A Hill-Moylan Lemma for Equilibrium-Independent Dissipativity

John W. Simpson-Porco

***Abstract*— Equilibrium-independent dissipativity (EID) is a recently introduced system property which requires a system to be dissipative with respect to any forced equilibrium configuration. This paper provides an algebraic characterization of EID for a class of control-affine nonlinear systems, in the spirit of the Hill-Moylan lemma. We apply the results to an equilibrium-independent version of the absolute stability problem, and to a simple second-order dynamic system.**

I. Introduction

The state-space theory of dissipativity provides a generalization of Lyapunov theory to systems with inputs and outputs. Introduced by Williems in [1], dissipativity is an input-output system-theoretic property, and includes classical input-output properties such as finite $\mathscr{L}_2$-gain, passivity, and conicity as special cases [2]. A key result by Hill and Moylan [3], [4] characterized dissipativity for nonlinear control-affine systems in terms of a system of partial differential equations. Dissipative systems theory and associated control design techniques are now fairly mature, with several reference books available [5]–[7]. These techniques have proved particularly useful for studying the stability of interconnected dynamical systems: if subsystems are shown to satisfy certain dissipation inequalities, then suitable stability conditions on the interconnection pattern can often be derived [8].

Classical dissipation inequalities are implicitly referenced to a chosen equilibrium input-state-output configuration $(\bar{u}, \bar{x}, \bar{y})$, often taken to be the origin. This sometimes proves problematic, for two reasons. First, a storage function used to certify that the system is dissipative with respect to one equilibrium configuration need not successfully certify dissipativity with respect to another, different equilibrium configuration; shifting the storage function does not work in general. It can therefore be difficult to certify dissipativity for multiple operating points, or the in the presence of a constant disturbance which shifts the system's equilibrium. The second issue concerns the study interconnections. If several dissipative systems with equilibria at the origin are interconnected with one another, the origin is an equilibrium point for the closed-loop system, and dissipativity theory provides tools for assessing its stability [7]. In general however, the very act of interconnection between subsystems will induce a new closed-loop equilibrium set, determined by the simultaneous solution of all subsystem equilibrium equations and all interconnection constraints. When many uncertain nonlinear systems are interconnected, explicitly calculating this equilibrium set may prove infeasible. It then becomes challenging to construct classical storage functions for the subsystems in order to verify internal stability and/or I/O properties of the interconnection.

One remedy to these issues is termed *incremental dissipativity*, which requires that a dissipation inequality hold along any two arbitrary trajectories of a forced system [9]; a closely related property termed differential dissipativity is discussed in [14], [15]. Under appropriate assumptions, incremental dissipativity implies the existence of a unique equilibrium trajectory, and has proven useful for studying output regulation [9], [10] and synchronization of interconnected systems [11]–[13], where all subsystem trajectories converge to a common global steady-state trajectory. Incremental dissipativity however is quite demanding as a system property, since often we wish only to establish stability/dissipativity of trajectories with respect to one or more equilibrium configurations, and not between all possible trajectories.

As an intermediate property between classical and incremental dissipativity, *equilibrium-independent dissipativity* (EID) has been recently been introduced [16]–[18], requiring a dissipation inequality to hold between any system trajectory and any forced equilibrium point. This property has been used for the control of port-Hamiltonian systems [19], [20], for performance certification of interconnected systems [21], [22], for congestion control [23], for stability analysis of various power system models [24]–[26], and for analysis of optimization algorithms [27]. Particularly relevant to this paper is [19], where a Lyapunov construction based on the *Bregman divergence* was used to establish equilibrium-independent passivity. The theory of EID systems presented in [16]–[18] has not however been developed to the level of the classical dissipativity literature [5]–[7]. Notably absent is an algebraic characterization of EID, analogous to the Hill-Moylan lemma for standard dissipative systems.

Contributions: We consider continuous-time nonlinear control-affine systems with constant input and throughput matrices.[1] We show in Section III that for such systems, EID can be characterized in terms of an appropriately modified Hill-Moylan lemma [4]. Roughly speaking, the results can be interpreted as saying that dissipativity plus an appropriate incremental stability-like condition yields EID. We then apply our results to an equilibrium-independent variant of the absolute stability problem, and to a second-order system.

J. W. Simpson-Porco is with the Department of Electrical and Computer Engineering, University of Waterloo, ON, Canada. Email: jwsimpson@uwaterloo.ca. This work was supported in part by the NSERC Discovery Grant RGPIN-2017-04008 and by University of Waterloo start-up funding.

[1] We consider this particular subclass of control-affine systems because (i) it is sufficient for the applications we have considered, and (ii) it permits an intuitive and simple extension of Hill-Moylan conditions for classical dissipativity to EID.

An extended version of this paper presents many additional results [28]; see our conclusions in Section V for details.

Notation: The set $\mathbb{R}$ (resp. $\mathbb{R}_{\geq 0}$) is the set of real (resp. nonnegative) numbers. The $n \times n$ identity matrix is I_n, $\mathbb{0}$ is a matrix of zeros of appropriate dimension, while $\mathbb{0}_n$ is the n-vector of all zeros. Throughout, $\|x\|_2 = (x^{\mathsf{T}} x)^{1/2}$ denotes the 2-norm of $x \in \mathbb{R}^n$. The set of real-valued square-integrable signals $v : [0, \infty) \to \mathbb{R}^m$ is denoted by $\mathscr{L}_2^m[0, \infty)$, with $\mathscr{L}_{2e}^m[0, \infty)$ denoting the associated extended signal space [7, Chapter 1]. For a twice-differentiable function $V : \mathbb{R}^n \to \mathbb{R}$, $\nabla V : \mathbb{R}^n \to \mathbb{R}^n$ is its gradient while $\nabla^2 V : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ is its Hessian.

II. Nonlinear Dissipative Systems

A. Control-Affine Systems and Forced Equilibria

Consider the continuous-time nonlinear control-affine systems with constant input and throughput matrices

$$\Sigma : \begin{cases} \dot{x} = f(x) + Gu \\ y = h(x) + Ju \end{cases} \tag{1}$$

with state $x(t) \in \mathcal{X} := \mathbb{R}^n$, input $u(t) \in \mathcal{U} := \mathbb{R}^m$ and output $y(t) \in \mathcal{Y} := \mathbb{R}^p$ where $m, p \leq n$. The maps $f : \mathcal{X} \to \mathbb{R}^n$ and $h : \mathcal{X} \to \mathcal{Y}$ are assumed to be sufficiently smooth such that trajectories are forward complete for all initial conditions $x(0) \in \mathcal{X}$ and all input functions $u(\cdot) \in \mathscr{L}_{2e}^m[0, \infty)$, with corresponding output trajectories $y(\cdot) \in \mathscr{L}_{2e}^p[0, \infty)$. The input matrix $G \in \mathbb{R}^{n \times m}$ is constant and has rank m. The throughput matrix $J \in \mathbb{R}^{p \times m}$ is constant. We will be interested in forced equilibria of (1), determined by

$$\begin{aligned} \mathbb{0}_n &= f(\bar{x}) + G\bar{u} \\ \bar{y} &= h(\bar{x}) + J\bar{u}\,. \end{aligned} \tag{2}$$

When $m = n$, the system is fully actuated and for any desired equilibrium point $\bar{x} \in \mathcal{X}$, $\bar{u} = -G^{-1} f(\bar{x})$ is the associated constant input. When $m < n$, let $G^{\perp} \in \mathbb{R}^{(n-m) \times n}$ be a full-rank left annihilator of G, that is, $G^{\perp} G = \mathbb{0}$ and $\operatorname{rank}(G^{\perp}) = n - m$ [29, Lemma 2]. It follows then that

$$\mathcal{E}_{\Sigma} \triangleq \begin{cases} \mathcal{X} & \text{if} \quad m = n \\ \{\bar{x} \in \mathcal{X} \mid G^{\perp} f(\bar{x}) = \mathbb{0}_{n-m}\} & \text{if} \quad m < n \end{cases}$$

is the set of assignable equilibrium points. For every $\bar{x} \in \mathcal{E}_{\Sigma}$, we have the associated unique constant input and output vectors

$$\begin{aligned} \bar{u} &= k_u(\bar{x}) \triangleq -(G^{\mathsf{T}} G)^{-1} G^{\mathsf{T}} f(\bar{x})\,, \\ \bar{y} &= k_y(\bar{x}) \triangleq h(\bar{x}) - J(G^{\mathsf{T}} G)^{-1} G^{\mathsf{T}} f(\bar{x})\,. \end{aligned} \tag{3}$$

B. Classical Dissipativity of Control-Affine Systems

We provide brief review of dissipativity theory for control-affine nonlinear systems; see [5]–[7] for various overviews of dissipativity and related concepts. In this subsection, we make the additional assumptions for (1) that $f(\mathbb{0}_n) = \mathbb{0}_n$ and $h(\mathbb{0}_n) = \mathbb{0}_p$, so that $(\bar{u}, \bar{x}, \bar{y}) = (\mathbb{0}_m, \mathbb{0}_n, \mathbb{0}_p)$ is an equilibrium configuration. Let $\mathsf{w} : \mathcal{U} \times \mathcal{Y} \to \mathbb{R}$ be a continuous function of the input u and output y, called the *supply rate*. The system Σ in (1) is *dissipative* with respect to the supply rate $\mathsf{w}(u, y)$ if there exists a continuously differentiable *storage function* $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$ with $V(\mathbb{0}_n) = 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) \triangleq \nabla V(x)^{\mathsf{T}} (f(x) + Gu) \leq \mathsf{w}(u(t), y(t)) \tag{4}$$

for all $t \geq 0$ and all measurable inputs $u(\cdot) \in \mathscr{L}_{2e}^m[0, \infty)$. The inequality (4) is called a *dissipation inequality*, the interpretation of which is that the rate of change of energy $V(x(t))$ held by the system is less than the supplied power $\mathsf{w}(u(t), y(t))$. We focus exclusively on quadratic supply rates

$$\mathsf{w}(u, y) = \begin{bmatrix} y \\ u \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} Q & S \\ S^{\mathsf{T}} & R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}, \tag{5}$$

where $Q = Q^{\mathsf{T}}$, S, and $R = R^{\mathsf{T}}$ are matrices of appropriate dimensions. The supply rate (5) contains some common I/O system properties as special cases, including passivity $(Q, S, R) = (\mathbb{0}, \frac{1}{2} I_m, \mathbb{0})$ and finite L_2-gain $(Q, S, R) = (-I_p, \mathbb{0}, \gamma^2 I_m)$ for $\gamma \geq 0$. The key characterization of quadratically dissipative continuous-time control-affine systems is due to Hill and Moylan.

Lemma 2.1: **(Hill-Moylan Conditions, [4]):** The control-affine system Σ in (1) is dissipative with respect to the supply rate (5) with continuously-differentiable storage function $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$ if and only if there exists an integer $k > 0$, a matrix $W \in \mathbb{R}^{k \times m}$ and a continuous function $l : \mathcal{X} \to \mathbb{R}^k$ such that

$$\nabla V(x)^{\mathsf{T}} f(x) = h^{\mathsf{T}}(x) Q h(x) - l^{\mathsf{T}}(x) l(x) \tag{6a}$$

$$\frac{1}{2} \nabla V(x)^{\mathsf{T}} G = h^{\mathsf{T}}(x)(QJ + S) - l^{\mathsf{T}}(x) W \tag{6b}$$

$$W^{\mathsf{T}} W = R + J^{\mathsf{T}} S + S^{\mathsf{T}} J + J^{\mathsf{T}} Q J \tag{6c}$$

In most applications, the first equation in (6) enforces some type of stability, while the remaining equations ensure a proper matching of inputs and outputs to generate the supply rate (5). When specialized to LTI systems $\dot{x} = Fx + Gu\,, y = Hx + Ju$, with quadratic storage functions $V(x) = x^{\mathsf{T}} P x$, $P = P^{\mathsf{T}} \succeq \mathbb{0}$, Lemma 2.1 states that dissipativity with respect to the quadratic supply rate (5) is equivalent to the existence of an integer $k > 0$ and matrices $L \in \mathbb{R}^{k \times n}, W \in \mathbb{R}^{k \times m}$ solving the linear matrix equality

$$\begin{bmatrix} F^{\mathsf{T}} P + PF & PG \\ G^{\mathsf{T}} P & \mathbb{0} \end{bmatrix} - \begin{bmatrix} H & J \\ \mathbb{0} & I_m \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} Q & S \\ S^{\mathsf{T}} & R \end{bmatrix} \begin{bmatrix} H & J \\ \mathbb{0} & I_m \end{bmatrix} + \begin{bmatrix} L^{\mathsf{T}} \\ W^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} L & W \end{bmatrix} = \mathbb{0}\,.$$

III. Equilibrium-Independent Dissipativity for Continuous-Time Control-Affine Systems

The concept of equilibrium-independent dissipativity (EID) requires dissipativity of a system with respect to any viable equilibrium configuration [16], [18], [22]. Our definition roughly follows [18], [22].

Definition 3.1: **(Equilibrium-Independent Dissipativity):** The control-affine system (1) is equilibrium-independent dissipative (EID) with supply rate $\mathsf{w} : \mathcal{U} \times \mathcal{Y} \to \mathbb{R}$ if, for every equilibrium $\bar{x} \in \mathcal{E}_\Sigma$, there exists a continuously-differentiable storage function $V_{\bar{x}} : \mathcal{X} \to \mathbb{R}_{\geq 0}$ such that $V_{\bar{x}}(\bar{x}) = 0$ and

$$\dot{V}_{\bar{x}}(x(t)) := \nabla V_{\bar{x}}(x)^{\mathsf{T}}(f(x) + Gu) \leq \mathsf{w}(u - \bar{u}, y - \bar{y})\,, \quad (7)$$

for all $t \geq 0$ and all measurable inputs $u(\cdot) \in \mathscr{L}_{2e}^m[0,\infty)$, where $\bar{u} = k_u(\bar{x})$, $\bar{y} = k_y(\bar{x})$. A set of storage functions $\{V_{\bar{x}}(x)\,,\ \bar{x} \in \mathcal{E}_\Sigma\}$ satisfying (7) is an *EID storage function family*.

Note that in Definition 3.1, the supply rate $\mathsf{w}(\cdot,\cdot)$ does not depend on $\bar{x}$. In other words, EID as defined requires a uniformity in the supply rate across all assignable equilibrium points. For memoryless nonlinearities $\psi : \mathcal{D} \subset \mathbb{R}^m \to \mathbb{R}^p$, all storage functions in Definition 3.1 are taken as zero and ψ is EID if

$$\begin{bmatrix}\psi(z_2) - \psi(z_1) \\ z_2 - z_1\end{bmatrix}^{\mathsf{T}} \begin{bmatrix} Q & S \\ S^{\mathsf{T}} & R\end{bmatrix} \begin{bmatrix}\psi(z_2) - \psi(z_1) \\ z_2 - z_1\end{bmatrix} \geq 0 \quad (8)$$

for every $z_1, z_2 \in \mathcal{D}$. In the square case where $m = p$, inequality (8) describes several classes of mappings associated with gradients of convex functions [30], including

(i) *monotone:* $Q = \mathbb{0}$, $S = \frac{1}{2}I_m$, $R = \mathbb{0}$,
(ii) *ν-strongly monotone:* $Q = \mathbb{0}$, $S = \frac{1}{2}I_m$, $R = -\nu I_m$,
(iii) *ρ-cocoercive:* $Q = -\rho I_m$, $S = \frac{1}{2}I_m$, $R = \mathbb{0}$,

where $\rho, \nu > 0$, as well as *γ-Lipschitz* mappings with $Q = -I_m$, $S = \mathbb{0}$, and $R = \gamma^2 I_m$.

A. Hill-Moylan Conditions for EID

Our first result gives a version of Lemma 2.1 appropriate for EID systems. The Lyapunov construction is inspired by [19], and provides a convenient parameterization of the EID storage function family $\{V_{\bar{x}}(x), \bar{x} \in \mathcal{E}_\Sigma\}$ in terms of a chosen function $V(x)$. The following preliminary result is helpful.

Lemma 3.2: **(Bregman Divergence Properties):** Let $V : \mathbb{R}^n \to \mathbb{R}$ be differentiable and for $z \in \mathbb{R}^n$ let $V_z(x) \triangleq V(x) - V(z) - \nabla V(z)^{\mathsf{T}}(x - z)$. If V is (strictly, μ-strongly) convex, then

(i) $V_z(x) \geq 0$ (resp. $V_z(x) > 0$, $V_z(x) \geq \frac{\mu}{2}\|x - z\|_2^2$) for all $x \neq z$;
(ii) $x \mapsto V_z(x)$ is (strictly, strongly) convex;

Proof: Clearly $V_z(z) = 0$. That $V_z(x) \geq 0$ for $x \neq z$ follows immediately from convexity, since $V_z(x) = V(x) - [V(z) + \nabla V(z)^{\mathsf{T}}(x - z)]$ is the difference between $V(x)$ and its linear approximation at z, with strict inequality if V is strictly convex. Strong convexity of $V(x)$ is equivalent to

$$V(x) - V(z) \geq \nabla V(z)^{\mathsf{T}}(x - z) + \frac{\mu}{2}\|x - z\|_2^2$$

which immediately shows that $V_z(x) \geq \frac{\mu}{2}\|x - z\|_2^2$. Convexity of $x \mapsto V_z(x)$ follows by directly checking that $V_z(x) - V_z(x') - \nabla V_z(x')^{\mathsf{T}}(x - x') \geq 0$ for all $x, x' \in \mathbb{R}^n$, with strict inequality when V is strictly convex, and with zero replaced by $\frac{\mu}{2}\|x - x'\|_2^2$ when V is μ-strongly convex. ■

Lemma 3.3: **(Hill-Moylan Conditions for EID):** Consider the control-affine system Σ in (1). Let $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$ be continuously differentiable and convex, and for $\bar{x} \in \mathcal{E}_\Sigma$, let

$$V_{\bar{x}}(x) := V(x) - V(\bar{x}) - \nabla V(\bar{x})^{\mathsf{T}}(x - \bar{x})\,. \quad (9)$$

The system Σ is EID with respect to the quadratic supply rate $\mathsf{w}(u, y)$ in (5) with storage function family $\{V_{\bar{x}}(x)\,,\ \bar{x} \in \mathcal{E}_\Sigma\}$ if and only if there exists an integer $k > 0$, a matrix $W \in \mathbb{R}^{k \times m}$, and a function $\ell : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^k$ such that

$$\begin{aligned}&[\nabla V(x) - \nabla V(\bar{x})]^{\mathsf{T}}[f(x) - f(\bar{x})] \\ &\quad = [h(x) - h(\bar{x})]^{\mathsf{T}} Q [h(x) - h(\bar{x})] - \|\ell(x, \bar{x})\|_2^2\end{aligned} \quad (10a)$$

$$\begin{aligned}\frac{1}{2}[\nabla V(x) - \nabla V(\bar{x})]^{\mathsf{T}} G = [h(x) - h(\bar{x})]^{\mathsf{T}}(QJ + S) \\ - \ell(x, \bar{x})^{\mathsf{T}} W\end{aligned} \quad (10b)$$

$$W^{\mathsf{T}} W = R + J^{\mathsf{T}} S + S^{\mathsf{T}} J + J^{\mathsf{T}} Q J \quad (10c)$$

for all $(x, \bar{x}) \in \mathcal{X} \times \mathcal{E}_\Sigma$. The function $\ell(x, \bar{x})$ appearing in (10a)–(10b) may always be chosen to have the form

$$\ell(x, \bar{x}) = l(x) - l(\bar{x}) + Tq(x, \bar{x})\,,$$

where $l : \mathcal{X} \to \mathbb{R}^k$, the columns of $T \in \mathbb{R}^{k \times r}$ with $r = \dim(\ker(W^{\mathsf{T}}))$ form a basis for $\ker(W^{\mathsf{T}})$, and $q : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^r$ satisfies $q(x, x) = \mathbb{0}_r$ for all $x \in \mathcal{X}$.

Lemma 3.3 says that if one can find a convex function $V(x)$ along with $\ell(x, \bar{x})$ and W satisfying (10a)–(10c), then (9) parameterizes the entire EID storage function family certifying EID with quadratic supply rate (5).

Proof of Lemma 3.3: Sufficiency: Let $\bar{x} \in \mathcal{E}_\Sigma$ be arbitrary, with associated equilibrium inputs/outputs given by (3). Consider the storage function candidate (9). It follows from Lemma 3.2 that $V_{\bar{x}}(\bar{x}) = 0$ and $V_{\bar{x}}(x) \geq 0$ for all $x \neq \bar{x}$. We compute that along system trajectories

$$\begin{aligned}\dot{V}_{\bar{x}} &= [\nabla V(x) - \nabla V(\bar{x})]^{\mathsf{T}}[f(x) + Gu] \\ &= [\nabla V(x) - \nabla V(\bar{x})]^{\mathsf{T}}[f(x) - f(\bar{x})] \\ &\quad + [\nabla V(x) - \nabla V(\bar{x})]^{\mathsf{T}} G(u - \bar{u})\end{aligned} \quad (11)$$

where we have used that $f(\bar{x}) + G\bar{u} = \mathbb{0}_n$ and, for notational simplicity, suppressed the time-dependence. Adding the non-negative quantity $\|\ell(x, \bar{x}) + W(u - \bar{u})\|_2^2$ to the right-hand side of the dissipation rate, we obtain

$$\begin{aligned}\dot{V}_{\bar{x}} &\leq [\nabla V(x) - \nabla V(\bar{x})]^{\mathsf{T}}[f(x) - f(\bar{x})] \\ &\quad + \|\ell(x, \bar{x})\|_2^2 + [\nabla V(x) - \nabla V(\bar{x})]^{\mathsf{T}} G(u - \bar{u}) \\ &\quad + 2\ell(x, \bar{x})^{\mathsf{T}} W(u - \bar{u}) + (u - \bar{u})^{\mathsf{T}} W^{\mathsf{T}} W(u - \bar{u})\,.\end{aligned}$$

Inserting (10a) and (10c), we obtain

$$\begin{aligned}\dot{V}_{\bar{x}} &\leq [h(x) - h(\bar{x})]^{\mathsf{T}} Q [h(x) - h(\bar{x})] \\ &\quad + [\nabla V(x) - \nabla V(\bar{x})]^{\mathsf{T}} G(u - \bar{u}) \\ &\quad + 2\ell(x, \bar{x})^{\mathsf{T}} W(u - \bar{u}) + (u - \bar{u})^{\mathsf{T}} \widehat{R}(u - \bar{u})\,,\end{aligned}$$

where $\widehat{R} = R + J^{\mathsf{T}} S + S^{\mathsf{T}} J + J^{\mathsf{T}} Q J$. Inserting (10b) into the dissipation inequality, we find

$$\begin{aligned}\dot{V}_{\bar{x}} &\leq [h(x) - h(\bar{x})]^{\mathsf{T}} Q [h(x) - h(\bar{x})] \\ &\quad + (u - \bar{u})^{\mathsf{T}} J^{\mathsf{T}} Q J(u - \bar{u}) \\ &\quad + 2[h(x) - h(\bar{x})]^{\mathsf{T}}(QJ + S)(u - \bar{u}) \\ &\quad + 2(u - \bar{u})^{\mathsf{T}} S^{\mathsf{T}} J(u - \bar{u}) + (u - \bar{u})^{\mathsf{T}} R(u - \bar{u})\,.\end{aligned}$$

Inserting $h(x) = y - Ju$ and $h(\bar{x}) = \bar{y} - J\bar{u}$, collecting terms, and simplifying, one arrives at $\dot{V}_{\bar{x}} \leq \mathsf{w}(u - \bar{u}, y - \bar{y})$ which shows the system is EID.

Necessity: Assume Σ is EID with supply rate $\mathsf{w}(u, y)$ and storage function (9), i.e., for each $\bar{x} \in \mathcal{E}_\Sigma$ it holds that $\dot{V}_{\bar{x}} \leq \mathsf{w}(u - \bar{u}, y - \bar{y})$. Defining $\mathsf{d}_{\bar{x}}(x, u) := -\dot{V}_{\bar{x}} + \mathsf{w}(u - \bar{u}, y - \bar{y})$, we find that

$$\begin{aligned} 0 \leq \mathsf{d}_{\bar{x}}(x, u) &= -[\nabla V(x) - \nabla V(\bar{x})]^\mathsf{T}[f(x) + Gu] \\ &\quad + (y - \bar{y})^\mathsf{T} Q (y - \bar{y}) + (u - \bar{u})^\mathsf{T} R (u - \bar{u}) \\ &\quad + 2(y - \bar{y})^\mathsf{T} S (u - \bar{u}) \end{aligned}$$

Substituting for y and $\bar{y}$, after some manipulation one obtains

$$\mathsf{d}_{\bar{x}}(x, u) = \begin{bmatrix} 1 \\ u - \bar{u} \end{bmatrix}^\mathsf{T} \underbrace{\begin{bmatrix} a(x, \bar{x}) & b(x)^\mathsf{T} - b(\bar{x})^\mathsf{T} \\ b(x) - b(\bar{x}) & \widehat{R} \end{bmatrix}}_{:=\mathcal{D}(x,\bar{x})} \begin{bmatrix} 1 \\ u - \bar{u} \end{bmatrix}$$

where

$$\begin{aligned} a(x, \bar{x}) &= -[\nabla V(x) - \nabla V(\bar{x})]^\mathsf{T}[f(x) - f(\bar{x})]\,, \\ &\quad + [h(x) - h(\bar{x})]^\mathsf{T} Q [h(x) - h(\bar{x})] \\ b(x)^\mathsf{T} &= -\frac{1}{2}\nabla V(x)^\mathsf{T} G + h(x)^\mathsf{T}(QJ + S)\,, \end{aligned} \tag{12}$$

and $\widehat{R}$ is as before. Since $\mathsf{d}_{\bar{x}}(x, u) \geq 0$ for all u, we in fact have that $\mathcal{D}(x, \bar{x}) \succeq \mathbb{0}$ for all $(x, \bar{x})$ [7, Lemma 4.1.3]; in particular then $a(x, \bar{x}) \geq 0$ and $\widehat{R} \succeq \mathbb{0}$. For each pair $(x, \bar{x})$, the matrix $\mathcal{D}(x, \bar{x})$ may be factorized as

$$\mathcal{D}(x, \bar{x}) = \begin{bmatrix} \ell(x, \bar{x})^\mathsf{T} \\ W^\mathsf{T} \end{bmatrix} \begin{bmatrix} \ell(x, \bar{x}) & W \end{bmatrix} \tag{13}$$

where $\ell : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^k$ and $W \in \mathbb{R}^{k \times m}$ for some positive integer k. The proof that W may always be taken as a constant matrix instead of a function $W(x, \bar{x})$ is omitted, and may be found in the extended version [28]. It follows by equating blocks of $\mathcal{D}(x, \bar{x})$ that

$$\ell(x, \bar{x})^\mathsf{T} \ell(x, \bar{x}) = a(x, \bar{x}) \tag{14a}$$
$$W^\mathsf{T} \ell(x, \bar{x}) = b(x) - b(\bar{x}) \tag{14b}$$
$$W^\mathsf{T} W = \widehat{R} \tag{14c}$$

for all pairs $(x, \bar{x})$. Substitution of the expressions for $a(x, \bar{x})$, $b(x)$ and $\widehat{R}$ into (14a)–(14c) immediately leads to the three equations (10a)–(10c). The proof of the final statement concerning the form of $\ell(x, \bar{x})$ may be found in the extended version [28]. □

As a simple comparison between dissipativity and EID, consider the case of a passivity supply rate $\mathsf{w}(u, y) = y^\mathsf{T} u$ for systems without feedthrough ($J = \mathbb{0}$), as studied in [16]–[19]. In this case, one may take $W = \mathbb{0}$ in both Lemma 2.1 and Lemma 3.3. The remaining Hill-Moylan conditions (6a)–(6b) quickly reduce to

$$\nabla V(x)^\mathsf{T} f(x) = -\|l(x)\|_2^2 \tag{15a}$$
$$G^\mathsf{T} \nabla V(x) = h(x)\,, \tag{15b}$$

while the remaining EID conditions (10a)–(10b) reduce to

$$[\nabla V(x) - \nabla V(\bar{x})]^\mathsf{T}[f(x) - f(\bar{x})] = -\|\ell(x, \bar{x})\|_2^2 \tag{16a}$$
$$G^\mathsf{T} \nabla V(x) = h(x)\,, \tag{16b}$$

First, observe that (15b) and (16b) are identical; this is a consequence of our assumption that that the input and throughput matrices G and J are independent of x. In contrast with the stability condition (15a), (16a) is an incremental-stability-like condition on the vector field f.

For scalar SISO systems ($n = m = p = 1$) without feedthrough, it has previously been shown (e.g., [22, Example 3.1]) that a system will be equilibrium-independent passive if $f : \mathbb{R} \to \mathbb{R}$ is decreasing and $h : \mathbb{R} \to \mathbb{R}$ is increasing. This is shown by using a Popov-type storage function family, parameterized as

$$V_{\bar{x}}(x) = \int_{\bar{x}}^{x} [h(z) - h(\bar{x})]\, \mathrm{d}z\,. \tag{17}$$

Since h is continuous and increasing, there exists a continuously differentiable convex function $V : \mathbb{R} \to \mathbb{R}$ such that $h(x) = \nabla V(x)$, and (17) can be seen as a special case of the Bregman construction (9) used in Lemma 3.3. Concerning the requirement that $f(\cdot)$ be decreasing, Lemma 3.3 allows us to give the following generalization of this result to MIMO systems.

Corollary 3.4: **(Equilibrium-Independent Passive Systems):** Consider the square control-affine nonlinear system

$$\dot{x} = f(x) + Gu\,, \quad y = G^\mathsf{T} \nabla V(x) \tag{18}$$

where $V : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable and strongly convex. If the mapping $-f \circ \nabla V^{-1}$ is monotone, then (18) is equilibrium-independent passive with storage function (9).

Proof: Since V is continuously differentiable and strongly convex, $x \mapsto \nabla V(x)$ is both maximally and strongly monotone, and is therefore a bijection on $\mathcal{X}$ [30, Example 22.9]. Therefore, $\tilde{f} := -f \circ \nabla V^{-1}$ is indeed well-defined, and by assumption satisfies

$$(x_1 - x_2)^\mathsf{T}(\tilde{f}(x_1) - \tilde{f}(x_2)) \geq 0\,, \qquad x_1, x_2 \in \mathcal{X}\,. \tag{19}$$

For the system (18), (16b) automatically holds, so (16) holds if and only if

$$[\nabla V(x) - \nabla V(\bar{x})]^\mathsf{T}[f(x) - f(\bar{x})] \leq 0 \tag{20}$$

for all $(x, \bar{x}) \in \mathcal{X} \times \mathcal{E}_\Sigma$. Setting $x_1 := \nabla V(x)$, $x_2 = \nabla V(\bar{x})$, we see that (19) implies (20), which shows the result. ■

Remark 3.5: **(Computational Verification of EID):** To computationally verify the EID property for a given nonlinear system (1) using Lemma 3.3, one would search for a differentiable function $V(x)$ such that

$$[\nabla V(x) - \nabla V(\bar{x})]^\mathsf{T}(x - \bar{x}) \geq 0$$
$$[\nabla V(x) - \nabla V(\bar{x})]^\mathsf{T}[f(x) + Gu] \leq \mathsf{w}(u - \bar{u}, y - \bar{y})$$

for all $(x, \bar{x}, u)$ with corresponding values for $(y, \bar{y}, \bar{u})$. For LTI systems with quadratic storage functions, these constraints reduce to linear matrix inequalities. When $f(x)$ and

$h(x)$ are polynomial functions, the search for a polynomial function $V(x)$ certifying EID can be cast as a sum-of-squares feasibility problem and solved via semidefinite programming; see [22] for further discussion. □

IV. Equilibrium-Independent Absolute Stability

We begin with an appropriate definition of observability.

Definition 4.1: **(Equilibrium-Independent Observability):** The system Σ in (1) is equilibrium-independent observable if, for every $\bar{x} \in \mathcal{E}_\Sigma$ with associated constant input/output vectors $\bar{u} = k_u(\bar{x})$ and $\bar{y} = k_y(\bar{x})$, no trajectory of $\dot{x} = f(x) + G\bar{u}$ can remain within the set $\{x \mid h(x) + J\bar{u} = \bar{y}\}$ other than the equilibrium trajectory $x(t) = \bar{x}$.

Definition 4.1 is the natural extension of zero-state observability to EID systems, requiring that every forced system be "zero-state" observable. Compared to the general discussion of forced equilibria in Section II-A, Definition 4.1 rules out the possibility that two distinct achievable equilibria $\bar{x}, \tilde{x} \in \mathcal{E}_\Sigma$ yield the same input/output pairs through (3).

We now consider the feedback system shown in Figure 1, consisting of a square ($\mathcal{U} = \mathcal{Y} = \mathbb{R}^m$) system Σ in feedback with a static nonlinear element $\psi : \mathbb{R}^m \to \mathbb{R}^m$; we assume ψ is sufficiently smooth to ensure well-defined closed-loop trajectories.

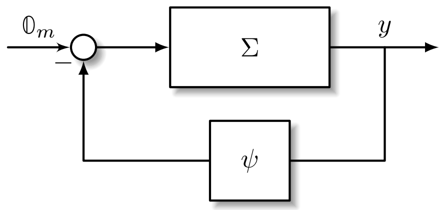

Fig. 1: System with static feedback nonlinearity.

Classically, the absolute stability problem is to determine conditions under which the feedback system is internally stable for all memoryless nonlinearities ψ satisfying a sector condition. In the standard formulation, Σ is assumed to have an equilibrium point at the origin, and ψ is assumed to satisfy $\psi(\mathbb{0}_m) = \mathbb{0}_m$; these assumptions ensure that the feedback interconnection has an unforced equilibrium point at the origin.[2] The development of equilibrium-independent dissipativity allows us to consider a sensible variant on this problem, where rather than being assumed, the existence of a closed-loop equilibrium point is inferred from the EID properties of the subsystems. For simplicity, we assume that $J = \mathbb{0}$ (Σ has no feedthrough).

Theorem 4.2: **(Equilibrium-Independent Circle Criterion):** Consider the feedback system in Figure 1, where Σ is square ($m = p$) and is equilibrium-independent observable. Assume that

(i) the nonlinearity $\psi : \mathbb{R}^m \to \mathbb{R}^m$ satisfies the incremental dissipation inequality (8), with parameters[3]

$$(Q_\psi, S_\psi, R_\psi) = \left(-I_m, \frac{K_1 + K_2}{2}, -K_1 K_2\right), \quad (21)$$

where K_1, K_2 are diagonal and $K = K_2 - K_1 \succ \mathbb{0}$;

(ii) the system

$$\Sigma' : \begin{cases} \dot{x} = f(x) - GK_1 h(x) + Gu_\ell \\ y_\ell = Kh(x) + u_\ell \end{cases} \quad (22)$$

is EID, satisfying Lemma 3.3 with $V(x)$ strictly convex and supply rate (5), with parameters

$$(Q_{\Sigma'}, S_{\Sigma'}, R_{\Sigma'}) = \left(-\varepsilon I_m, \frac{1}{2} I_m, \mathbb{0}\right) \quad (23)$$

for some $\varepsilon > 0$.

Then the closed-loop system possesses a unique and locally asymptotically stable equilibrium point. If $V(x)$ is strongly convex, then the equilibrium is globally asymptotically stable.

Proof of Theorem 4.2: We only sketch the proof here; the full proof may be found in the extended version [28]. To begin, a standard loop transformation (see, e.g., [31, Pg. 233]) turns the closed-loop system of Figure 1 into the negative feedback interconnection of the system Σ' from (22) and an incrementally passive nonlinearity ψ'. A straightforward algebraic argument then shows that the equilibrium sets of the two closed-loop systems are identical. The system Σ' is EID with supply rate (23), from which one can argue that the equilibrium input-output map $k_{u \to y} := k_y \circ k_u^{-1}$ of the system is ε-cocoercive and therefore maximally monotone [28, Lemma 3.3, Lemma A.1]. From here, one can deduce the existence of unique closed-loop equilibrium inputs and outputs $(\bar{u}, \bar{y})$ [28, Lemma A.3], and hence the existence of an equilibrium $\bar{x} \in \mathcal{E}_\Sigma$ satisfying (3). Closed-loop stability and uniqueness of the equilibrium follows by using the EID storage function $V_{\bar{x}}(x)$ as a Lyapunov function for $\bar{x}$, and invoking the observability assumption. When $V(x)$ is strongly convex, $V_{\bar{x}}(x)$ is radially unbounded and the stability result is global. □

Example 4.3: **(Second-Order System):** Consider the second-order system model

$$\Sigma : \begin{cases} \dot{x}_1 = x_2 \,, \\ M\dot{x}_2 = d - Dx_2 - \nabla U(x_1) + u \,, \\ y = x_2 \end{cases}$$

where $x_1, x_2 \in \mathbb{R}^n$, $M = M^\mathsf{T} \succ \mathbb{0}$, $D = D^\mathsf{T} \succ \mathbb{0}$, $d \in \mathbb{R}^n$, and $U : \mathbb{R}^n \to \mathbb{R}$ is μ-strongly convex. Consider now the feedback interconnection $u = -\psi(y)$ where $\psi : \mathbb{R}^n \to \mathbb{R}^n$ satisfies (21) with $K_1 = \alpha I_n$ and $K_2 = \beta I_n$ for scalars α, β

[2] Typically Σ is further assumed to be an LTI system; here we do not make this restriction.

[3] Equivalently, ψ satisfies the incremental sector condition

$$[\psi(z_2) - \psi(z_1) - K_1(z_2 - z_1)]^\mathsf{T}[\psi(z_2) - \psi(z_1) - K_2(z_2 - z_1)] \leq 0 \,.$$

satisfying $\beta > 0$ and $\alpha < \beta$. Following Theorem 4.2, we examine the loop-transformed system (22) given by

$$\Sigma' : \begin{cases} \dot{x}_1 = x_2 \,, \\ M\dot{x}_2 = d - (D+\alpha)x_2 - \nabla U(x_1) + u_\ell \,, \\ y_\ell = (\beta - \alpha)x_2 + u_\ell \,. \end{cases}$$

Define

$$V(x) \triangleq U(x_1) + \frac{1}{2}x_2^\mathsf{T} M x_2 \,,$$

which is strongly convex in (x_1, x_2). A computation using (9) shows that

$$V_{\bar{x}}(x) = \frac{1}{2}(x_2 - \bar{x}_2)^\mathsf{T} M (x_2 - \bar{x}_2) + U_{\bar{x}_1}(x_1) \,,$$

where $U_{\bar{x}_1}(x_1) = U(x_1) - U(\bar{x}_1) - \nabla U(\bar{x}_1)^\mathsf{T}(x_1 - \bar{x}_1)$. Along trajectories of Σ', a straightforward calculation shows that

$$\dot{V}_{\bar{x}} \leq -(\lambda_{\min}(D) + \alpha)\|x_2 - \bar{x}_2\|_2^2 + (x_2 - \bar{x}_2)^\mathsf{T}(u_\ell - \bar{u}_\ell) \,.$$

Substituting $x_2 = (y_\ell - u_\ell)/(\beta - \alpha)$ and a similar expression for $\bar{x}_2$, and working through some algebra, one finds that Σ' is quadratically dissipative with parameters (23), where

$$\varepsilon = \frac{\lambda_{\min}(D) + \alpha}{2\lambda_{\min}(D) + \beta + \alpha} \,.$$

If $\alpha > -\lambda_{\min}(D)$, then $\varepsilon > 0$ and all assumptions are satisfied. We conclude that in this case, the closed-loop system possesses a unique and globally exponentially equilibrium point. □

V. Conclusions

This paper has presented an algebraic characterization of equilibrium-independent dissipativity for a common class of continuous-time control-affine nonlinear systems. The family of storage functions required to establish EID is parameterized using the Bregman divergence of a chosen convex function, and the resulting algebraic characterization closely resembles the classical Hill-Moylan lemma for dissipative systems. An extended version of this paper [28] contains the proof of Theorem 4.2, as well as additional examples, results on maximal monotonicity of EID input/output relations, feedback stability theorems, and discrete-time versions of these results with accompanying examples.

References

[1] J. C. Willems, "Dissipative dynamical systems–Part I: General theory," *Archive for Rational Mechanics and Analysis*, vol. 45, no. 5, pp. 321–351, 1972.

[2] G. Zames, "On the input-output stability of time-varying nonlinear feedback systems – Part I: Conditions derived using concepts of loop gain, conicity, and positivity," *IEEE Transactions on Automatic Control*, vol. 11, no. 2, pp. 228–238, 1966.

[3] P. Moylan, "Implications of passivity in a class of nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 19, no. 4, pp. 373–381, 1974.

[4] D. Hill and P. Moylan, "The stability of nonlinear dissipative systems," *IEEE Transactions on Automatic Control*, vol. 21, pp. 708–711, 1976.

[5] R. Sepulchre, M. Janković, and P. V. Kokotović, *Constructive Nonlinear Control*. Springer, 1997.

[6] B. Brogliato, R. Lozano, B. Maschke, and O. Egeland, *Dissipative Systems Analysis and Control*. Springer, 2007.

[7] A. J. van der Schaft, *L2-Gain and Passivity Techniques in Nonlinear Control*, 3rd ed. Springer, 2016.

[8] M. Vidyasagar, *Input-Output Analysis of Large-Scale Interconnected Systems: Decomposition, Well-Posedness and Stability*. Springer, 1981.

[9] A. Pavlov and L. Marconi, "Incremental passivity and output regulation," *Systems & Control Letters*, vol. 57, no. 5, pp. 400 – 409, 2008.

[10] M. Bürger and C. D. Persis, "Dynamic coupling design for nonlinear output agreement and time-varying flow control," *Automatica*, vol. 51, pp. 210 – 222, 2015.

[11] G. B. Stan and R. Sepulchre, "Analysis of interconnected oscillators by dissipativity theory," *IEEE Transactions on Automatic Control*, vol. 52, no. 2, pp. 256–270, 2007.

[12] T. Liu, D. J. Hill, and J. Zhao, "Incremental-dissipativity-based output synchronization of dynamical networks with switching topology," in *IEEE Conf. on Decision and Control*, Los Angeles, CA, USA, Dec. 2014, pp. 4591–4596.

[13] H. Pang and J. Zhao, "Incremental (Q, S, R)-dissipativity and incremental stability for switched nonlinear systems," *Journal of the Franklin Institute*, vol. 353, no. 17, pp. 4542 – 4564, 2016.

[14] F. Forni and R. Sepulchre, "On differentially dissipative dynamical systems," *IFAC Symposium on Nonlinear Control Systems*, vol. 46, no. 23, pp. 15 – 20, 2013.

[15] A. van der Schaft, "On differential passivity," *IFAC Symposium on Nonlinear Control Systems*, vol. 46, no. 23, pp. 21 – 25, 2013.

[16] G. H. Hines, M. Arcak, and A. K. Packard, "Equilibrium-independent passivity: A new definition and numerical certification," *Automatica*, vol. 47, pp. 1949–1956, 2011.

[17] M. Bürger, D. Zelazo, and F. Allgöwer, "On the steady-state inverse-optimality of passivity-based cooperative control," *IFAC Workshop on Distributed Estimation and Control in Networked Systems*, vol. 46, no. 27, pp. 138 – 143, 2013.

[18] ——, "Duality and network theory in passivity-based cooperative control," *Automatica*, vol. 50, no. 8, pp. 2051 – 2061, 2014.

[19] B. Jayawardhana, R. Ortega, E. Garcia-Canseco, and F. Castanos, "Passivity of nonlinear incremental systems: Application to pi stabilization of nonlinear rlc circuits," *Systems & Control Letters*, vol. 56, no. 9, pp. 618 – 622, 2007.

[20] S. Aranovskiy, R. Ortega, and R. Cisneros, "Robust PI passivity-based control of nonlinear systems: Application to port-Hamiltonian systems and temperature regulation," in *American Control Conference*, Chicago, IL, USA, 2015, pp. 434–439.

[21] C. Meissen, L. Lessard, M. Arcak, and A. K. Packard, "Compositional performance certification of interconnected systems using ADMM," *Automatica*, vol. 61, pp. 55 – 63, 2015.

[22] M. Arcak, C. Meissen, and A. Packard, *Networks of Dissipative Systems: Compositional Certification of Stability, Performance, and Safety*. SpringerBriefs in Control, Automation and Robotics, 2016.

[23] J. T. Wen and M. Arcak, "A unifying passivity framework for network flow control," *IEEE Transactions on Automatic Control*, vol. 49, no. 2, pp. 162–174, 2004.

[24] S. Trip, M. Bürger, and C. D. Persis, "An internal model approach to (optimal) frequency regulation in power grids with time-varying voltages," *Automatica*, vol. 64, pp. 240 – 253, 2016.

[25] S. Trip and C. D. Persis, "Distributed optimal load frequency control with non-passive dynamics," *IEEE Transactions on Control of Network Systems*, vol. PP, no. 99, pp. 1–1, 2017.

[26] C. D. Persis and N. Monshizadeh, "Bregman storage functions for microgrid control," *IEEE Transactions on Automatic Control*, vol. 63, no. 1, pp. 53–68, 2018.

[27] J. W. Simpson-Porco, "Input/output analysis of primal-dual gradient algorithms," in *Allerton Conf. on Communications, Control and Computing*, Monticello, IL, USA, Sep. 2016, pp. 219–224.

[28] ——, "Equilibrium-Independent Dissipativity with Quadratic Supply Rates," *IEEE Transactions on Automatic Control*, 2017, submitted, available at https://arxiv.org/abs/1709.06986.

[29] R. Ortega, A. van der Schaft, F. Castanos, and A. Astolfi, "Control by interconnection and standard passivity-based control of port-Hamiltonian systems," *IEEE Transactions on Automatic Control*, vol. 53, no. 11, pp. 2527–2542, 2008.

[30] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, 1st ed. Springer, 2011.

[31] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Prentice Hall, 2002.