarily chosen (often taken to be 0). The error in the state estimate, defined as $e = x - \hat{x}$, is governed by the equation

$$\dot{e} = (A - LC)e\,, \qquad e(0) = x_0 - \hat{x}_0\,. \tag{5.6}$$

Observe that if the eigenvalues of $A - LC$ all lie in the left half plane, then regardless of $e(0)$, $e(t) \to 0$ exponentially and the goal of accurate state estimation is achieved. The question of the speed of convergence of $\hat{x}$ to x is precisely the dual of the pole assignment problem.

Theorem 5.2.1. *There exists an $n \times p$ matrix L such that $det(sI - A + LC) = r(s)$, where $r(s)$ is any nth degree monic polynomial with real coefficients, if and only if (5.1), (5.2) is observable.*

Proof. By duality, (C, A) is observable if and only if (A^T, C^T) is controllable. By the pole assignment theorem, this is equivalent to the existence of a matrix L^T such that $det(sI - A^T + C^T L^T)$ is any pre-assigned monic polynomial. Since $det(sI - A^T + C^T L^T) = det(sI - A + LC)$, the result follows. □

The state estimator (5.5) is called a *full state observer*. Its dimension is equal to n. However it is not hard to see that there is redundancy in the observer design. Indeed, the output y measures exactly a part of the state so that we really only need to estimate the remaining part. This leads to the idea of *minimal order observers* or *reduced order observers*.

5.3 Minimal Order Observers

We assume that the matrix C is of full rank p and that a basis has been chosen so that

$$C = [I_p \ \ 0]\,, \qquad I_p \text{ the } p \times p \text{ identity matrix}$$

There is no loss of generality in making these assumptions. Partition $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ so that $y = x_1$. Partition A and B correspondingly. Equation (5.1) can then be written as

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + B_1 u \\ \dot{x}_2 &= A_{22}x_2 + A_{21}x_1 + B_2 u\,. \end{aligned}$$

Then we have

Lemma 5.3.1. *The system $\dot{x} = Ax + Bu$, $y = Cx$ is observable if and only if $\dot{x}_2 = A_{22}x_2 + v$, $w = A_{12}x_2$ is observable.*

Proof. Since x_1 and u are known exactly, we may write (5.1) as

$$\underbrace{\dot{x}_1 - A_{11}x_1 - B_1u}_{w} = A_{12}x_2 \tag{5.7}$$

$$\dot{x}_2 = A_{22}x_2 + \underbrace{A_{21}x_1 + B_2u}_{v}. \tag{5.8}$$

The lemma now follows from the definition of observability. □

The same result can be proved more formally using the PBH test.

To estimate x_2, we imitate the development in the previous section to write

$$\dot{\hat{x}}_2 = A_{22}\hat{x}_2 + A_{21}x_1 + B_2u + L(A_{12}x_2 - A_{12}\hat{x}_2). \tag{5.9}$$

Then the estimation error $e_2 = x_2 - \hat{x}_2$ will satisfy

$$\dot{e}_2 = (A_{22} - LA_{12})e_2.$$

By observability, the eigenvalues of $A_{22} - LA_{12}$ may be pre-assigned. Notice that (5.9) on the surface would have to be implemented in the form

$$\dot{\hat{x}}_2 = A_{22}\hat{x}_2 + A_{21}x_1 + B_2u + L(\dot{x}_1 - A_{11}x_1 - B_1u - A_{12}\hat{x}_2)$$

which calls for differentiating $x_1 = y$. This problem is removed by writing

$$z = \hat{x}_2 - Lx_1.$$

Then z satisfies the equation

$$\dot{z} = (A_{22} - LA_{12})z + (A_{22} - LA_{12})Ly + (A_{21} - LA_{11})y + (B_2 - LB_1)u.$$

The state estimate is then given by

$$\hat{x} = \begin{bmatrix} y \\ z + Ly \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ L & I_{n-p} \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}.$$

5.4 Feedback Compensation

Suppose we would like to use feedback to control system (5.1). A natural approach is to separate the task of state estimation and control: first estimate the state using an observer, then use the state estimate, instead of the actual state, in the feedback controller. This suggests the closed loop control law given by

$$u = K\hat{x} \tag{5.10}$$

where

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \tag{5.11}$$

in the full observer case. The closed loop system is given by

$$\frac{d}{dt}\begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} A + BK & -BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \tag{5.12}$$

Note that the poles of the closed loop system are exactly the union of the eigenvalues of $A + BK$ and $A - LC$, which may be pre-assigned if controllability of (A, B) and observability of (C, A) holds. This fact is called the *separation principle* and it is one of the most useful results of linear system theory. In the case of the minimal order observer, we have

$$\dot{z} \;=\; Fz \;+\; Gu \;+\; Hy$$

$$\hat{x} = M \begin{bmatrix} y \\ z \end{bmatrix}$$

with

$$\hat{x}_1 = y = x_1$$

Again, we have

$$\dot{e}_2 \;=\; Fe_2$$

where the eigenvalues of F may be pre-assigned if (C, A) is observable. Hence, if we put $u = K\hat{x}$, we find, on partitioning $K \;=\; [K_1 \;\; K_2]$

$$u = K_1\hat{x}_1 + K_2\hat{x}_2 = K_1x_1 + K_2\hat{x}_2$$

$$= Kx - K_2e_2$$

The closed loop system is then

$$\dot{x} \;=\; (A + BK)x \;-\; BK_2e_2$$

$$\dot{e}_2 \;=\; Fe_2$$

Again, the closed loop system may take any pre-assigned set of poles provided (A, B) is controllable and (C, A) is observable.

Note that the controller is of the output feedback form. On substituting the control law (5.10) into (5.11), we obtain

$$\dot{\hat{x}} = (A + BK - LC)\hat{x} + Ly\,. \tag{5.13}$$

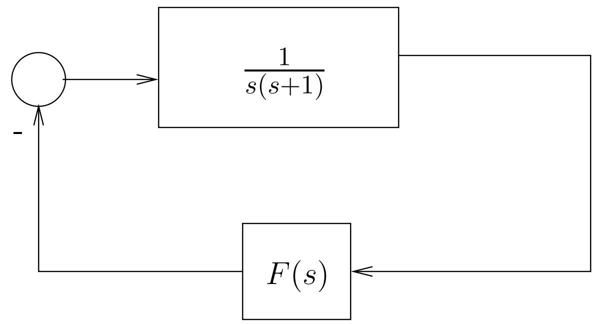
So the control law (5.10) can be described as the output of the system (5.13) driven by the measured system output y.

Example 5.4.1.

As an example, consider the problem of designing a compensator for the system with transfer function $\frac{1}{s(s+1)}$, so that all states of the closed loop system are regulated to zero with transients decaying at least at the rate of e^{-2t}. We would like to accomplish this using the feedback compensator design with a minimal order observer

Since we require a decay rate of at least e^{-2t}, the closed loop poles must have real part ≤ -2. Also, the dimension of the minimal order observer is 1, so that the closed loop characteristic polynomial must have degree 3 and whose roots must have real part ≤ -2. To be specific, let us choose the desired closed loop characteristic polynomial $r(s)$ to be $(s + 2)^3$. This will now be the compensator design specification.

The design is accomplished by the following steps.



1. State space realization of the open loop transfer function: To simplify the design procedure, we can choose a convenient realization such as the controllable canonical form. This gives the following realization for the transfer function $\frac{1}{s(s+1)}$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0]x$$

2. State feedback design: We design a feedback law so that if the state were actually available for measurement, the design specifications would be met. In this case, we want to find a vector k so that the control law $u = k^T x$ results in two closed loop poles at -2. Thus k should be chosen to achieve

$$det(sI - A - bk^T) = r_1(s) = (s+2)^2 = s_2 + 4s + 4$$

Since the system is in controllable canonical form, k is obtained immediately

$$k = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$$

3. Minimal order observer design: Since x_2 is not available for measurement, we design an observer to estimate x_2. The dynamics of the observer should given an error system with a pole at -2 to meet the design specifications. Now (1) may be written more explicitly as

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_2 + u$$

Thus the minimal order observer is

$$\begin{aligned} \dot{\hat{x}}_2 &= -\hat{x}_2 + u + L(\dot{x}_1 - \hat{x}_2) \\ &= -(1+L)\hat{x}_2 + u + L\dot{x}_1 \end{aligned}$$

This determines $L = 1$. Defining $z = \hat{x}_2 - x_1 = \hat{x}_2 - y$, we get

$$\dot{z} = -2\hat{x}_2 + u = -2z - 2y + u \tag{5.14}$$

$$\hat{x}_2 = z + y \tag{5.15}$$

4. Compensator design: On putting parts (2) and (3) together, we obtain the following controller

$$u = -4y - 3\hat{x}_2 = -7y - 3z$$

with z satisfying (5.14).

The compensator transfer function $F(s)$ can be obtained as follows. From (5.14) and (4),

$$s\hat{z}(s) \;=\; -5\hat{z}(s) \;-\; 9\hat{y}(s)$$

so that

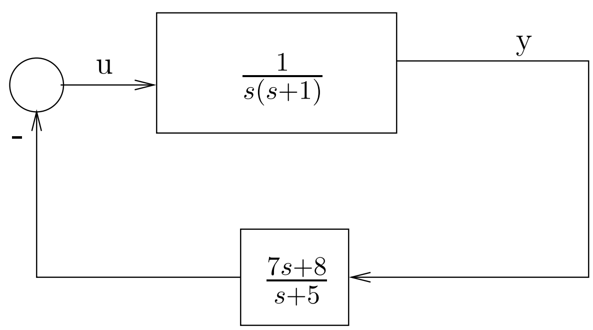
$$\frac{\hat{z}(s)}{\hat{y}(s)} \;=\; -\,\frac{9}{s+5}$$

Hence

$$\hat{u}(s) \;=\; -F(s)\hat{y}(s) \;=\; -7\hat{y}(s) \;+\; \frac{27}{s+5}\,\hat{y}(s)$$

$$= \;-\,\frac{7s+8}{s+5}\,\hat{y}(s)$$

The closed loop system is given by



with transfer function

$$\frac{\frac{1}{s(s+1)}}{1+\frac{1}{s(s+1)}\frac{7s+8}{s+5}} \;=\; \frac{s+5}{s^3+6s^2+12s+8} \;=\; \frac{s+5}{(s+2)^3}$$

5.5 Detectability

There is a property for (C, A) which is analogous to the property of stabilizability for (A, B).

Definition 5.5.1. A pair (C, A) is said to be *detectable* if there exists a matrix L such that $A - LC$ is stable.

Since the eigenvalues of $A - LC$ are the same as those of $(A - LC)^T = A^T - C^T L^T$, we immediately see that (C, A) is detectable if and only if (A^T, C^T) is stabilizable. Thus all properties about detectability can be inferred from those of stabilizability. In particular, to ensure that the state estimation error converges to 0, we really only need detectability of (C, A). However, we would then not be able to guarantee that the error goes to 0 at a certain rate.

5.6 Observable Canonical Form

The pair (C, A) is said to be in *observable canonical form* iff (A^T, C^T) is in controllable canonical form. Suppose we have a single output system which is complete observable. Then we can apply the coordinate transformation of Theorem 3.5.1 to put (A^T, c^T) in controllable canonical form. The result is observable canonical form given by

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\alpha_0 \\ 1 & 0 & \cdots & 0 & -\alpha_1 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & -\alpha_{n-1} \, . \end{bmatrix}$$

$$c = [0 \ \cdots \ 0 \ 1] \ .$$

It is easy to verify that the observability matrix for this pair (c, A) always has rank n, regardless of the values of the coefficients α_j. This form does not have the usefulness of controllable canonical form, so it is not further discussed.

5.7 Kalman Decomposition

A system can be decomposed both with respect to controllability and observability, and this is called a *Kalman decomposition.* Suppose we have determined that neither $\mathcal{R}(Q_c)$ nor $\mathcal{N}(Q_o)$ are trivial subspaces. We define a set of subspaces as follows:

1. Define $\mathcal{X}_2 = \mathcal{R}(Q_c) \cap \mathcal{N}(Q_o)$, the controllable, unobservable subspace.
2. Set $\mathcal{X}_1$ to be the complementary subspace of $\mathcal{X}_2$ in $\mathcal{R}(Q_c)$; that is, $\mathcal{X}_1 \oplus \mathcal{X}_2 = \mathcal{R}(Q_c)$.
3. Set $\mathcal{X}_4$ to be the complementary subspace of $\mathcal{X}_2$ in $\mathcal{N}(Q_o)$; that is, $\mathcal{X}_4 \oplus \mathcal{X}_2 = \mathcal{N}(Q_o)$.
4. Let $\mathcal{X}_3$ be a complement of $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3$; that is, $\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \mathcal{X}_3 \oplus \mathcal{X}_4 = \mathbb{R}^n$.

Let T be the matrix whose columns are the basis vectors of these subspaces (in numerical order). Then

$$\overline{A} = T^{-1}AT, \qquad \overline{B} = T^{-1}B, \qquad \overline{C} = CT$$

gives

$$\overline{A} = \begin{bmatrix} A_{co} & 0 & A_{13} & 0 \\ A_{21} & A_{c\overline{o}} & A_{23} & A_{24} \\ 0 & 0 & A_{\overline{c}o} & 0 \\ 0 & 0 & A_{43} & A_{\overline{co}} \end{bmatrix}, \qquad \overline{B} = \begin{bmatrix} B_{co} \\ B_{c\overline{o}} \\ 0 \\ 0 \end{bmatrix}.$$

$$\overline{C} = [C_{co} \ 0 \ C_{\overline{c}o} \ 0] \ .$$

Lemma 5.7.1. *The system*

$$\dot{x} = Ax + Bu \, , \quad y = Cx + Du$$

and the system

$$\dot{z} = A_{co}z + B_{co}u \, , \quad y = C_{co}z + Du$$

have the same transfer matrix.

The system $(A_{co}, B_{co}, C_{co}, D)$ has the minimum state dimension of all input-output equivalent state models. For that reason it is called the *minimal realization* of the transfer matrix. The dimension of this system is $\dim(\mathcal{R}(Q_c)) - \dim(\mathcal{R}(Q_c) \cap \mathcal{N}(Q_o))$.

Chapter 6

Linear Quadratic Optimal Control

In this chapter, we study a different control design methodology, based on optimization. Control design objectives are formulated in terms of a cost criterion. An optimal control law is one that minimizes the cost criterion. One of the most successful results of linear control theory is that if the cost criterion is quadratic and the optimization is over an infinite horizon, the resulting optimal control law is a linear feedback with many nice properties, including closed loop stability. These results are intimately connected to the system theoretic properties of stabilizability and detectability.

6.1 Optimal Control Problem

Consider the linear system in state space form

$$\dot{x} = Ax + Bu\,, \qquad x(0) = x_0\,. \tag{6.1}$$

We define the class of admissible controls $\mathcal{U}$ to be of the form $u = \phi(t)$ such that the following conditions are satisfied: (i) ϕ is a continuous function, (ii) the closed loop system has a unique solution, and (iii) the closed loop system results in $\lim_{t\to\infty} x(t) = 0$. The control objective is to find, in the class of admissible controls a control law that minimizes the *cost function*

$$J(x_0, \phi) = \int_0^\infty \left[x^T(t)Qx(t) + \phi^T(t)R\phi(t)\right] dt\,, \tag{6.2}$$

where Q is a symmetric positive semidefinite matrix and R is a symmetric positive definite matrix. We have indicated explicitly the dependence of the cost criterion on the initial condition and the choice of control law. We refer to the control problem as the *linear quadratic optimal control problem* and the control law which solves this optimization problem is the *optimal control*.

We can interpret the cost criterion as follows. Since Q is positive semidefinite, $x^T(t)Qx(t) \geq 0$ and represents the penalty incurred at time t for state trajectories that deviate from 0. Similarly, since R is positive definite, $\phi^T(t)R\phi(t) > 0$ unless $\phi(t) = 0$. It represents the control effort at time t in trying to regulate $x(t)$ to 0. The entire cost criterion reflects the cumulative penalty incurred over the infinite horizon. The admissible control requirement (iii) ensures that state regulation occurs as $t \to \infty$. The choice of the weighting matrices Q and R reflects the tradeoff between the requirements of regulating the state to 0 and minimizing the control effort. For example, a diagonal matrix

$$Q = \begin{bmatrix} q_1 & 0 & 0 & \cdots & 0 \\ 0 & q_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & q_n \end{bmatrix}$$

gives the quadratic form

$$x^T(t)Qx(t) = \sum_{i=1}^{n} q_i x_i^2(t)\,.$$

The relative magnitudes of q_i's indicate the relative importance that the designer attaches to $x_i(t)$ being away from 0.

To ensure the control problem is well-posed, we make the following standing assumption throughout this chapter: (A, B) *is stabilizable*. By stabilizability, there exists a feedback gain K such that the closed loop system

$$\dot{x} = (A + BK)x \tag{6.3}$$

is stable. The feedback law $u = Kx$ is clearly admissible. The solution of (6.3) is $x(t) = e^{(A+BK)t}x_0$, satisfying condition (ii). The cost function is given by

$$J(x_0, Kx) = x_0^T \int_0^\infty e^{(A+BK)^T t}(Q + K^T RK)e^{(A+BK)t} dt x_0$$

which is finite. Hence the optimization problem is well-posed.

6.2 Dynamic Programming

To solve the optimal control problem, we use *dynamic programming*. Define the *instantaneous cost*

$$L(x, u) = x^T Qx + u^T Ru\,.$$

For the initial state $x_0 = x$, define the optimal cost or *value function*

$$V(x) = \inf_{\phi \in \mathcal{U}} J(x, \phi)$$

where inf denotes infimum or the greatest lower bound. If the minimum is achieved using some control law ϕ, the infimum is then actually the minimum. Note that we have used the variable x as the argument. We shall derive a differential equation for $V(x)$.

We argue intuitively as follows. Suppose we consider the control as being applied first over an interval $[0\ \tau]$ and then over $[\tau\ \infty]$. Let $u(t), 0 \le t \le \tau$ be the control applied over $[0\ \tau]$, leading to the state $x(\tau)$ at time τ. It is intuitively clear that in order for the system to have optimal behaviour over $[0\ \infty)$, regardless of what its behaviour over $[0\ \tau]$ is, the system must behave optimally from τ onwards. Such optimal behaviour over $[\tau\ \infty)$ gives the cost $V(x(\tau))$, so that the total cost is given by

$$J = \int_0^\tau L(x(t), u(t))dt + V(x(\tau))\,.$$

Since $u(t)$ is arbitrary, the optimal cost satisfies the equation

$$V(x) = \min_{u(t), 0 \le t \le \tau} \left[\int_0^\tau L(x(t), u(t))dt + V(x(\tau)) \right]. \tag{6.4}$$

For small τ, we can perform an expansion of (6.4) in terms of τ. We use the notation $o(\tau)$ to denote a quantity that has the property $\frac{o(\tau)}{\tau} \to 0$ as $\tau \to 0$. First we approximate the integral by

$$\int_0^\tau L(x(t), u(t))dt = \tau L(x, u) + o(\tau)\,.$$

Then we obtain

$$V(x(\tau)) = V(x) + \tau \frac{\partial V}{\partial x}(x)(Ax + Bu) + o(\tau)\,,$$

where $\frac{\partial V}{\partial x}$ is the gradient of V with respect to x (a $1 \times n$ row vector). Substituting into (6.4), we obtain the *Hamilton Jacobi Bellman* (HJB) equation for V: satisfied by $V(x)$

$$\min_{u \in \mathbb{R}^m} \left\{ \frac{\partial V}{\partial x}(x)(Ax + Bu) + L(x, u) \right\} = 0\,. \tag{6.5}$$

To determine the minimizing element u in (6.5), we observe that, if $R > 0$, we can complete the square in the following quadratic form

$$u^T Ru + 2\alpha^T u + \beta = (u + R^{-1}\alpha)^T R(u + R^{-1}\alpha) + \beta - \alpha^T R^{-1}\alpha$$

so that

$$\min_u (u^T Ru + 2\alpha^T u + \beta) = \beta - \alpha^T R^{-1}\alpha$$

with the minimizing u given by

$$u = -R^{-1}\alpha\,.$$

Using these results in (6.5), we see that the minimizing u for (6.5) is

$$u = -\frac{1}{2} R^{-1} B^T \frac{\partial V}{\partial x}(x)\,, \tag{6.6}$$

resulting in the equation

$$\frac{\partial V}{\partial x}^T (x)(Ax) + x^T Qx - \frac{1}{4}\frac{\partial V}{\partial x}(x) BR^{-1}B^T \frac{\partial V}{\partial x}^T (x) = 0\,. \tag{6.7}$$

To solve (6.7), we use the trial solution

$$V(x) = x^T Px$$

for some $P \geq 0$. Note that

$$\begin{aligned}
\frac{\partial}{\partial x_k}(x^T Px) &= \frac{\partial}{\partial x_k} \sum_{i,j=1}^n x_i P_{ij} x_j \\
&= \sum_j P_{kj} x_j + \sum_i x_i P_{ik} \\
&= (Px)_k + (P^T x)_k \\
&= 2(Px)_k\,.
\end{aligned}$$

Substituting into (6.7), we get

$$x^T (A^T P + PA - PBR^{-1}B^T P + Q)x = 0\,.$$

Since this is true for all x, P must satisfy the matrix quadratic equation

$$A^T P + PA - PBR^{-1}B^T P + Q = 0\,. \tag{6.8}$$

Equation (6.8) is called the *algebraic Riccati equation*, and is one of the most famous equations in linear control theory.

In terms of P, the minimizing u would be given by

$$u = -R^{-1}B^T P x \tag{6.9}$$

so that this would be our candidate for the optimal control law. To establish that this is indeed the optimal control law, we have to verify that the control law (6.9) is admissible, and that $J(x_0, \phi)$ is minimized using $\phi(x(t)) = -R^{-1}B^T Px(t)$. We make use of the following fundamental result, whose proof may be found in *W.M. Wonham, Linear Multivariable Control: A Geometric Approach.*

Theorem 6.2.1 (Riccati)**.** *Assume (A, B) is stabilizable, and $(\sqrt{Q}, A)$ is detectable. Then there exists a unique solution P, in the class of positive semidefinite matrices, to the algebraic Riccati equation (6.8). Furthermore, the closed-loop system matrix $A - BR^{-1}B^T P$ is stable.*

Armed with this theorem, we see immediately that if (A, B) is stabilizable, and $(\sqrt{Q}, A)$ is detectable, the control law (6.9) will be admissible as it is stabilizing. We now verify that it is optimal, again by completion of squares. For any admissible u,

$$\begin{aligned}
J(x_0, u) &= \int_0^\infty \left[x^T(t)Qx(t) + u^T(t)Ru(t)\right] dt \\
&= \int_0^\infty \left[x^T(t)PBR^{-1}B^T Px(t) + u^T(t)Ru(t) - x^T(t)(A^T P + PA)x(t)\right] dt \\
&= \int_0^\infty \left[(u(t) + R^{-1}B^T Px(t)]^T R[(u(t) + R^{-1}B^T Px(t)\right] dt \\
&\quad - \int_0^\infty \left[u^T(t)B^T Px(t) + x^T(t)PBu(t) + x^T(t)A^T Px(t) + x^T(t)PAx(t)\right] dt \\
&= \int_0^\infty \left[[(u(t) + R^{-1}B^T Px(t)]^T R[(u(t) + R^{-1}B^T Px(t)] - \dot{x}^T(t)Px(t) - x^T(t)P\dot{x}(t)\right] dt \\
&= \int_0^\infty [(u(t) + R^{-1}B^T Px(t)]^T R[(u(t) + R^{-1}B^T Px(t)] - \int_0^\infty \frac{d}{dt}(x^T(t)Px(t))dt \\
&= x_0^T Px_0 + \int_0^\infty [(u(t) + R^{-1}B^T Px(t)]^T R[(u(t) + R^{-1}B^T Px(t)]dt\,.
\end{aligned}$$

Since $x_0^T Px_0$ is a constant unaffected by choice of u, and since $u = -R^{-1}B^T Px$ is admissible and $R > 0$, it is clear that the optimal control law is indeed given by

$$u(t) = -R^{-1}B^T Px(t)$$

with the optimal cost given by

$$V(x) = x^T Px\,. \tag{6.10}$$

Theorem 6.2.1 is a deep result. The algebraic Riccati equation is a quadratic matrix equation. As such, it may have no positive semidefinite solutions, or even real solutions. It may also have an infinite number of solutions. However, under verifiable system theoretic properties of stabilizability and detectability, a unique positive semidefinite solution is guaranteed and it also gives the optimal control. We therefore have a complete solution to the optimal control problem.

Note that the conditions of stabilizability and detectability are sufficient conditions for the existence and uniqueness of solution of (6.8). Stabilizability is clearly necessary for the solution of the optimal control problem, as has already been mentioned. Without stabilizability, the class of admissible control laws would be empty. It is of interest to examine what can happen if detectability fails. We illustrate with several examples.

Example 6.2.1.

Consider the system

$$\dot{x} = u$$

with cost criterion

$$J(x_0, u) = \int_0^\infty u^2(t)dt\,.$$

Here $Q = 0$ and $(\sqrt{Q}, A)$ is not detectable. The solution to the algebraic Riccati equation is $P = 0$. However the resulting control law $u = 0$ is not admissible, even though it gives $J = 0$, since $x(t) \not\to 0$ as $t \to \infty$. Now consider the admissible control law $\phi_\epsilon(x) = -\epsilon x$ for $\epsilon > 0$. This gives the closed loop system

$$\dot{x} = -\epsilon x$$

with the solution $x(t) = e^{-\epsilon t}x_0$. The corresponding cost is

$$J(x_0, \phi_\epsilon(x)) = \int_0^\infty \epsilon^2 e^{-2\epsilon t} x_0^2 dt = \frac{\epsilon x_0^2}{2}\,.$$

$J(x_0, \phi_\epsilon(x))$ can be made arbitrarily small by decreasing ϵ. However, $\inf_\phi J(x_0, \phi) = 0$ cannot be attained with any admissible control. Hence the optimal control does not exist, although there is a unique positive semidefinite solution to the algebraic Riccati equation.

Example 6.2.2.

Consider the system

$$\dot{x} = x + u$$

with cost criterion

$$J(x_0, u) = \int_0^\infty u^2(t)dt\,.$$

Here $Q = 0$ and $(\sqrt{Q}, A)$ is not detectable. The algebraic Riccati equation is given by

$$2P - P^2 = 0$$

so that there are 2 positive semidefinite solutions, 0 and 2. Once again, the solution $P = 0$ results in the control law $u = 0$, which is not admissible. On the other hand, $P = 2$ results in $u = -2x$, which is stabilizing and hence admissible. Therefore the optimal control is given by $u = -2x$, even though the solution to the algebraic Riccati equation is not unique.

Example 6.2.3 (Double Integrator)**.**

Consider the system $\ddot{y} = u$ with cost criterion

$$J = \int_0^\infty [y^2(t) + ru^2(t)]dt\,, \qquad r > 0\,.$$

A state space representation of this system is

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x\,. \end{aligned}$$

The cost function can be rewritten as

$$J = \int_0^\infty [x^T(t)Qx(t) + ru^2(t)]dt$$

where $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. It is easy to verify that $(\sqrt{Q}, A)$ is detectable, and that (A, B) is stabilizable. We proceed to solve the algebraic Riccati equation. Let $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$, where we have explicitly used the fact that P is symmetric. The elements of P satisfy the following equations:

$$\begin{aligned} -\frac{1}{r}p_2^2 + 1 &= 0 \\ p_1 - \frac{1}{r}p_2 p_3 &= 0 \\ 2p_2 - \frac{1}{r}p_3^2 &= 0 \,. \end{aligned}$$

The first gives the solutions $p_2 = \pm\sqrt{r}$. The third equation gives $p_3 = \pm(2rp_2)^{\frac{1}{2}}$. This implies $p_2 = \sqrt{r}$. Furthermore, for P to be positive semidefinite, all its diagonal entries must be nonnegative. Hence $p_3 = \sqrt{2}r^{\frac{3}{4}}$. Finally, $p_1 = \frac{1}{r}p_2 p_3 = \sqrt{2}r^{\frac{1}{4}}$ so that

$$P = \begin{bmatrix} \sqrt{2}r^{\frac{1}{4}} & \sqrt{r} \\ \sqrt{r} & \sqrt{2}r^{\frac{3}{4}} \end{bmatrix} .$$

P is in fact positive definite since $P_{11} > 0$ and $detP > 0$ (these are the necessary and sufficient conditions for a 2x2 matrix to be > 0).

The optimal closed loop system is given by

$$\begin{aligned} \dot{x} &= (A - BR^{-1}B^T P)x \\ &= \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{r} \begin{bmatrix} \sqrt{r} & \sqrt{2}r^{\frac{3}{4}} \end{bmatrix} \right) x \\ &= \begin{bmatrix} 0 & 1 \\ -r^{-\frac{1}{2}} & -\sqrt{2}r^{-\frac{1}{4}} \end{bmatrix} x \,. \end{aligned}$$

The poles of the closed loop system are given by the roots of the polynomial $s^2 + \sqrt{2}r^{-\frac{1}{4}}s + r^{-\frac{1}{2}}$. This is in the form of the standard second order system characteristic polynomial $s^2 + 2\zeta\omega_0 s + \omega_0^2$, with $\omega_0 = r^{-\frac{1}{4}}$, and $\zeta = \frac{1}{\sqrt{2}}$. The damping ratio of $\frac{1}{\sqrt{2}}$ of the optimal closed loop system is often referred to as the best compromise between small overshoot and good speed of response, and it is independent of r. For a fixed damping ratio, the larger the natural frequency ω_0, the faster the speed of response (recall that the peak time is inversely proportional to ω_0). Thus, we see that if r decreases, the speed of response becomes faster. Since a small r implies small control penalty and hence allows larger control inputs, this behaviour gives a good interpretation of the role of the quadratic weights in the cost criterion.

Example 6.2.4 (Servomotor)**.**

Consider the servomotor system given by the transfer function

$$y(s) = \frac{1}{s(s+1)}u(s)$$

A state space representation of this system is

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \,. \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \,. \end{aligned}$$

The cost function is

$$J = \int_0^\infty [y^2(t) + ru^2(t)]dt = \int_0^\infty [x^T(t)Qx(t) + ru^2(t)]dt\,,$$

where $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and $r > 0$. Again, it is easy to verify that $(\sqrt{Q}, A)$ is detectable, and that (A, B) is stabilizable. We proceed to solve the algebraic Riccati equation. Let $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$. The elements of P satisfy the following equations:

$$\begin{aligned} -\frac{1}{r}p_2^2 + 1 &= 0 \\ p_1 - p_2 - \frac{1}{r}p_2 p_3 &= 0 \\ 2(p_2 - p_3) - \frac{1}{r}p_3^2 &= 0\,. \end{aligned}$$

Solving these equations, we get

$$\begin{aligned} p_2 &= \sqrt{r} \\ p_3 &= r\sqrt{1 + 2r^{-\frac{1}{2}}} - r \\ p_1 &= \sqrt{r + 2r^{\frac{1}{2}}}\,, \end{aligned}$$

so that

$$P = \begin{bmatrix} \sqrt{r + 2r^{\frac{1}{2}}} & \sqrt{r} \\ \sqrt{r} & r\sqrt{1 + 2r^{-\frac{1}{2}}} - r \end{bmatrix}.$$

The optimal closed loop system matrix is given by

$$\begin{aligned} A - BR^{-1}B^T P &= \begin{bmatrix} 0 & 1 \\ -\frac{1}{r}p_2 & -1 - \frac{1}{r}p_3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -r^{-\frac{1}{2}} & -\sqrt{1 + 2r^{-\frac{1}{2}}} \end{bmatrix}. \end{aligned}$$

The characteristic polynomial of the closed loop system is given by $s^2 + \sqrt{1 + 2r^{-\frac{1}{2}}}s + r^{-\frac{1}{2}}$, with poles located at $\dfrac{-\sqrt{1 + 2r^{-\frac{1}{2}}} \pm \sqrt{1 - 2r^{-\frac{1}{2}}}}{2}$.

6.3 Detectability and Closed Loop Stability

While the complete proof of Theorem (6.2.1) is beyond the scope of this course, we can give an indication of the role played by detectability on closed loop stability. We first consider the *Lyapunov equation*

$$A^T P + PA + Q = 0\,. \tag{6.11}$$

Pre-multiplying by $e^{A^T s}$, post-multiplying by e^{As}, and integrating from 0 to t, we get, for each t,

$$P = e^{A^T t} P e^{At} + \int_0^t e^{A^T s} Q e^{As} ds \tag{6.12}$$

(6.12) is not a solution to (6.11) since P appears on both sides of (6.12); it is merely an alternative representation for P. However, if A is stable, by letting $t \to \infty$, we get the solution

$$P = \int_0^\infty e^{A^T s} Q e^{As} ds$$

We now discuss the converse result. Let $Q \geq 0$, and set

$$Q(t) = \int_0^t e^{A^T s} Q e^{As} ds \,.$$

Suppose there is a solution $P \geq 0$ to the Lyapunov equation (6.11). From (6.12), we get that $0 \leq Q(t) \leq P$, so that $Q(t)$ is bounded as $t \to \infty$. Here is where detectability comes in.

Lemma 6.3.1. *Assume $(\sqrt{Q}, A)$ is detectable. Then boundedness of $Q(t)$ implies A is stable.*

Proof. First note that according to the PBH test, detectability is equivalent to the following: For any μ which is an eigenvalue of A with $Re\,\mu \geq 0$, $\begin{bmatrix} \sqrt{Q} \\ A - \mu I \end{bmatrix} v = 0$ if and only if $v = 0$. Now suppose $Q(t)$ is bounded but that A is unstable. There exists an eigenvalue μ of A with $Re\,\mu \geq 0$, and with corresponding eigenvector v. Let v^* denote the conjugate transpose of the vector v. Then

$$\begin{aligned} v^* Q(t) v &= \int_0^t v^* e^{A^T s} Q e^{As} v ds \\ &= \int_0^t v^* e^{\mu^* s} Q e^{\mu s} v ds \\ &= \int_0^t e^{2(Re\,\mu)s} ds v^* Q v \,. \end{aligned}$$

The only way that $v^* Q(t) v$ can be bounded is $v^* Q v = 0$. But then this gives

$$\begin{bmatrix} \sqrt{Q} \\ A - \mu I \end{bmatrix} v = 0$$

for a nonzero v, which contradicts the assumption of detectability. This proves A is stable. ☐

Combining the above discussion, we see that if $Q \geq 0$ and $(\sqrt{Q}, A)$ is detectable, then a solution $P \geq 0$ to the Lyapunov equation (6.11) exists if and only if A is stable. Now rewrite the algebraic Riccati equation in the following form

$$(A - BR^{-1}B^T P)^T P + P(A - BR^{-1}B^T P) + PBR^{-1}B^T P + Q = 0 \tag{6.13}$$

Putting $F = A - BR^{-1}B^T P$, (6.13) can be written as

$$F^T P + PF + PBR^{-1}B^T P + Q = 0 \tag{6.14}$$

This is almost in the form of the lemma, except that detectability would involve $(\sqrt{Q + PBR^{-1}B^T P}, F)$, which is not the detectability assumption of Theorem 6.2.1. However, we make use of the following result on detectability: For any K and $R > 0$, $(\sqrt{Q}, A)$ is detectable if and only if $(\sqrt{Q + KRK}, A - BK)$ is detectable. So putting $K = R^{-1}B^T P$, we see that $F = A - BK$ and $\sqrt{Q + PBR^{-1}B^T P} = \sqrt{Q + KRK}$, and we can conclude that $(\sqrt{Q}, A)$ is detectable if and only if $(\sqrt{Q + PBR^{-1}B^T P}, F)$ is detectable. Hence the existence of a positive semidefinite solution P to the algebraic Riccati equation, equivalently to (6.14), guarantees $F = A - BR^{-1}B^T P$ is stable.

6.4 An Optimal Observer

From the results on observers in Chapter 5, we know that pole placement in state feedback control has a dual problem of pole placement of observer error dynamics. Since linear quadratic optimal control produces an optimal state feedback law, it is natural to ask if there is a corresponding optimal observer. To formulate the problem precisely requires the introduction of stochastic processes and estimation theory. We shall only discuss the construction of an "optimal" observer purely by analogy.

Consider a full order observer given by the equation

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(y - C\hat{x}) \\ y &= Cx\,. \end{aligned}$$

The estimation error $e = x - \hat{x}$ satisfies the equation $\dot{e} = (A - LC)e$. If we think in terms of an optimal feedback law L^T to place the poles of $A^T - C^T L^T$, we would be in an analogous situation as linear quadratic optimal control. If we proceed formally, we can identify A^T as the counterpart of A in linear quadratic optimal control, and C^T as the counterpart of B. Although there is no obvious analogues of Q and R matrices, let us take a matrix $W \geq 0$ to correspond to Q, and a matrix $V > 0$ to correspond to R. Then the feedback gain L^T should correspond to

$$L^T = V^{-1}CP$$

or

$$L = PC^T V^{-1} \tag{6.15}$$

where P should satisfy the Riccati equation

$$AP + PA^T - PC^T V^{-1} CP + W = 0\,. \tag{6.16}$$

The resulting observer, given by

$$\dot{\hat{x}} = A\hat{x} + Bu + PC^T V^{-1}(y - C\hat{x}) \tag{6.17}$$

where P satisfies (6.16) indeed has optimality properties. It is called the *steady state Kalman filter*. In terms of the Kalman filter, the matrix W has the interpretation of the intensity of the additive white noise w driving the system

$$\dot{x} = Ax + Bu + w\,.$$

The matrix V has the interpretation of the intensity of the additive white sensor noise v in the observation equation

$$y = Cx + v\,.$$

By the results of the control Riccati equation, we can immediately conclude that if (C, A) is detectable and $(A, \sqrt{W})$ is stabilizable, the Kalman filter matrix $A - PC^T V^{-1} C$ is stable, and the poles of the error dynamics will be "optimally" placed. Naturally, we can use the steady state Kalman filter in combination with linear quadratic optimal control to design an output feedback controller based on the separation principle. The resulting controller

$$u = -R^{-1}B^T P_c \hat{x} \tag{6.18}$$

where P_c satisfies the control algebraic Riccati equation (6.8), is called the *linear-quadratic-Gaussian* (LQG) controller. It has certain optimality properties, the detailed exposition of which is beyond the scope of this course.

Chapter 7

Output Regulation

In this chapter we extend the pole placement, observer-based output feedback design to solve tracking problems. By tracking we mean that the output is commanded to track asymptotically a desired reference trajectory. We will examine several variants of the problem. The simplest case is when there is full state information and there are no disturbances affecting the system. The next case is when there is partial state information, but the system is observable, and there are no disturbances. Applying the separation principle, we can solve this problem using observer-based output feedback design. Also, we study a special case when the reference trajectory to be tracked is a non-zero constant. Finally, the case when there are disturbances will be addressed in the next chapter.

First, we look at several examples as motivation and to illustrate two issues that will be faced in designing tracking controllers.

Example 7.0.1.

Suppose we want a small mobile robot to track a curve on the floor. We model the robot as a "kinematic unicycle", which is the simplest vehicle model that captures the no-sideslip constraint of wheeled vehicles. If the curve is a circle whose radius is not too small, intuition suggests that it should be feasible to design steering and velocity inputs to achieve the circular path exactly. If the initial condition of the robot is not aligned with the circle, then one would design a feedback controller that makes the robot approach the circular path asymptotically. Suppose instead that the path is a circle, combined with a straight line path that emanates from the circle at a right angle. If we require the robot to follow this path with unit speed, then because no vehicle can turn a right corner instantaneously, this path is infeasible. While exact tracking is not possible, one may be able to find a controller that keeps the trajectory of the robot as near as possible to the desired path. In summary, there are two issues in designing a tracking controller: feasibility of *exact tracking* of the desired output and *asymptotic convergence* to the desired output.

Example 7.0.2.

Consider the linear system

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \,,$$

and suppose we want this system to track a desired trajectory $x_d = (\mathrm{e}^{-t}, 0)$ which starts at initial condition $(1, 0)$ and ends at the origin. The first question to ask is: can the system track the desired trajectory assuming that the initial condition is $x(0) = (1, 0)$? Second: can we design an asymptotic controller that tracks the desired trajectory even if the initial condition is not $(1, 0)$? It is not difficult to see that the system cannot achieve exact tracking of the desired trajectory because the closed loop system can never point to the left along the positive x_1 axis for any control value. We would like to mathematically formalize this observation to obtain checkable conditions on the feasibility of exact tracking. Once it is determined that exact tracking can be achieved, we want a systematic procedure to design asymptotic tracking controllers.

One case in which exact tracking is trivial is SISO systems in controllable canonical form. Consider the system in Section 3.5. If we set

$$u = \alpha_0 x_1 + \alpha_1 x_2 + \cdots + \alpha_{n-1} x_n + v$$

where $v \in \mathbb{R}$ is a new input, then the system is converted to a chain of integrators with input v and output y. In order to achieve exact tracking of a desired output $y_d(t)$ we simply require that $v = y_d^{(n)}(t)$, i.e. v is the nth time derivative of the desired output signal. Also initial conditions must match exactly; namely, $x_i(0) = y_d^{(i-1)}(0)$, $i = 1, \ldots, n$.

7.1 Output Regulation with Full State Information

Consider the linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx\,. \end{aligned} \tag{7.1}$$

Let y_d denote the desired signal for the output $y(t)$ to track asymptotically. We assume $y_d \neq 0$ and we assume it is generated as the output of an *exosystem* given by

$$\begin{aligned} \dot{w} &= Sw \\ y_d &= C_d w\,. \end{aligned}$$

The vector $w \in \mathbb{R}^q$ is the state of the exosystem. The tracking error is

$$e = y - y_d = Cx - C_d w\,.$$

The control objective is to design a feedback law $u(t) = F_1 x + F_2 w$ such that

(AS) $(A + BF_1)$ is stable, and

(R) $e(t) \to 0$ as $t \to \infty$.

The first requirement is that the closed loop system be stable when $w = 0$. The second requirement is that regulation of the output is achieved. A controller that satisfies the above objectives is called a *regulator*. In this section we make the following two assumptions:

- (A, B) is stabilizable.
- Both x and w are measurable.

7.1.1 Exact Tracking

First we consider the question of feasibility of the exact tracking problem. This problem can be stated as follows. Given the system (7.1) and a desired output $y_d(t)$ find a control $u(t)$ and an initial condition $x(0)$ such that $e(t) = 0$, $\forall t \geq 0$. We are interesting in studying this problem for theoretical interest and not practical motivations. This problem can be seen as a necessary first step in solving the asymptotic tracking problem.

Lemma 7.1.1. *Suppose there exists $u = \overline{F}_2 w$, where $\overline{F}_2 \in \mathbb{R}^{m\times q}$, and a map $\Pi : \mathbb{R}^q \to \mathbb{R}^n$ such that*

$$\Pi S = A\Pi + B\overline{F}_2 \tag{7.2}$$
$$C\Pi = C_d\,. \tag{7.3}$$

If $x(0) = \Pi w(0)$, then $e(t) = 0$ for all $t \geq 0$.

Proof. Suppose there exists $u = \overline{F}_2 w$ and Π such that (7.2)-(7.3) hold. Then define $z = x - \Pi w$. We have

$$\begin{aligned} \dot z &= \dot x - \Pi \dot w \\ &= Ax + B\overline{F}_2 w - \Pi S w \\ &= Ax - A\Pi w + A\Pi w + B\overline{F}_2 w - \Pi S w \\ &= Az \qquad\qquad \text{by (7.2)}\,. \end{aligned}$$

Since $z(0) = 0$ by assumption, the unique solution of $\dot z = Az$ is $z(t) = 0$, for all $t \geq 0$. That is, $x = \Pi w$ for all $t \geq 0$. Then, using (7.3) we obtain

$$e(t) = Cx(t) - C_d w(t) = C\Pi z(t) + C\Pi w(t) - C_d w(t) = 0\,.$$

□

The equations (7.2)-(7.3) are known as the *regulator* or *FBI* equations (after B. Francis, C. Byrnes, and A. Isidori).

Based on the proof of Lemma 7.1.1 we define the *tracking subspace* $\mathcal{T} \subset \mathbb{R}^{q+n}$

$$\mathcal{T} = \{(x, w) \mid x - \Pi w = 0\}.$$

If the regulator equations hold, then the tracking subspace is invariant under the closed loop dynamics. That is, $\mathcal{T}$ is A_{cl}-invariant where

$$A_{cl} = \begin{bmatrix} A & B\overline{F}_2 \\ 0 & S \end{bmatrix}.$$

Lemma 7.1.1 shows that the regulator equations and proper choice of initial conditions are sufficient for exact tracking. To what extent are the regulator equations also necessary for exact tracking? For this we require some extra conditions.

Lemma 7.1.2. *Assume that (C, A) is observable and $eig(S) \cap eig(A) = \emptyset$. Also assume that $(S, w(0))$ is controllable. Suppose that there exists an initial condition $x(0)$ and a control $u = \overline{F}_2 w$, where $\overline{F}_2 \in \mathbb{R}^{m\times q}$, such that the closed-loop system satisfies $e(t) = 0$ for all $t \geq 0$. Then there exists a map $\Pi : \mathbb{R}^q \to \mathbb{R}^n$ such that* (7.2)-(7.3) *hold. Moreover $x(0) = \Pi w(0)$.*

Proof. Since $eig(S) \cap eig(A) = \emptyset$, by Sylvester's Theorem (Gantmacher, *Theory of Matrices, vol. 1*, p. 225) there exists a unique solution Π of (7.2). Define $z(t) = x(t) - \Pi w(t)$. Using (7.2), we obtain $\dot z = Az$. Now consider

$$e(t) = Cz(t) + (C\Pi - C_d)w(t) = 0\,, \qquad \forall t \geq 0\,.$$

Since $eig(S) \cap eig(A) = \emptyset$, we know $Cz(t) = 0$ and $(C\Pi - C_d)w(t) = 0$, for all $t \geq 0$. Since (C, A) is observable, the first equation $Ce^{At}z(0) = 0$ yields $z(0) = 0$, or $x(0) = \Pi w(0)$, as desired. Next consider

$$(C\Pi - C_d)e^{St}w(0) = 0\,, \qquad \forall t \geq 0\,.$$

By controllability of $(S, w(0))$, we obtain $C\Pi = C_d$, which gives (7.3). □

Example 7.1.1.

Consider again the example 7.0.2. We want to determine if the reference signal $y_d(t) = [\, e^{-t}\ 0\,]^T$ is feasible. First we observe that the signal is generated by an exosystem

$$\dot{w} = -w\,, \qquad w(0) = 1\,, \qquad y_d = \begin{bmatrix} 1 \\ 0 \end{bmatrix} w = C_d w\,.$$

Next we check the regulator equations:

$$\begin{bmatrix} \Pi_1 \\ \Pi_2 \end{bmatrix}(-1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \Pi_1 \\ \Pi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\overline{f}_2\,.$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \Pi_1 \\ \Pi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Simplifying, we obtain the equations

$$\begin{aligned} \Pi_1 &= \Pi_2 \\ -\Pi_2 &= \Pi_1 + \overline{f}_2 \\ \Pi_1 &= 1 \\ \Pi_2 &= 0\,, \end{aligned}$$

for which there is no solution. Finally, we can verify that (C, A) is observable and $(S, w(0))$ is controllable. From Lemma 7.1.2 we conclude that the exact tracking problem is infeasible, as expected.

7.1.2 Asymptotic Tracking

The regulator equations tell us a relationship between x and w, namely $x = \Pi w$ and a feedforward (open-loop) control $u = \overline{F}_2 w$ for exacting tracking of the desired output. What if $x(0) \neq \Pi w(0)$? To deal with a mismatch in initial conditions we need a feedback correction term in the control. The modified control for asymptotic tracking is:

$$u = \overline{F}_2 w + F_1(x - \Pi w) \triangleq F_1 x + F_2 w\,. \tag{7.4}$$

Let $\mathbb{C}^+$ denote the closed right-half complex plane.

Lemma 7.1.3. *Suppose that $eig(S) \in \mathbb{C}^+$ and $\overline{A} = A + BF_1$ is Hurwitz. A regulator $u = \overline{F}_2 w + F_1(x - \Pi w)$ exists if and only if there exist maps $\Pi : \mathbb{R}^q \to \mathbb{R}^n$ and $\overline{F}_2 : \mathbb{R}^q \to \mathbb{R}^m$ satifying* (7.2)-(7.3).

Proof. ($\Longleftarrow$) Suppose there exist $(\Pi, \overline{F}_2)$ satisfying the regulator equations. Define $z = x - \Pi w$.

$$\begin{aligned} \dot{z} &= \dot{x} - \Pi \dot{w} \\ &= \overline{A} z + [A\Pi + B\overline{F}_2 - \Pi S] w \\ &= \overline{A} z\,. \end{aligned}$$

Since $\overline{A}$ is Hurwitz, $z(t) \to 0$. Then

$$\begin{aligned} y(t) = Cx &= Cx - C\Pi w + C\Pi w \\ &= Cz + y_d\,. \end{aligned}$$

Hence, $y(t) - y_d(t) = Cz \to 0$ as $t \to \infty$.

($\Longrightarrow$) Since $eig(S) \cap eig(\overline{A} = \emptyset$, by Sylvester's Theorem (Gantmacher, *Theory of Matrices, vol. 1*, p. 225), there exists a unique solution Π satisfying

$$\Pi S - \overline{A}\Pi = BF_2\,.$$

Letting $\overline{F}_2 = F_2 + F_1 \Pi$, we obtain (7.2). As shown above, $\dot{z} = \overline{A} z$. Hence $z(t) \to 0$ as $t \to \infty$. Also as above, $e(t) = Cz + (C\Pi - C_d) w$. By assumption $e(t) \to 0$ and since $z(t) \to 0$, it must be that $(C\Pi - C_d) w \to 0$ for all initial conditions $w(0)$. Since $eig(S) \in \mathbb{C}^+$, $w(t) \not\to 0$. Hence, $C\Pi = C_d$. □

Example 7.1.2.

Consider again the example 7.0.2, but with a less ambitious tracking problem. Suppose that $y = x_1$ and $y_d(t) = \mathrm{e}^{-t}$. The exosystem is

$$\dot{w} = -w\,, \qquad w(0) = 1\,, \qquad y_d = w\,.$$

$$\begin{bmatrix} \Pi_1 \\ \Pi_2 \end{bmatrix} (-1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \Pi_1 \\ \Pi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \overline{f}_2\,.$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \Pi_1 \\ \Pi_2 \end{bmatrix} = 1\,.$$

We obtain $\Pi_1 = 1$, $\Pi_2 = 1$, and $\overline{f}_2 = -2$. For exact tracking we use $u = -2w$, and the initial conditions are $x_1(0) = \Pi_1 w(0) = 1$ and $x_2(0) = \Pi_2 w(0) = 1$.

Next we design an asymptotic controller. Let $u = -2w + F_1(x - \Pi w)$, where $F_1 \in \mathbb{R}^{1\times 2}$. Since (A, B) is controllable we can design F_1 such that $A + BF_1$ has any desired closed-loop eigenvalues. If we want the eigenvalues to be $-1, -1$ then $F_1 = [0 - 2]$.

7.2 Special Case - Constant Reference Signals

While the tracking problem has been solved for much more general classes of reference inputs, in this section we focus on constant step reference inputs. This is an important special case, since the most common tracking problem is that of set point tracking. For convenience, we use $y_d(t) = y_d$ to denote the constant desired reference trajectory. Also, for simplicity, we shall assume that the number of inputs is equal to the number of outputs. Since $y_d \neq 0$, the steady state value of $x(t)$ cannot be 0. Also the exosystem is $\dot{w} = 0$, with $w(0) = 1$, so the regulator equations are

$$\begin{aligned} 0 &= A\Pi + B\overline{F}_2 \\ C\Pi &= y_d \,. \end{aligned}$$

Rearranging this equation we have

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} \Pi \\ \overline{F}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ y_d \end{bmatrix} . \tag{7.5}$$

By assumption of equal number of inputs and outputs, $\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$ is a square matrix. Since we would like to track any set point changes, y_d is arbitrary. Equation (7.5) can be solved uniquely for $\begin{bmatrix} \Pi \\ \overline{F}_2 \end{bmatrix}$ if and only if

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \quad \text{is nonsingular.} \tag{7.6}$$

Let us give an interpretation to condition (7.6). Suppose the input to (7.1) is given by $e^{\lambda t}\theta$. For zero initial conditions, the solution $x(t)$ is given by $e^{\lambda t}\xi$. Substituting into (7.1), we get the following equation

$$\lambda\xi = A\xi + B\theta \,. \tag{7.7}$$

Suppose this input results in no output. Then we must also have

$$C\xi = 0 \,. \tag{7.8}$$

In the single-input single-output case (i.e., θ is a scalar), this gives the condition

$$C(\lambda I - A)^{-1}B = 0 \,. \tag{7.9}$$

Such a λ is therefore a zero of the transfer function

$$G(s) = C(sI - A)^{-1}B$$

. For the multivariable case, we can combine equations (7.7) and (7.8) into

$$\begin{bmatrix} (A - \lambda I) & B \\ C & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \theta \end{bmatrix} = 0 \,. \tag{7.10}$$

(7.10) can be solved for nonzero $\begin{bmatrix} \xi \\ \theta \end{bmatrix}$ if and only if

$$\begin{bmatrix} (A - \lambda I) & B \\ C & 0 \end{bmatrix} \quad \text{is not full rank} \,. \tag{7.11}$$

By analogy with the single-input single-output case, we call such a λ a *transmission zero* of the system.

From this discussion, we see that condition (7.6) corresponds to having no transmission zero at the origin. We can now state the conditions for solvability of the tracking problem:

1. (A, B) stabilizable
2. The system (7.1) has no transmission zeros at 0 .

If, in addition, (A, B) is in fact controllable, then the rate of convergence to 0 of the tracking error can be pre-assigned.

Example 7.2.1.

Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 24 & -10 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

The characteristic polynomial of the plant is given by

$$det(sI - A) = s^3 + 10s^2 - 24s = s(s^2 + 10s - 24) = s(s + 12)(s - 2)$$

The transfer function is given by

$$G(s) = \frac{1}{s(s + 12)(s - 2)}$$

so that there are no transmission zeros at 0. To solve for the steady state values Π and $\overline{F}_2$, we set

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 24 & -10 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Pi \\ \overline{F}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ y_d \end{bmatrix} .$$

Successively from the equation for each row, we see that $\Pi_2 = 0$, $\Pi_3 = 0$, $\overline{F}_2 = 0$, and $\Pi_1 = y_d$. Note that (A, B) is controllable. Suppose we choose the closed loop poles to be at -1, $-1 \pm i$. This corresponds to the desired characteristic polynomial

$$r(s) = (s^2 + 2s + 2)(s + 1) = s^3 + 3s^2 + 4s + 2\,.$$

Since (A, B) is in controllable canonical form, we immediately obtain

$$F_1 = \begin{bmatrix} -2 & -28 & 7 \end{bmatrix} .$$

The asymptotic tracking controller is

$$u = \begin{bmatrix} -2 & -28 & 7 \end{bmatrix} \begin{bmatrix} x_1 - y_d \\ x_2 \\ x_3 \end{bmatrix} \tag{7.12}$$

$$= 2(x_1 - y_d) - 28x_2 + 7x_3\,. \tag{7.13}$$

To determine the transfer function from the reference input $y_d(t)$ to the output $y(t)$, first note that

$$A + BF_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 24 & -10 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & -28 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix} . \tag{7.14}$$

Writing

$$u = F_1(x - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} y_d)$$

we can write the closed loop system as

$$\begin{aligned}\dot{x} &= (A + BF_1)x - BF_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} y_d \\ &= (A + BF_1)x + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} y_d \end{aligned} \tag{7.15}$$

Noting that (7.15) is again in controllable canonical form, we can immediately write down the transfer function from y_d to y as

$$\begin{aligned} y(s) &= C(sI - A - BF_1)^{-1} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} y_d(s) \\ &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \frac{\begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix}}{s^3 + 3s^2 + 4s + 2} 2y_d(s) \\ &= \frac{2}{s^3 + 3s^2 + 4s + 2} y_d(s)\,. \end{aligned} \tag{7.16}$$

Since $y_d(s) = \frac{y_d}{s}$ and the closed loop system is stable, the steady state value of y can be determined from the final value theorem of Laplace transforms

$$\begin{aligned} \lim_{t \to \infty} y(t) &= \lim_{s \to 0} \frac{2}{s^3 + 3s^2 + 4s + 2} y_d \\ &= y_d \end{aligned}$$

so that asymptotic tracking is indeed achieved. In this example, no additional feedforward control is needed since there is a pole at the origin for the open loop plant. From classical control theory, we know that for such (type-1) systems, asymptotic step tracking is guaranteed using unity feedback as long as the closed loop system is stable. The state space formulation gives exactly this structure.

Example 7.2.2.

As another example, consider the linear system (7.1), but with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

The open loop characteristic polynomial is given by

$$det(sI - A) = s^3 - 2s^2 - s + 2 = (s-1)(s+1)(s-2)$$

Again to solve for the steady state values of Π and $\overline{F}_2$, put

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Pi \\ \overline{F}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} y_d$$

First, second, and 4th rows give respectively $\Pi_2 = 0$, $\Pi_3 = 0$, $\Pi_1 = y_d$, while the 3rd row gives $\overline{F}_2 = 2y_d$.

Suppose we would like to place the closed loop poles at -2, $-1 \pm i$, so that the desired characteristic polynomial is

$$r(s) = (s^2 + 2s + 2)(s + 2) = s^3 + 4s^2 + 6s + 4\,.$$

This results in

$$F_1 = \begin{bmatrix} -2 & -7 & -6 \end{bmatrix}.$$

The control law is given by

$$\begin{aligned} u &= \overline{F}_2 w + F_1(x - \Pi w) \\ &= 4y_d - 2x_1 - 7x_2 - 6x_3\,. \end{aligned}$$

The transfer function from y_d to y is easily evaluated to be

$$y(s) = \frac{4}{s^3 + 4s^2 + 6s + 4} y_d(s)\,.$$

7.3 Output Regulation with Partial State Information

The extension to observer-based output regulation is straightforward using the separation principle. Whether one uses the full-order or reduced-order observer, the observer estimation error $x(t) - \hat{x}(t)$ satisfies a homogeneous stable equation. By replacing the control law (7.4) with

$$u = F_1 \hat{x} + F_2 w \tag{7.17}$$

we are guaranteed a solution of the tracking problem with a controller which is based on output feeback combined with a feedforward term.

To illustrate the procedure, we re-visit Example 7.2.2, using a reduced-order observer to estimate x_2 and x_3 and employing the feedback law (7.17). The decomposed system equations are given by

$$\begin{aligned} \begin{bmatrix} \dot{x_2} \\ \dot{x_3} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ -2 \end{bmatrix} y \\ \dot{x}_1 &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}. \end{aligned}$$

Hence the reduced-order observer is given by

$$\begin{bmatrix} \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ -2 \end{bmatrix} y + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \left(\dot{y} - \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \end{bmatrix}\right). \tag{7.18}$$

The system matrix for the observer is given by

$$H = \begin{bmatrix} -l_1 & 1 \\ 1 - l_2 & 2 \end{bmatrix},$$

where l_1 and l_2 are to be chosen to place the poles of the observer. Its characteristic polynomial is given by

$$det(sI - H) = s^2 + (l_1 - 2)s + (l_2 - 1 - 2l_1)$$

Let us choose the observer poles to be at $-4, -4$. The desired observer characteristic polynomial is given by

$$r_o(s) = s^2 + 8s + 16$$

On matching coefficients, we see that $l_1 = 10$ and $l_2 = 37$. Thus the reduced-order observer is given by

$$\begin{bmatrix} \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \end{bmatrix} = \begin{bmatrix} -10 & 1 \\ -36 & 2 \end{bmatrix} \begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ -2 \end{bmatrix} y + \begin{bmatrix} 10 \\ 37 \end{bmatrix} \dot{y} \tag{7.19}$$

We shall use (7.19) for derivation of the controller and so we shall not perform the transformation to remove $\dot{y}$. The control law is given by

$$\begin{aligned} u &= \overline{F}_2 w + F_1(\hat{x} - \Pi w) \\ &= 4y_d - 2\hat{x}_1 - 7\hat{x}_2 - 6\hat{x}_3 \\ &= 4y_d - 2y - 7\hat{x}_2 - 6\hat{x}_3 \,. \end{aligned} \tag{7.20}$$

Substituting the control law into (7.19), we find

$$\begin{aligned} \begin{bmatrix} \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \end{bmatrix} &= \begin{bmatrix} -10 & 1 \\ -36 & 2 \end{bmatrix} \begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (4y_d - 2y - \begin{bmatrix} 7 \\ 6 \end{bmatrix} \begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \end{bmatrix}) + \begin{bmatrix} 0 \\ -2 \end{bmatrix} y + \begin{bmatrix} 10 \\ 37 \end{bmatrix} \dot{y} \\ &= \begin{bmatrix} -10 & 1 \\ -43 & -4 \end{bmatrix} \begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} 4y_d + \begin{bmatrix} 0 \\ -4 \end{bmatrix} y + \begin{bmatrix} 10 \\ 37 \end{bmatrix} \dot{y} \,. \end{aligned} \tag{7.21}$$

Using (7.21), we can determine the transfer functions from y and y_d to $\begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \end{bmatrix}$ as

$$\begin{bmatrix} \hat{x}_2(s) \\ \hat{x}_3(s) \end{bmatrix} = \begin{bmatrix} s+10 & -1 \\ 43 & s+4 \end{bmatrix}^{-1} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} 4y_d + \left(\begin{bmatrix} 0 \\ -4 \end{bmatrix} + \begin{bmatrix} 10s \\ 37s \end{bmatrix} \right) y \right) .$$

Finally, substituting into (7.20), we obtain

$$\begin{aligned} u &= -\begin{bmatrix} 7 \\ 6 \end{bmatrix} \begin{bmatrix} s+10 & -1 \\ 43 & s+4 \end{bmatrix}^{-1} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} 4y_d + \left(\begin{bmatrix} 0 \\ -4 \end{bmatrix} + \begin{bmatrix} 10s \\ 37s \end{bmatrix} \right) y \right) + 4y_d - 2y \\ &= \frac{s^2+8s+16}{s^2+14s+83} 4y_d - \frac{2s^2+4s-102}{s^2+14s+83} y - \frac{292s^2+179s}{s^2+14s+83} y \\ &= -\frac{294s^2+183s-102}{s^2+14s+83} y + \frac{s^2+8s+16}{s^2+14s+83} 4y_d \,. \end{aligned} \tag{7.22}$$

If we express (7.22) in the form

$$u(s) = -F(s)y(s) + C(s)y_d(s)$$

the closed loop transfer function from y_d to y is given by

$$y(s) = \frac{G(s)}{1+G(s)F(s)} C(s) y_d(s) \,.$$

Substituting, we finally get

$$\begin{aligned} y(s) &= \frac{4(s^2+8s+16)}{s^5+12s^4+54s^3+116s^2+128s+64} y_d(s) \\ &= \frac{4(s^2+8s+16)}{(s+4)^2(s+2)(s^2+2s+2)} y_d(s) \,. \end{aligned} \tag{7.23}$$

In the final transfer function (7.23), the observer poles are in fact cancelled, leaving

$$y(s) = \frac{4}{(s+2)(s^2+2s+2)} y_d(s) \,. \tag{7.24}$$

Chapter 8

Output Regulation with Disturbances

In this chapter we extend the output regulation methodology of the previous chapter to the case when the system is affected by an unmeasurable disturbance signal whose frequency content is known. It will be seen that the regulator equations again play a central role in solving this problem.

Disturbance signals are sometimes modelled as unmeasurable white noise inputs with certain statistical properties. See Section 6.4 on Kalman filter design. On the other hand, much may be know about a disturbance signal, even it is not directly measurable. A more realistic assumption is that the frequency content of the disturbance signal is known. There are many applications of this situation. For example:

- Landing a helicopter on a rolling ship.
- Rejecting engine vibration in autopilot actuators.
- Velocity tracking for UTIAS rovers subject to uneven weight distribution in the wheels due to the batteries.
- Control of a lift bridge in the presence of wind disturbances and a constant disturbance due to uncentered center of mass of the bridge span.

8.1 Problem Statement

Consider the linear system

$$\begin{aligned} \dot{x} &= Ax + Bu + Dw \\ y &= Cx + C_w w \, . \end{aligned} \tag{8.1}$$

The term Dw represents an unmeasurable disturbance signal which is generated by an exosystem whose dynamics are known. The output y represents the measurements of both the system and exosystem states. Let $y_d(t) \neq 0$ denote the desired signal to track asymptotically. We assume that both the disturbance and y_d are generated by an exosystem given by

$$\begin{aligned} \dot{w} &= Sw \\ y_d &= E_d w \end{aligned}$$

where $w \in \mathbb{R}^q$. The tracking error is

$$e = Ex - E_d w \, .$$

The control objective is to design a dynamic feedback of the form

$$\begin{aligned} \dot{\xi} &= G\xi + Hy \quad (8.2)\\ u &= F\xi \quad (8.3) \end{aligned}$$

such that

(AS) If $w(t) = 0$, then the closed-loop system

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} A & BF \\ HC & G \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix},$$

is asymptotically stable.

(R) For all initial conditions $(x(0), \xi(0), w(0))$, the closed-loop system satisfies $e(t) \to 0$ as $t \to \infty$.

A controller that satisfies the above objectives is called a *regulator*. In this section we assume neither x nor w are fully measurable. In addition, we require:

- (A, B) is stabilizable.
- (C_c, A_c) is detectable, where

$$A_c = \begin{bmatrix} A & D \\ 0 & S \end{bmatrix}, \qquad C_c = \begin{bmatrix} C & C_w \end{bmatrix}. \quad (8.4)$$

The detectability assumption says that both x and w can be recovered from the measurement y. The strategy for designing the regulator (8.2)-(8.3) will be to build an observer for x and w based on y.

8.2 Disturbance Decoupling

Before proceeding we discuss a simpler problem called *disturbance decoupling*. Suppose that we do not know the frequency content of the disturbance signal, so we do not have an exosystem model for it. We want to design a feedback control $u = Fx$ such that the disturbance does not appear in the output of the system. Consider the closed-loop system

$$\dot{x} = (A + BF)x + Dw\,, \qquad y = Cx\,.$$

The solution is

$$y(t) = Ce^{(A+BF)t}x(0) + \int_0^t Ce^{(A+BF)(t-\tau)}Dw(\tau)d\tau\,.$$

The requirement that y is unaffected by w implies that for all signals w,

$$C\int_0^t e^{(A+BF)(t-\tau)}Dw(\tau)d\tau = 0\,. \quad (8.5)$$

Let $\tilde{Q}_c$ be the controllability matrix of $(A + BF, D)$. Then condition (8.5) is equivalent to $C\mathcal{R}(\tilde{Q}_c) = 0$, or

$$\mathcal{R}(\tilde{Q}_c) \subset \mathcal{N}(C)\,. \quad (8.6)$$

Thus, the controller must be designed to guarantee the geometric condition (8.6) holds. See *W.M. Wonham, Linear Multivariable Control: A Geometric Approach* for the details.

8.3 Disturbance Rejection

Now we return to our output regulation problem with disturbance rejection. The main result is the following.

Theorem 8.3.1. *Suppose that* $eig(S) \in \mathbb{C}^+$*,* (A, B) *is stabilizable, and* (C_c, A_c) *is detectable. A regulator of the form* (8.2)-(8.3) *exists if and only if there exist maps* $\Pi : \mathbb{R}^q \to \mathbb{R}^n$ *and* $\overline{F}_2 : \mathbb{R}^q \to \mathbb{R}^m$ *satisfying*

$$\begin{aligned} \Pi S &= A\Pi + B\overline{F}_2 + D && (8.7) \\ E\Pi &= E_d\,. && (8.8) \end{aligned}$$

The main idea is to construct an observer for the composite state (x, w). To this end, define $\eta = (x, w) \in \mathbb{R}^{n+q}$. The composite system is

$$\dot{\eta} = A_c\eta + B_c u\,, \qquad y = C_c\eta$$

where

$$B_c = \begin{bmatrix} B \\ 0 \end{bmatrix}.$$

An observer for the composite system is

$$\dot{\hat{\eta}} = A_c\hat{\eta} + B_c u + L(y - \hat{y})\,, \qquad \hat{y} = C_c\hat{\eta}\,. \tag{8.9}$$

The estimator error $\delta\eta = \eta - \hat{\eta}$ has dynamics $\dot{\delta\eta} = (A_c - LC_c)\delta\eta$. Since (C_c, A_c) is detectable we can choose L such that $A_c - LC_c$ is Hurwitz. We will construct a feedback

$$u = F\hat{\eta} = F_1\hat{x} + F_2\hat{w} \stackrel{\triangle}{=} \overline{F}_2\hat{w} + F_1(\hat{x} - \Pi\hat{w}). \tag{8.10}$$

Since (A, B) is stabilizable we can choose F_1 such that $\overline{A} \stackrel{\triangle}{=} A + BF_1$ is Hurwitz.

Proof. ($\Longleftarrow$) Suppose $(\Pi, \overline{F}_2)$ is a solution of (8.7)-(8.8). Consider the controller described by (8.9)-(8.10). We will show this controller is a regulator with $\xi = \hat{\eta}$. First check the the asymptotic stability requirement. Suppose $w(t) = 0$. Then the dynamics of the closed-loop system are given by

$$\begin{bmatrix} \dot{x} \\ \dot{\delta\eta} \end{bmatrix} = \begin{bmatrix} \overline{A} & -BF \\ 0 & (A_c - LC_c) \end{bmatrix} \begin{bmatrix} x \\ \delta\eta \end{bmatrix}.$$

Since the system matrix is Hurwitz we have $x(t) \to 0$ as desired.

Next we consider the regulation requirement. Define $z = x - \Pi w$. We have

$$\begin{aligned} \dot{z} &= Ax + B(F_1\hat{x} + F_2\hat{w}) + Dw - \Pi S w \\ &= \overline{A}z + [\overline{A}\Pi + BF_2 + D - \Pi S]w - BF_1\delta x - BF_2\delta w \\ &= \overline{A}z - BF_1\delta x - BF_2\delta w\,. \end{aligned}$$

Combining with the dynamics of $\delta\eta$ we have the composite dynamics

$$\begin{bmatrix} \dot{z} \\ \dot{\delta\eta} \end{bmatrix} = \begin{bmatrix} \overline{A} & -BF \\ 0 & (A_c - LC_c) \end{bmatrix} \begin{bmatrix} z \\ \delta\eta \end{bmatrix}.$$

Hence $z(t) \to 0$. Next we have

$$e(t) = Ex - E_d w = Ez + (E\Pi - E_d)w = Ez\,.$$

Hence, $e(t) \to 0$ as desired.

($\Longrightarrow$) Suppose we have a regulator of the form (8.2)-(8.3). When $w(t) = 0$, the closed-loop dynamics are

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} A & BF \\ HC & G \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix}.$$

By assumption, the closed-loop dynamics are asymptotically stable. Now consider the Sylvester equation

$$\begin{bmatrix} \Pi \\ \Sigma \end{bmatrix} S = \begin{bmatrix} A & BF \\ HC & G \end{bmatrix} \begin{bmatrix} \Pi \\ \Sigma \end{bmatrix} + \begin{bmatrix} D \\ HC_w \end{bmatrix}. \tag{8.11}$$

Because the eigenvalues of S and of the closed-loop system matrix are disjoint, by Sylvester's theorem there exists a unique solution for (Π, Σ). In particular, setting $\overline{F}_2 = F\Sigma$, we obtain (8.7). Next, let $z_1 = x - \Pi w$ and $z_2 = \xi - \Sigma w$. Then using (8.11) we obtain

$$\dot{z} = \begin{bmatrix} A & BF \\ HC & G \end{bmatrix} z$$

so $z(t) \to 0$. Finally, we get $e(t) = Ez(t) + (E\Pi - E_d)w(t)$. By assumption $e(t) \to 0$. Also, $z(t) \to 0$. Since $eig(S) \subset \mathbb{C}^+$, for all $w(0)$, $w(t) \to 0$. This implies $E\Pi = E_d$, as desired. □