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Performance Limitations of the Servomechanism Problem when the Number of Tracking/Disturbance Poles Increases

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***Abstract*— In this paper, we study the cheap control problem and determine what some of the inherent system limitations are in achieving high performance for LTI systems. In particular, we observe that a fundamental difficulty in designing a high performance controller for a system may occur, which is related to the infinite transmission zero structure of the system. A continuous measure, called the *Toughness Index*, is introduced to characterize such limitations. We then apply these results to the robust servomechanism problem (RSP), and show that the Toughness Index of the RSP becomes worst as the number of tracking/disturbance poles to be tracked/regulated increases. This implies that high performance control in the RSP cannot be obtained for a large number of tracking/disturbance poles, even for minimum phase systems.**

I. INTRODUCTION

It is well known in both classical and modern linear control theory that a closed-loop system can often achieve high performance if the feedback controller gains are allowed to be sufficiently large. In fact, under certain criteria, "perfect control" ([1], [2]) can be achieved if the control effort is allowed to be arbitrarily large. Of course in practice, one cannot implement extremely large controller gains, and practical restrictions such as controller actuator sizing limits, impose a limitation on the fastest response achievable by the closed-loop system. Some systems, however, may be more difficult to control than others, and we are interested in studying and being able to characterize what are some of the inherent difficulties a system may have in achieving high performance.

In our approach, we consider high performance controllers obtained via the *cheap control problem*. In particular, consider the following stabilizable and detectable linear time-invariant (LTI) system as modeled by

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = x_0, \\ y &= Cx, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^r$ are the system states, inputs, and outputs, respectively. The cheap control problem consists of finding a stabilizing feedback controller that minimizes the following quadratic performance index:

$$J_\epsilon = \min_u \int_0^\infty \left(y^T Q y + \epsilon u^T R u\right) dt, \tag{2}$$

where $\epsilon > 0$ is a small positive scalar, Q is positive definite, and R is positive definite. In the case as $\epsilon \to 0$, the cost of the control effort decreases, subsequently allowing larger controller gains, and resulting in faster output response times.

The cheap LQR problem has been extensively studied in the past (e.g., see [2]–[7]). One of the most well-known results is that perfect regulation (i.e., $J_{\epsilon\to 0} = 0$) can be achieved if and only if the system is minimum phase and right invertible (e.g., see [2], [4], [6]). If the system is nonminimum phase, then there exists a fundamental performance limitation in controller design, which for some cases such as the servomechanism problem, can be characterized in terms of the number and locations of the nonminimum phase zeros [8]. Another system limitation is the rate at which a closed-loop system's faraway poles approach infinity as $\epsilon \to 0$, which is the focus of our study. For the cheap control problem, it is well known that as $\epsilon \to 0$, the closed-loop poles behave in such a way that i) some poles asymptotically approach the system's stable finite transmission zeros, ii) some approach the mirror-image of the unstable transmission zeros, and iii) all the other poles approach infinity in various Butterworth patterns at a rate relative to $\frac{1}{\epsilon^k}$ for some $k > 0$ (e.g., see [4]). Since the closed-loop poles have a direct effect on the system's response time, the rate at which the faraway poles approach infinity as $\epsilon \to 0$ poses a limitation on system performance. Other inherent difficulties a system may have in transient control include unbounded peaking as $\epsilon \to 0$ ([9], [10]), and singular initial behaviour [6].

As mentioned earlier, the focus of this paper is to investigate system limitations due to the faraway closed-loop poles. In particular, it is noted that if the rate at which the faraway poles approach infinity is slow, then in order to achieve high performance, one is required to choose ϵ to be extremely small, resulting in impractically large controller gains. So subsequently if one has limited controller gain or control effort, then a slow rate poses a fundamental performance limitation, as we will see in some examples later on. In this paper, we introduce a measure based on this observation, called the Toughness Index, to characterize a system's ability to achieve high performance using cheap control.

This paper is organized as follows. First, some topics related to cheap control are reviewed in Section II. Section III proposes a method for determining the asymptotic rates at which the faraway poles approach infinity for the cheap control problem, and defines a Toughness Index for the system based on these rates. Section IV applies the Toughness Index to study the robust servomechanism problem [11], and shows that as the number of tracking/disturbance poles increases, there are fundamental limits in obtaining high performance in the resultant closed-loop system, even if the

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system is minimum phase. Finally, a few numerical examples are looked at in Section V.

II. Preliminaries

In this paper, a transfer matrix realization, $H(s)$, of the system (1) is given by

$$H(s) = C\,(sI - A)^{-1}\,B. \tag{3}$$

The finite transmission zeros [12] of (1) are the points in the complex plane that satisfy

$$\operatorname{rank}\left(\begin{bmatrix} A - sI & B \\ C & 0 \end{bmatrix}\right) < n + \min(m, r),$$

and are the roots of the numerator polynomials of the nonzero elements of the Smith-McMillan form of (3). System (1) or (3) is said to be *minimum phase* if it has no transmission zeros in the closed right-half part of the complex plane, otherwise it is said to be *non-minimum phase*. The transfer matrix $H(s)$ is said to have an infinite transmission zero of order k (i.e., $1/s^k$) if $H(1/s)$ has a finite transmission zero of precisely that order at $s = 0$.

A. Asymptotic locations of the optimal closed-loop poles

Recall that given a LTI system (1), the optimal control law that minimizes the performance index (2) is

$$u = -\frac{1}{\epsilon} B^T P_\epsilon x, \tag{4}$$

where P_ϵ is the unique positive semidefinite solution to the algebraic Riccati equation (ARE)

$$A^T P_\epsilon + P_\epsilon A + C^T Q C - \frac{1}{\epsilon} P_\epsilon B R^{-1} B^T P_\epsilon = 0. \tag{5}$$

The optimal closed-loop poles of $A - \frac{1}{\epsilon} B R^{-1} B^T P_\epsilon$ are the stable eigenvalues of the Hamiltonian matrix (e.g., see [4])

$$Z = \begin{bmatrix} A & -\frac{1}{\epsilon} B R^{-1} B^T \\ -C^T Q C & -A^T \end{bmatrix}, \tag{6}$$

or equivalently, the stable roots of

$$det\left(I + \frac{1}{\epsilon} R^{-1} H^T(-s)\, Q H(s)\right) = 0, \tag{7}$$

where $H(s)$ is the transfer matrix of (1) . In the case when system (1) is square (i.e., $m = r$), $\det(H(s))$ is given by

$$\det(H(s)) = \frac{\alpha \prod_{i=1}^{p} (s - z_i)}{\prod_{i=1}^{n} (s - p_i)}$$

where α is a constant, p_i, for $i = 1, \ldots, n$, are the eigenvalues of system (1), and z_i, for $i = 1, \ldots, p$, are the finite transmission zeros of (1). We now recall the following result [4].

Lemma 2.1: ([4]) Given (1), assume that $m = r$; then it follows from (7) that as $\epsilon \to 0$, the following is true:

- p of the optimal closed-loop poles approach $\hat{z}_i$, where

$$\hat{z}_i = \begin{cases} z_i & \text{if } \operatorname{Re} z_i \le 0 \\ -z_i & \text{if } \operatorname{Re} z_i > 0 \end{cases};$$

- the remaining (faraway) closed-loop poles approach infinity in various Butterworth configurations of different orders and radii; and
- a "rough estimate" of the faraway poles' distance to the origin as $\epsilon \to 0$ is given by

$$\left(\frac{\alpha^2}{\epsilon^m}\right)^{1/(2(n-p))}. \tag{8}$$

III. Main Results

In this section, we study how to compute the *exact* asymptotic rates at which the individual faraway closed-loop poles approach infinity as $\epsilon \to 0$. As shown in the previous section, a "rough estimate" of the rates is given by (8); however, the estimate is rather crude (e.g., see [4, Example 3.21]), and also does not provide an estimate for each individual pole. To compute the individual faraway poles for a given ϵ, one can directly solve the Hamiltonian matrix (6) for the stable eigenvalues. However, for systems with large dimensions (e.g., $n > 50$), this method may have very severe numerical problems when ϵ is chosen to be small (e.g., $\epsilon = 10^{-12}$), and so cannot be used. Instead, we use an alternative method to determine the faraway poles by computing the closed-loop eigenvalues of a reduced model based on the system's infinite transmission zeros. In the remainder of this paper, we assume, unless specified otherwise, that the LTI system has equal inputs and outputs.

A. Reduced model based on $H(s)$ approximation

Let $H(s)$ be a strictly proper transfer matrix realization of (1), and expand $H(s)$ in a Laurent series about the origin $s = 0$ as follows:

$$H(s) = \sum_{k=1}^{\infty} \frac{C A^{k-1} B}{s^k} \tag{9}$$

Assuming that the infinite transmission zeros of $H(s)$ are $[1/s^{p_1}, \ldots, 1/s^{p_k}]$, denote $p^* = \max(p_1, \ldots, p_k)$. Now approximate $H(s)$ by the truncated series $\hat{H}(s)$ given by

$$\hat{H}(s) = \sum_{k=1}^{p^*} \frac{C A^{k-1} B}{s^k}, \tag{10}$$

where the approximation is valid for large s. It can easily be shown that $\hat{H}(s)$ and $H(s)$ have identical infinite transmission zeros. Also, as $\epsilon \to 0$, the faraway closed-loop poles approach infinity and the approximation $\hat{H}(s)$ for large s improves; so on replacing $H(s)$ with $\hat{H}(s)$ in (7) for $\epsilon \to 0$, we obtain

$$\begin{aligned} 0 &= \det\left(I + \frac{1}{\epsilon} R^{-1} \hat{H}^T(-s)\, Q \hat{H}(s)\right) \\ &= \det\left(s^{2p^*} I + \frac{1}{\epsilon} \sum_{k=0}^{2p^*-2} (-1)^k M_k s^k\right), \end{aligned} \tag{11}$$

where M_k, for $k = 0, \ldots, 2p^* - 2$, is given by

$$M_k = \sum_{j=0}^{k} R^{-1} \left(C A^{p^*-1-j} B\right)^T Q C A^{p^*-1-k+j} B,$$

and where for notational convenience, we assume that $A^i = 0$ for $i < 0$. In this case, on applying controller (4), the faraway poles for a given ϵ can be obtained by computing the stable roots of the matrix polynomial (11), or equivalently, the stable eigenvalues of the matrix $\mathcal{A}_\epsilon$, where

$$\mathcal{A}_\epsilon = \begin{bmatrix} 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & I \\ \frac{1}{\epsilon}M_0 & -\frac{1}{\epsilon}M_1 & \frac{1}{\epsilon}M_2 & \cdots & \frac{1}{\epsilon}M_{2(p^*-1)} & 0 \end{bmatrix}. \tag{12}$$

B. Toughness Index

Using the same derivation for obtaining the estimate (8), one can show that the distances of the faraway poles of (11) from the origin, or equivalently, the distances of the faraway eigenvalues of (12), asymptotically approach the following as $\epsilon \to 0$:

$$\left(\frac{\alpha_i}{\epsilon}\right)^{\beta_i} \tag{13}$$

for some constant $\alpha_i > 0$ and $\beta_i > 0$, for $i = 1, 2, \ldots, p^*$. To compute α_i and β_i, the following procedure is used.

First compute the eigenvalues of $\mathcal{A}_\epsilon$ for $\epsilon = \epsilon_1$ and $\epsilon = \epsilon_2$, where $\epsilon_1 > \epsilon_2$, and ϵ_1 and ϵ_2 are chosen sufficiently small such that the distances of the faraway eigenvalues for ϵ_1 and ϵ_2 are closely described by (13). Assume the subsequent eigenvalues are given by

$$\text{For } \epsilon = \epsilon_1: \qquad \lambda(\mathcal{A}_{\epsilon_1}) = \left\{ s_1^{(\epsilon_1)}, s_2^{(\epsilon_1)}, \ldots, s_{p^*}^{(\epsilon_1)} \right\},$$

$$\text{For } \epsilon = \epsilon_2: \qquad \lambda(\mathcal{A}_{\epsilon_2}) = \left\{ s_1^{(\epsilon_2)}, s_2^{(\epsilon_2)}, \ldots, s_{p^*}^{(\epsilon_2)} \right\}.$$

Then, compute

$$\beta_i = \frac{\log_{10}\left(\left|s_i^{(\epsilon_2)}\right| / \left|s_i^{(\epsilon_1)}\right|\right)}{\log_{10}(\epsilon_1/\epsilon_2)}, \quad \text{for } i = 1, 2, \ldots, p^* \tag{14}$$

and

$$\alpha_i = \epsilon_1 \left(\left|s_i^{(\epsilon_1)}\right|\right)^{1/\beta_i}, \quad \text{for } i = 1, 2, \ldots, p^*. \tag{15}$$

We can now make the following definition.

Definition 3.1 (Toughness Indices): Given (C, A, B) of system (1), positive definite Q, positive definite R, and $0 < \epsilon \ll 1$, let p^* denote the number of optimal faraway closed-loop poles. Then for a given $\epsilon > 0$, define the set of *Toughness Indices* for system (1) with the performance index (2) as follows:

$$Index_i = \left(\frac{\alpha_i}{\epsilon}\right)^{\beta_i}, \quad \text{for } i = 1, 2, \ldots, p^*, \tag{16}$$

where α_i and β_i, for $i = 1, \ldots, p^*$, are given by (14) and (15). Also, for a given $\epsilon > 0$, define the *Toughness Index* for the overall system (1) as

$$ToughnessIndex = \min_i \left(\frac{\alpha_i}{\epsilon}\right)^{\beta_i}. \tag{17}$$

The Toughness Index as defined in (17) gives an indication of a system's difficulty in achieving high performance by cheap control. In particular, the Toughness Index (17) corresponds to the dominant optimal faraway closed-loop pole's distance from the origin for $\epsilon \to 0$. So if the Toughness Index is small, even for small ϵ, then it implies that the dominate faraway pole is very close to the origin even when the control effort is allowed to be relatively cheap. This in turn implies that the closed-loop system will have a slow response time, even when large controller gains are used.

IV. Application: Robust Servomechanism Problem

In this section, we review and study the robust servomechanism problem (RSP) [11] for the case when there is a large number of tracking/disturbance poles to be tracked/regulated.

Consider the following LTI system with disturbances:

$$\begin{aligned} \dot{x} &= Ax + Bu + Ew \\ y &= Cx + Du + Fw \\ e &= y_{ref} - y, \end{aligned} \tag{18}$$

where $x \in \mathbb{R}^n$ is the state of the system, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^r$ is a measurable output, $w \in \mathbb{R}^\Omega$ is an unmeasurable disturbance, $e \in \mathbb{R}^r$ is the error, and $y_{ref} \in \mathbb{R}^r$ is a specified tracking signal. Denote λ_i as a tracking/disturbance pole [11], where $\operatorname{Re}\lambda_i \geq 0$, for $i = 1, \ldots, p$, and let the coefficients of the tracking/disturbance polynomial be

$$\prod_{i=1}^{p} (\lambda - \lambda_i) = \lambda^p + \delta_p \lambda^{p-1} + \cdots + \delta_2 \lambda + \delta_1. \tag{19}$$

The control objective is then to solve the RSP [11] for the system (18) with respect to the class of tracking/disturbance signals described by (19); i.e., find a LTI controller such that

i) the closed-loop system is asymptotically stable;
ii) asymptotic tracking/regulation occurs; i.e.,

$$\lim_{t\to\infty} e(t) = 0$$

for all initial conditions; and
iii) conditions (i)-(ii) hold for any arbitrary perturbations in the plant model (18) that do not cause the resultant perturbed closed-loop system to be unstable.

The following existence conditions for a solution to the RSP to exist are obtained from [11].

Lemma 4.1 ([11]): There exists a solution to the RSP for (18) with respect to the tracking/disturbance poles of (19) if and only if the following conditions are all true:

i) (C, A, B) is stabilizable and detectable;
ii) the number of inputs is more than or equal to the number of outputs; i.e., $m \geq r$; and
iii) the transmission zeros of (C, A, B, D) exclude the tracking/disturbance poles (19).

Assume now that the existence conditions of Lemma 4.1 are satisfied; then this implies that there exists $K_0 \in \mathbb{R}^{m\times n}$ and $K_1 \in \mathbb{R}^{m\times rp}$, and a LTI controller of the form

$$\hat{v} = K_0\hat{x} + K_1\hat{\eta}, \tag{20}$$

where $\hat{x} \in \mathbb{R}^n$ and $\hat{\eta} \in \mathbb{R}^{rp}$ are the inputs, and $\hat{v} \in \mathbb{R}^m$ are the outputs of the controller, such that (20) will stabilize the following *plant-servocompensator augmented system* [11]:

$$\begin{aligned} \begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{\eta}} \end{bmatrix} &= \begin{bmatrix} A & 0 \\ \tilde{\mathcal{B}}C & \tilde{\mathcal{C}} \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{\eta} \end{bmatrix} + \begin{bmatrix} B \\ \tilde{\mathcal{B}}D \end{bmatrix} \hat{v} \qquad (21) \\ z &= \begin{bmatrix} 0 & \tilde{\mathcal{D}} \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{\eta} \end{bmatrix}, \end{aligned}$$

where $\tilde{\mathcal{C}}$, $\tilde{\mathcal{B}}$, and $\tilde{\mathcal{D}}$ are given by

$$\tilde{\mathcal{C}} := \text{block diag}(\underbrace{\mathcal{C}, \dots, \mathcal{C}}_{r}), \quad \tilde{\mathcal{B}} := \text{block diag}(\underbrace{\mathcal{B}, \dots, \mathcal{B}}_{r}),$$

$$\mathcal{C} := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\delta_1 & -\delta_2 & -\delta_3 & \cdots & -\delta_p \end{bmatrix}, \quad \mathcal{B} := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

and for $\mathcal{D} = [1, 0, 0, \dots, 0] \in \mathbb{R}^{1 \times p}$,

$$\tilde{\mathcal{D}} := \text{block diag}\left(\underbrace{\mathcal{D}, \dots, \mathcal{D}}_{r}\right).$$

Using the above results, a controller that solves the RSP for system (18) is then given by

$$\dot{\eta} = \tilde{\mathcal{C}}\eta + \tilde{\mathcal{B}}e \qquad (22a)$$

$$u = K_0 x + K_1 \eta, \qquad (22b)$$

where $e \in \mathbb{R}^r$ and $u \in \mathbb{R}^m$ are the controller inputs and outputs, respectively, K_0 and K_1 are the same as given in (20), the *servocompensator* [11] of the controller is given in (22a), and the state x can be estimated using an observer.

The augmented system (21) has the following property.

Lemma 4.2: Let $[1/s^{p_1}, \dots, 1/s^{p_k}]$ be the infinite transmission zeros of (C, A, B, D) in (18); then $\left[1/s^{(p_1+N)}, \dots, 1/s^{(p_k+N)}\right]$ are the infinite transmission zeros of $\left(\begin{bmatrix} 0 & \tilde{\mathcal{D}} \end{bmatrix}, \begin{bmatrix} A & 0 \\ \tilde{\mathcal{B}}C & \tilde{\mathcal{C}} \end{bmatrix}, \begin{bmatrix} B \\ \tilde{\mathcal{B}}D \end{bmatrix}\right)$, where N is the number of tracking/disturbance poles.

Proof: See Appendix. ■

Assume now that the controller (20) is found to minimize the following cheap performance index:

$$J = \int_0^\infty \left(z^T z + \epsilon \hat{v}^T \hat{v}\right) d\tau \qquad (23)$$

for some given $0 < \epsilon \ll 1$. From Lemma 4.2, we see that if the Toughness Index of the original plant (C, A, B) is found to be $\left(\frac{\alpha}{\epsilon}\right)^{(1/(2k))}$, for some $\alpha > 0$ and $k > 0$, then the Toughness Index of the plant-servocompensator augmented system (21) is $\left(\frac{\gamma}{\epsilon}\right)^{(1/(2k+2N))}$ for some $\gamma > 0$. *So if one is to design a RSP controller with a large number of tracking/disturbance poles (as is often the case), then one may encounter difficulty trying to achieve high performance, as measured by the Toughness Index.* We will see some examples of this latter point in the following section.

V. Numerical Examples

We now study four examples.

A. Example 1: Longitudinal control of an airplane

Consider the following LTI system found in Example 3.21 of [4] given by

$$A = \begin{bmatrix} -0.01580 & 0.02633 & -9.810 & 0 \\ -0.1571 & -1.030 & 0 & 120.5 \\ 0 & 0 & 0 & 1 \\ 5.274_{10^{-4}} & -0.01652 & 0 & -1.466 \end{bmatrix}, \qquad (24)$$

$$B = \begin{bmatrix} 6.056_{10^{-4}} & 0 \\ 0 & -9.496 \\ 0 & 0 \\ 0 & -5.565 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The system (24) has a finite transmission zero at $s = -1.002$ and infinite transmission zeros $\left[1/s, 1/s^2\right]$.

Using the approach in Section III, the magnitudes of the stable eigenvalues of (12) for $Q = I$, $R = I$, and ϵ equal to 10^{-9}, 10^{-10}, 10^{-11}, and 10^{-12} are given below:

	$\epsilon_1 = 10^{-9}$	$\epsilon_2 = 10^{-10}$	$\epsilon_3 = 10^{-11}$	$\epsilon_4 = 10^{-12}$
$\lvert s_i^{(\epsilon)} \rvert$	$4.197_{10^{+2}}$	$7.464_{10^{+2}}$	$1.327_{10^{+3}}$	$2.360_{10^{+3}}$
	$4.197_{10^{+2}}$	$7.464_{10^{+2}}$	$1.327_{10^{+3}}$	$2.360_{10^{+3}}$
	$1.913_{10^{+1}}$	$6.050_{10^{+1}}$	$1.913_{10^{+2}}$	$6.050_{10^{+2}}$

and on calculating the Toughness Indices of Section III-B, we obtain

	$\epsilon_2 = 10^{-10}$	$\epsilon_3 = 10^{-11}$	$\epsilon_4 = 10^{-12}$
β_i	0.25	0.25	0.25
	0.25	0.25	0.25
	0.50	0.50	0.50

and

	$\epsilon_2 = 10^{-10}$	$\epsilon_3 = 10^{-11}$	$\epsilon_4 = 10^{-12}$
α_i	$3.103_{10^{+1}}$	$3.104_{10^{+1}}$	$3.106_{10^{+1}}$
	$3.103_{10^{+1}}$	$3.104_{10^{+1}}$	$3.106_{10^{+1}}$
	$3.660_{10^{-7}}$	$3.660_{10^{-7}}$	$3.659_{10^{-7}}$

.

Hence for a given $0 < \epsilon \ll 1$, the Toughness Indices (16), which correspond to the optimal closed-loop faraway poles' distances from the origin, are given by

$$\left(\frac{31.0}{\epsilon}\right)^{0.25}, \left(\frac{31.0}{\epsilon}\right)^{0.25}, \left(\frac{3.66_{10^{-7}}}{\epsilon}\right)^{0.5}, \qquad (25)$$

while the Toughness Index for the overall system is given by $\min\left(\left(\frac{31.0}{\epsilon}\right)^{0.25}, \left(\frac{3.66_{10^{-7}}}{\epsilon}\right)^{0.5}\right)$. To verify these results, we compare the faraway poles as given by (25) with the optimal closed-loop poles obtained by solving for the actual optimal controller (4), and we observe that there is a strong agreement, as shown below.

ϵ	$s \in \lambda\big(A - \frac{1}{\epsilon}BB^T P_\epsilon\big)$	$\lvert s\rvert$	$\big(\frac{\alpha_i}{\epsilon}\big)^{\beta_i}$
10^{-10}	$-5.278_{10+2} + j5.278_{10+2}$	7.464_{10+2}	7.466_{10+2}
	$-5.278_{10+2} - j5.278_{10+2}$	7.464_{10+2}	7.466_{10+2}
	-6.087_{10+1}	6.087_{10+1}	6.050_{10+1}
	-1.009	1.009	–
10^{-11}	$-9.385_{10+2} + j9.385_{10+2}$	1.327_{10+3}	1.327_{10+3}
	$-9.385_{10+2} - j9.385_{10+2}$	1.327_{10+3}	1.327_{10+3}
	-1.914_{10+2}	1.914_{10+2}	1.913_{10+2}
	-1.003	1.003	–
10^{-12}	$-1.669_{10+3} + j1.669_{10+3}$	2.360_{10+3}	2.361_{10+3}
	$-1.669_{10+3} - j1.669_{10+3}$	2.360_{10+3}	2.361_{10+3}
	-6.050_{10+2}	6.050_{10+2}	6.049_{10+2}
	-1.002	1.002	–

B. Example 2: A simple example

Given the following two unstable plants described by

$$H_1(s) = \frac{1}{s^{50}} \quad \text{and} \quad H_2(s) = \frac{(s-1)^{40}}{s^{50}},$$

it is desired to determine the asymptotic distances of the two systems' closed-loop poles controlled using optimal state feedback and minimizing the cheap control performance index (2) for $Q = I$, $R = I$, and some $0 < \epsilon \ll 1$.

In this case, the standard optimization algorithm `lqr.m` of MATLAB fails for all $\epsilon < 1$ due to numerical problems. Therefore it is not possible to study the asymptotic behaviour for $\epsilon \to 0$ by simply using `lqr.m` to compute the closed-loop poles. However, using the proposed numerical procedure of Section III, it is immediately obtained that the Toughness Index for $H_1(s)$ is given by $(1/\epsilon)^{0.01}$, and for $H_2(s)$ is given by $(1/\epsilon)^{0.05}$. So for $H_1(s)$ and a sufficiently small ϵ, at least one of the faraway closed-loop poles of the resultant closed-loop system (if implemented) would be at a distance D from the origin, where $D = (1/\epsilon)^{0.01}$; i.e., if $\epsilon = 10^{-10}$, then $D = 1.259$, and if $\epsilon = 10^{-100}$, then $D = 10$. Since a distance of $D = 10$ approximately relates to at least a time constant of 0.1, it is therefore virtually impossible to obtain a relatively fast settling time for $H_1(s)$ using cheap control. In comparison, the distance of the faraway closed-loop poles of $H_2(s)$ for $\epsilon = 10^{-10}$ and $\epsilon = 10^{-100}$ would be $D = 3.1623$ and $D = 10^5$, respectively.

One could also have studied the asymptotic distance of the closed-loop poles by computing the eigenvalues of the Hamiltonian matrix (6), instead of solving the full LQR problem. However, it should be noted that computing the eigenvalues of (12) in the proposed procedure of Section III is often more numerically sound than computing the eigenvalues of the Hamiltonian matrix (6), especially for systems with infinite transmission zeros of low order. In this example, for instance, a state-space realization of the second plant $H_2(s)$ has 50 states ($n = 50$), and infinite transmission zeros $\big[1/s^{10}\big]$. The dimension of the corresponding Hamiltonian matrix (6) is then of order 100. On the other hand, the dimension of the $\mathcal{A}_\epsilon$ matrix in (12) is only 20, so computing the eigenvalues of $\mathcal{A}_\epsilon$ is numerically more reliable than computing the eigenvalues of the Hamiltonian matrix.

TABLE I
SUMMARY OF MASS-SPRING EXAMPLE FOR TRACKING/DISTURBANCE POLES = $[0, \pm j1]$

ϵ	$\lVert [\,K_0 \quad K_1\,] \rVert$	trace(P_ϵ)	TZ_∞
10^{-4}	1.588_{10+2}	3.934	$\big[1/s^5, 1/s^5\big]$
10^{-8}	1.121_{10+4}	1.114	$\big[1/s^5, 1/s^5\big]$
10^{-12}	1.021_{10+6}	0.4140	$\big[1/s^5, 1/s^5\big]$

C. Example 3: RSP for a mass-spring system

Consider a mass-spring system, where two smaller masses are attached to larger mass via springs and dampers. The outputs are the outputs of the two smaller masses and the inputs are the forces applied to the two smaller masses. The system can be modeled by the following LTI system:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -0.2 & -0.02 & 0.1 & 0.01 & 0.1 & 0.01 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0.1 & -1 & -0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0.1 & 0 & 0 & -1 & -0.1 \end{bmatrix}, \tag{26}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T, C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The system (26) is open-loop unstable, has two minimum phase finite transmission zeros at $s = -0.0100 \pm j0.447$, and infinite transmission zeros $\big[1/s^2, 1/s^2\big]$.

Suppose we want to solve the RSP for (26) with tracking/disturbance poles $[0, \pm j1]$. In this case, the resultant augmented system to be stabilized has infinite transmission zeros $\big[1/s^5, 1/s^5\big]$. On applying the results of Section III, it is determined that the Toughness Index is given by

$$\left(\frac{1}{\epsilon}\right)^{1/10}, \tag{27}$$

which indicates that a relatively slow transient response will occur. On solving the RSP using state feedback and minimizing the cheap performance index (23) with $Q = I$, $R = I$, and $\epsilon = \big(10^{-4}, 10^{-8}, 10^{-12}\big)$, the results are obtained for a tracking reference signal of $y_{ref} = 1$ and is presented in Figures 1-2. From these figures[1], we see that for $\epsilon = 10^{-12}$, the system achieves a response with a settling time of ~ 1 sec. For these chosen ϵ, Table I summarizes the size of the corresponding controller gains (20), and the performance cost given by the trace of P_ϵ, where P_ϵ is the solution to the associated ARE (5), and TZ_∞ denotes infinite transmission zeros.

Now suppose we want to solve the RSP for (26) with tracking/disturbance poles $[0, \pm j1, \pm j2, \pm j4, \pm j8, \pm j10]$. The resultant augmented system now has infinite transmission zeros $\big[1/s^{13}, 1/s^{13}\big]$. Figures 3-4 display the input and output responses for $\epsilon = \big(10^{-12}, 10^{-18}, 10^{-24}\big)$, and Table II summarizes the corresponding controller gains and performance costs. From these figures, it can be seen that

[1]Note that in the figures, Output 1 and Output 2 actually coincide with each other very closely.

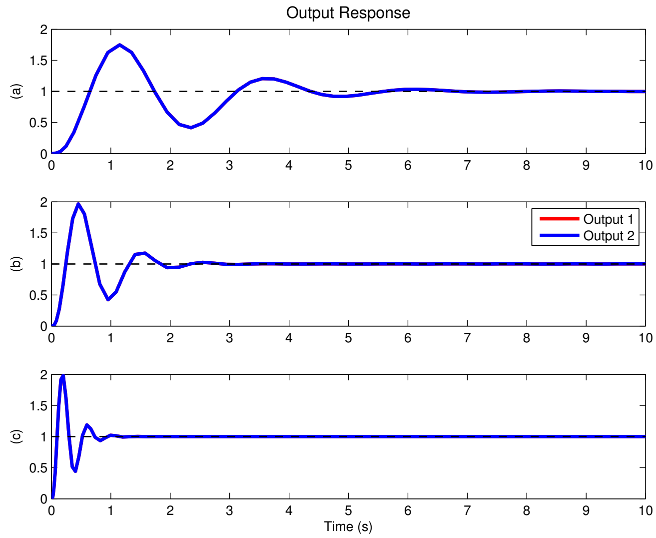

Fig. 1. Output response of the mass-spring system using cheap servomechanism control with tracking/disturbance poles $[0, \pm j1]$, and (a) $\epsilon = 10^{-4}$, (b) $\epsilon = 10^{-8}$, and (c) $\epsilon = 10^{-12}$.

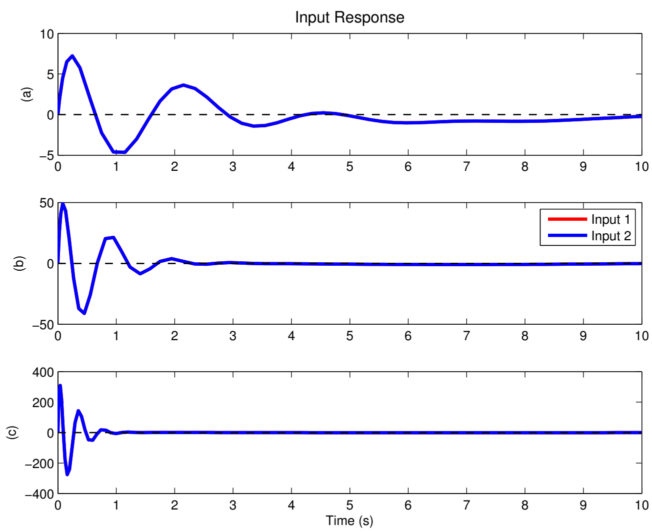

Fig. 2. Input response of the mass-spring system using cheap servomechanism control with tracking/disturbance poles $[0, \pm j1]$, and (a) $\epsilon = 10^{-4}$, (b) $\epsilon = 10^{-8}$, and (c) $\epsilon = 10^{-12}$.

with the additional tracking/disturbance poles, the system has difficulty achieving a fast response. In particular, even with a large controller gain corresponding to $\epsilon = 10^{-24}$ (and input magnitudes similar to the previous response shown in Figure 2), the settling time is about ~ 7 sec. On computing the Toughness Indices (16), it is determined that all the faraway poles approach infinity at a rate

$$\left(\frac{1}{\epsilon}\right)^{1/26}, \tag{28}$$

which is much slower than (27).

D. *Example 4: Control of a commercial hard disc drive*

The study of obtaining high performance control of a commercial hard disk drive system was carried out in [13]. The plant model describing the disk drive model in this case was a SISO LTI system of high order ($\gg 10$), and it was desired to reject unknown and unmeasurable disturbances using a RSP controller design. In this case the class of disturbances to be rejected were modeled by harmonic sinusoidal signals with five frequencies given by $180\pi \times (1, 2, 3, 5, 24)$ radians/sec.

TABLE II
SUMMARY OF MASS-SPRING EXAMPLE FOR TRACKING/DISTURBANCE POLES = $[0, \pm j1, \pm j2, \pm j4, \pm j8, \pm j10]$

ϵ	$\|[\ K_0\ \ K_1\ ]\|$	trace(P_ϵ)	TZ_∞
10^{-12}	3.207_{10+6}	3.479_{10+1}	$[1/s^{13}, 1/s^{13}]$
10^{-18}	2.159_{10+9}	6.249_{10+0}	$[1/s^{13}, 1/s^{13}]$
10^{-24}	1.428_{10+12}	2.430_{10+0}	$[1/s^{13}, 1/s^{13}]$

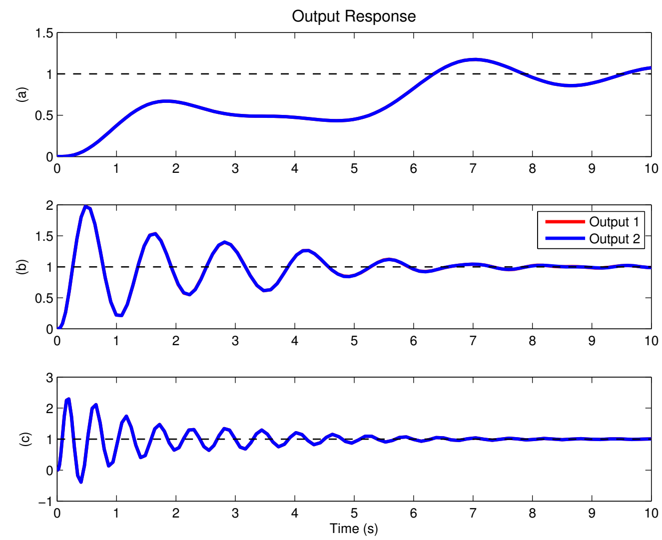

Fig. 3. Output response of the mass-spring system using cheap servomechanism control with tracking/disturbance poles $[0, \pm j1, \pm j2, \pm j4, \pm j8, \pm j10]$, and (a) $\epsilon = 10^{-12}$, (b) $\epsilon = 10^{-18}$, and (c) $\epsilon = 10^{-24}$.

The resultant RSP controller design was highly effective compared to conventional approaches, but the following observation was made in [13]:

> "For this initial servocompensator design, one can see in Figure 3 that an undesirably long settling time occurs due to the relatively slow error attenuation of the sinusoidal harmonic having frequency component ω_5."

This undesirable long settling time is precisely the type of result that would be predicted by the Toughness Index. Due to the large number of tracking/disturbance poles required to be rejected in this study (a total of 10), extremely high controller gains would be required to speed up the disturbance rejection properties of the servo-controller, but such high controller gains are completely unrealistic to use, and so some of the closed-loop poles of the system were sluggish.

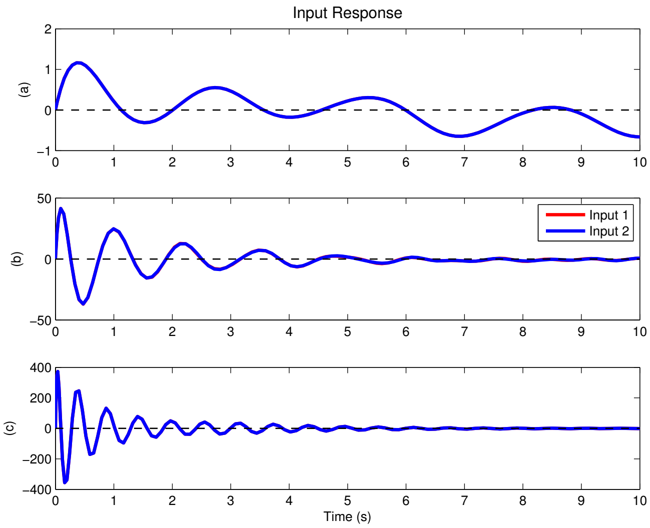

Fig. 4. Input response of the mass-spring system using cheap servomechanism control with tracking/disturbance poles $[0, \pm j1, \pm j2, \pm j4, \pm j8, \pm j10]$, and (a) $\epsilon = 10^{-12}$, (b) $\epsilon = 10^{-18}$, and (c) $\epsilon = 10^{-24}$.

VI. Conclusions

In this paper, a Toughness Index is proposed to characterize a LTI system's difficulty in using cheap control to achieve high performance. The Toughness Index measures the distances of the faraway optimal closed-loop poles from the origin, and indicates that there exists a performance limitation if the distances are small, even when the cost of the control effort is made relatively cheap. An important observation is then made, which is that in solving the RSP, the Toughness Index becomes worst as the number of tracking/disturbance poles increases. Thus even for minimum phase systems, there is a performance limitation that occurs when trying to solve the RSP for a large number of tracking/disturbance poles.

Appendix

Proof of Lemma 4.2

Denote

$$H(s) = \begin{bmatrix} 0 & \tilde{\mathcal{D}} \end{bmatrix} \left(sI - \begin{bmatrix} A & 0 \\ \tilde{\mathcal{B}}C & \tilde{\mathcal{C}} \end{bmatrix} \right)^{-1} \begin{bmatrix} B \\ \tilde{\mathcal{B}}D \end{bmatrix}.$$

From the diagonal structure of $\tilde{\mathcal{B}}$, $\tilde{\mathcal{C}}$, and $\tilde{\mathcal{D}}$ in (21), it can easily be shown that

$$H(s) = \text{diag}(\underbrace{H_2(s), \ldots, H_2(s)}_{r})H_1(s), \qquad (29)$$

where $H_1(s) := C(sI - A)^{-1} + D$ and $H_2(s) := \mathcal{D}(sI - \mathcal{C})^{-1}\mathcal{B}$. Now given that the infinite transmission zeros of $H_1(s)$ are $[1/s^{p_1}, \ldots, 1/s^{p_k}]$, there exists the following Smith-McMillan factorization at infinity [14] for (C, A, B, D):

$$H_1(s) = B_1(s) \begin{bmatrix} \Lambda(s) & 0 \\ 0 & 0 \end{bmatrix} B_2(s), \qquad (30)$$

where $\Lambda(s) = \text{diag}(s^{-p_1}, \ldots, s^{-p_k})$, and $B_1(s)$ and $B_2(s)$ are respectively $(r \times r)$ and $(m \times m)$ bicausal isomorphisms. Furthermore, it can easily be shown that the infinite transmission zeros of $H_2(s)$ are $\left[1/s^N\right]$, so there exists a following factorization at infinity:

$$H_2(s) = B_3(s) \left[\frac{1}{s^N} \right] B_4(s) =: B_5(s) \frac{1}{s^N}, \qquad (31)$$

where $B_3(s)$, $B_4(s)$, and $B_5(s) := B_3(s)B_4(s)$ are all (1×1) bicausal isomorphisms. From (29)-(31), it can be seen that $H(s)$ then has the following Smith-McMillan factorization at infinity:

$$H(s) = \text{diag}(\underbrace{B_5(s), \ldots, B_5(s)}_{r})B_1(s) \begin{bmatrix} \tilde{\Lambda}(s) & 0 \\ 0 & 0 \end{bmatrix} B_2(s),$$

where $\tilde{\Lambda}(s) = \text{diag}\big(s^{-(p_1+N)}, \ldots, s^{-(p_k+N)}\big)$; hence the infinite transmission zeros of $H(s)$ are given by $\left[1/s^{(p_1+N)}, \ldots, 1/s^{(p_k+N)}\right]$.

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