Forwarding for discrete-time linear systems: optimality and global stabilization under input saturation

Samuele Zoboli * **Daniele Astolfi** ** **Mattia Mattioni** ***
John W. Simpson-Porco **** **Nathan van de Wouw** †

* *Université de Toulouse III, LAAS-CNRS, UPR 8001, Toulouse, France.*
** *Université Lyon 1– CNRS, UMR 5007, LAGEPP, Villeurbanne, France*
*** *Universitá degli Studi di Roma* La Sapienza, *Rome, Italy*
**** *Department of Electrical and Computer Engineering, University of Toronto, Toronto, ON, Canada.*
† *Department of Mechanical Engineering, Eindhoven University of Technology, Eindhoven, 5612AZ,Noord-Brabant, TheNetherlands.*

Abstract: We revisit the forwarding approach for the feedback stabilization of discrete-time systems in feedforward form. We show that the resulting feedback design is parametrized by a change of coordinate which is defined via a Sylvester equation. Moreover, we investigate the optimality of such a feedback by explicitly computing a quadratic cost which is minimized by the closed loop trajectories. Finally, we apply the forwarding approach to solve the problem of global stabilization of feedforward form systems which are simply stable in the presence of input saturations. The proposed approach is also shown to be effective in tackling the stabilization of systems with input saturation and time-delays.

Keywords: Forwarding, discrete-time systems, saturation, inverse optimality, input delay

1. INTRODUCTION

Stabilization of systems in feedforward (or cascade) form is a common problem in control theory. Feedforward forms may naturally appear due to decomposition of system's dynamics, or may arise in output regulation problems (e.g., (Mantri et al., 1997)), where one of the subsystems describes the internal model. Typical examples are the well-known case of integral action-based controllers, see, e.g. (Simpson-Porco, 2021; Zoboli et al., 2023) and repetitive control, see. e.g. (Tomizuka et al., 1989; Aarnoudse et al., 2023). In this paper, we study the stabilization of discrete-time systems in feedforward form by revisiting the so-called *forwarding* approach. This methodology was initially developed for continuous-time systems in the 90's, see, e.g., (Praly, 2001), and it remains fairly unexplored in the discrete-time context, except for (Mazenc and Nijmeijer, 1998; Mattioni et al., 2019). Based on recent continuous-time results (Giaccagli et al., 2024), we propose a restructured version of existing discrete-time forwarding methods. Our findings aim to provide solid foundations for constructive nonlinear extensions encompassing the design of contractive discrete-time feedback laws (Tran et al., 2016, 2018), for which few approaches are presently available.

With the above goal in mind, we first focus on linear systems. We suppose that the system under investigation can be decomposed into two subsystems and that, while the first one can be directly controlled by the control action, the second one is affected solely by an output of the former. Our revisited forwarding approach is rooted in a change of coordinates that is uniquely determined via a Sylvester equation, under the mild assumption of disjoint spectra of the two subsystems (see, e.g. (Astolfi et al., 2024)). Such a change of coordinates decouples the dynamics of the two subsystems, while emphasizing the effect of the control input on the second one. As a consequence, it simplifies the derivation of a Control Lyapunov Function and the construction of a stabilizing controller. Inspired by results on the optimality of control laws for systems in feedforward form (see, e.g., (Ahmed-Ali et al., 1999)), we study the optimality properties of the proposed feedback law. Interestingly, we show that our control design is a Linear Quadratic Regulator (LQR) for the overall system, with a specific choice of the state weighting matrix and a structured solution to the Discrete Algebraic Riccati Equation (DARE). Hence, we present a result providing an explicit expression of the optimal cost minimized by our control law.

Finally, we show how the proposed solution can be employed to solve the global stabilization problem of systems in feedforward form in the presence of saturated control actions, e.g. (Lin et al., 1996; Tarbouriech et al., 2011; Yang et al., 2021). Assuming that the open-loop dynamics are Lyapunov stable, we propose two alternative designs which are derived from the general linear framework. The first one involves a feedback term that depends on both the subsystems' states. Differently from standard results, we obtain global stability guarantees in the presence of open-loop simply stable eigenvalues (i.e. the eigenvalues lie

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on the unitary disk) without need of relying on low-gain approaches. The second one is a low-gain feedback law that solely depends on the second subsystem's state, resulting in a less demanding control action. Both results can be also used in the context of systems subject to delayed and saturated control actions.

The rest of the article is organized as follows. Preliminaries are discussed in Section 2. The main results on linear forwarding and inverse optimality are presented in Section 3. The forwarding approach for input saturated systems is presented in Section 4. The case of saturated delayed inputs and an illustration is given in Section 5. Conclusions and perspectives are discussed in Section 6. A technical result is given in the Appendix.

Notation. We denote with $\mathbb{R}$, resp. $\mathbb{N}$ the set of real numbers, resp. non-negative integers. We denote with $|\cdot|$ the standard Euclidean norm of a vector and the induced norm matrix. For a quadratic matrix A, we denote $\mathrm{He}\{A\} := \frac{1}{2}(A + A^\top)$. We describe a discrete-time dynamics $x_{k+1} = Ax_k + Bu_k$, with x_k denoting the state at time $k \in \mathbb{N}$, with the more compact notation $x^+ = Ax + Bu$, where x^+ denotes the value of the discrete-time signal x at the next time step. For a Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, we compactly denote $\Delta V = V(x^+) - V(x)$.

2. PRELIMINARIES ON SYSTEMS IN FEEDFORWARD FORM

Consider a discrete-time linear time-invariant system in the feedforward form

$$\begin{aligned} x^+ &= Ax + Bu, \\ z^+ &= Fz + Gx \end{aligned} \tag{1}$$

where $(x, z) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$ is the state and $u \in \mathbb{R}^{n_u}$ is the control input. In compact notation, the overall system is

$$\xi^+ = \mathbf{F}\xi + \mathbf{G}u, \quad \xi = \begin{pmatrix} x \\ z \end{pmatrix}, \ \mathbf{F} := \begin{pmatrix} A & 0 \\ G & F \end{pmatrix}, \ \mathbf{G} := \begin{pmatrix} B \\ 0 \end{pmatrix}.$$

We assume the following mild properties.

Assumption 1. *The pair $(\mathbf{F}, \mathbf{G})$ is stabilizable, the spectra of A and F are disjoint, and F is invertible.*

A stabilizing feedback controller for (1) may of course be obtained by following standard LQR design (e.g. (Bu et al., 2019)), aimed at minimizing an infinite-horizon quadratic objective function of the form

$$J(\xi_0, u) = \lim_{N\to\infty} \sum_{k=0}^{N} \xi_k^\top \mathbf{X} \xi_k + u_k^\top \mathbf{R} u_k, \tag{2}$$

with $\mathbf{X} \succeq 0$, $(\mathbf{X}^{1/2}, \mathbf{F})$ detectable, and $\mathbf{R} \succ 0$. Although this procedure is quite appealing, it is not easy to extend LQR designs to nonlinear cascades. A different approach, and one which does extend to nonlinear systems, is to proceed by first decoupling the two subsystems via a change of coordinates. This allows the exploitation of forwarding-based designs (Mazenc and Praly, 1996; Mazenc and Nijmeijer, 1998; Mattioni et al., 2019), the aim of which is the design of a control Lyapunov function, which in turns provides a joint selection of the feedback law and the corresponding (possibly weak) Lyapunov function. This approach is similar to the change of coordinates used in analyzing singularly perturbed systems in so-called actuator or sensor form, see, e.g. (Kokotović et al., 1999, Chapter 2.2). The decoupling procedure simplifies the controller design, and the resulting designs will in fact satisfy certain inverse-optimality properties (Ahmed-Ali et al., 1999; Monaco and Normand-Cyrot, 2015). Moreover, this methodology allows for a constructive design that can be naturally extended to the nonlinear context, as done for the continuous-time case, see, e.g. (Mazenc and Praly, 1996).

To this end, in order to derive a stabilizing controller for system (1), we first propose a change of coordinates aimed at diagonalizing the dynamics. In particular, consider the Sylvester equation (Sylvester, 1884; Astolfi et al., 2024)

$$MA = FM + G \tag{3}$$

which has a unique solution $M \in \mathbb{R}^{n_z \times n_x}$ by Assumption 1, see, e.g., (Bhatia and Rosenthal, 1997) and references therein. Defining the change of coordinates

$$z \mapsto \eta := z - Mx, \tag{4}$$

simple computations show that in the new coordinates (x, η) the dynamics (1) read as

$$\begin{aligned} x^+ &= Ax + Bu, \\ \eta^+ &= F\eta - MBu. \end{aligned} \tag{5}$$

As done for the dynamics in original coordinates, we define the compact notation

$$\begin{aligned} &\zeta^+ = \mathbf{A}\zeta + \mathbf{B}u, \\ &\zeta = \begin{pmatrix} x \\ \eta \end{pmatrix}, \quad \mathbf{A} := \begin{pmatrix} A & 0 \\ 0 & F \end{pmatrix}, \quad \mathbf{B} := \begin{pmatrix} B \\ -MB \end{pmatrix}. \end{aligned} \tag{6}$$

Note that, given the block-diagonal structure of system (5) and Assumption 1, the pairs (A, B) and $(F, -MB)$ are stabilizable. In the new coordinates (5) a control law

$$u = Kx + L\eta = \mathbf{K}\zeta, \tag{7}$$

can be designed by following a forwarding approach, e.g., (Mattioni et al., 2019); this will be described in the next section. Furthermore, we will show that (7) is also an optimal control law in the new coordinates, minimizing a quadratic objective

$$J(\zeta_0, u) = \lim_{N\to\infty} \sum_{k=0}^{N} \zeta_k^\top \mathbf{Q} \zeta_k + u_k^\top \mathbf{R} u_k \tag{8}$$

for some particular positive definite matrices $\mathbf{R}, \mathbf{Q}$. In other words, we claim that $\mathbf{K}$ in (7) can be reinterpreted as an optimal LQR gain

$$\mathbf{K} = -(\mathbf{R} + \mathbf{B}^\top \mathbf{P} \mathbf{B})^{-1} \mathbf{B}^\top \mathbf{P} \mathbf{A}, \tag{9}$$

with $\mathbf{P}$ solution to the DARE

$$\mathbf{A}^\top \mathbf{P} \mathbf{A} - \mathbf{A}^\top \mathbf{P} \mathbf{B} (\mathbf{R} + \mathbf{B}^\top \mathbf{P} \mathbf{B})^{-1} \mathbf{B}^\top \mathbf{P} \mathbf{A} = \mathbf{P} - \mathbf{Q}. \tag{10}$$

Then, the objective (8) can be mapped into a quadratic cost (2) for the original system (1).

Hence, in the next section, we will show that under some assumptions the matrix $\mathbf{P}$ can be always selected in a block-diagonal form.

3. LINEAR FORWARDING

In this section we derive a constructive forwarding-based design for control of discrete-time linear systems feedforward form. In the first subsection, we present an explicit control law, distinct from that in existing works, e.g., (Mattioni et al., 2019) as already commented in the Introduction. Then, we devote the second subsection to the study of the inverse optimality of this control law.

3.1 Main Design

Forwarding-based control law design requires the following additional assumption on the state matrices of system (1).

Assumption 2. *The matrices A, F satisfy the following assumptions:*

i) The matrix A is Schur-Cohn stable, i.e. there exist $P = P^\top \succ 0$ and $\rho \in (0,1)$ such that
$$A^\top PA \preceq \rho P. \tag{11}$$
ii) The matrix F satisfies $F^\top F - \mathrm{I} \preceq 0$.

Remark 1. Item *i)* in Assumption 2 does not affect the generality of the result. Indeed, the matrix A can always be made Schur-Cohn stable via a preliminary state-feedback. With a slight abuse of notation and to simplify reading, we refer to the same matrix A whether this preliminary feedback has been applied or not.

Remark 2. Item *ii)* in Assumption 2 implies that F has semisimple eigenvalues inside the unitary disk. This can be generalized by asking for the existence of a positive definite matrix $S \succ 0$ satisfying
$$F^\top SF - S \preceq 0.$$
However, such a matrix S can always be selected as the identity by performing a preliminary change of coordinates defined as $z \mapsto \bar{z} := S^{-\frac{1}{2}} z$. All the arguments in the rest of the paper (e.g., the Lasalle arguments in the proof of Theorem 1) can be trivially repeated due to the invertibility of $S^{\frac{1}{2}}$. We therefore suppose z is already in such coordinates to simplify the notation.

Remark 3. Under Assumption 2, system (1) is not exponentially stable if F has eigenvalues on the unit circle. This case is particularly relevant in output regulation theory, where the z-dynamics represents an integral action (Simpson-Porco, 2021; Zoboli et al., 2023), a repetitive controller (Tomizuka et al., 1989; Aarnoudse et al., 2023) or more generically an internal model, see, e.g. (Mantri et al., 1997).

Instead of solving the DARE (10) by selecting a state-weighting matrix $\mathbf{Q}$ and computing the corresponding $\mathbf{P}$, we follow a different route by directly making a particular selection of such a matrix $\mathbf{P}$. Specifically, by relying on Assumption 2, we select
$$\mathbf{P} = \begin{pmatrix} P & 0 \\ 0 & \mathrm{I} \end{pmatrix} \tag{12}$$
with P given by (11). Indeed, it is possible to show that such a selection is a solution to
$$\mathbf{A}^\top\mathbf{P}\mathbf{A} - \mathbf{A}^\top\mathbf{P}\mathbf{B}(\mathbf{R} + \mathbf{B}^\top\mathbf{P}\mathbf{B})^{-1}\mathbf{B}^\top\mathbf{P}\mathbf{A} - \mathbf{P} \preceq 0,$$
and that the components of the gain $\mathbf{K}$ in (7) read
$$\begin{aligned} K &:= -(\mathbf{R} + B^\top(P + M^\top M)B)^{-1}B^\top PA, \\ L &:= (\mathbf{R} + B^\top(P + M^\top M)B)^{-1}B^\top M^\top F. \end{aligned} \tag{13}$$
Furthermore, the closed-loop matrix $(\mathbf{A} + \mathbf{BK})$ is Schur-Cohn stable. This is formalized in the following result.

Theorem 1. *Let Assumptions 1,2 hold. Let $\mathbf{R} \succ 0$, let $P \succ 0$ satisfy (11) and M be defined as in (3). Then, the control law*
$$u = Kx + L(z - Mx) \tag{14}$$
with K, L selectd as in (13) makes the origin of the closed-loop (1),(14) exponentially stable. Moreover, there exists a sufficiently small constant $\varepsilon > 0$ and a positive definite matrix $Q \succ 0$ such that the function
$$W(x,z) := x^\top Px + (z - Mx)^\top (\mathrm{I} + \varepsilon Q)(z - Mx) \tag{15}$$
is a strict Lyapunov function for (1) under (14), namely $\Delta W \leq -(1-\lambda)W(x,z)$ for some $\lambda \in (0,1)$.

The proof of Theorem 1 first proceeds using LaSalle arguments, as is typical in forwarding-like approaches (e.g., (Mazenc and Praly, 1996; Praly, 2001) for the continuous-time case and (Mazenc and Nijmeijer, 1998; Mattioni et al., 2019) for the discrete-time counterpart). However, we also propose a strict Lyapunov function exploiting the detectability properties of the system, using techniques from (Praly, 2019).

Proof. Define the short form notation
$$\Psi := MB, \qquad N := B^\top(P + M^\top M)B, \tag{16}$$
and consider the system (1). According to Section 2, the change of coordinates $\eta := z - Mx$ transforms (1) into (5) and (14) into (7). Consider now the candidate Lyapunov function
$$V(x,\eta) := x^\top Px + \eta^\top\eta. \tag{17}$$
with P as in (11) and let $\mathbf{P}$ be as in (12). Using (5) and (7), we compute that
$$V(x^+, \eta^+) = \begin{pmatrix} x^\top & \eta^\top \end{pmatrix} \begin{pmatrix} \bar{\mathbf{P}}_{xx} & \bar{\mathbf{P}}_{x\eta} \\ \bar{\mathbf{P}}_{x\eta}^\top & \bar{\mathbf{P}}_{\eta\eta} \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix}, \tag{18}$$
where we have defined
$$\begin{aligned} \bar{\mathbf{P}}_{xx} &:= A^\top PA + \mathrm{He}\{PBK\} + K^\top NK, \\ \bar{\mathbf{P}}_{\eta\eta} &:= F^\top F - \mathrm{He}\{\Psi L\} + L^\top NL, \\ \bar{\mathbf{P}}_{x\eta} &:= A^\top PBL - K^\top\Psi^\top F + K^\top NL. \end{aligned} \tag{19}$$
By the definitions of K, L in (13), we derive
$$B^\top PA = -(\mathbf{R} + N)K, \quad \Psi^\top F = (\mathbf{R} + N)L, \tag{20}$$
and thus the expressions (19) simplify to
$$\begin{aligned} \bar{\mathbf{P}}_{xx} &= A^\top PA - K^\top(2\mathbf{R} + N)K, \\ \bar{\mathbf{P}}_{\eta\eta} &= F^\top F - L^\top(2\mathbf{R} + N)L, \\ \bar{\mathbf{P}}_{x\eta} &= -K^\top(2\mathbf{R} + N)L. \end{aligned} \tag{21}$$
Hence, by (18) with (21), we have
$$V(x^+, \eta^+) = x^\top A^\top PAx + \eta^\top F^\top F\eta - u^\top(2\mathbf{R} + N)u.$$
By Assumption 2, using the matrix inequalities $A^\top PA \preceq \rho P$ and $F^\top F \preceq \mathrm{I}$ further yields
$$\Delta V \leq -(1-\rho)x^\top Px - u^\top(2\mathbf{R} + N)u. \tag{22}$$
Since $\rho \in (0,1)$, by discrete-time LaSalle arguments (Mei and Bullo, 2017), and recalling the definition of u, system (5) in closed-loop with (7) converges to the largest invariant set contained in
$$\{(x,\eta) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \colon x = 0, L\eta = 0\}. \tag{23}$$
Since the pair $(F, -MB)$ is stabilizable, it can be shown that the pair (F, L) is detectable by using the definition of L in (13) and applying Lemma 1 in the Appendix. As a consequence, the set (23) boils down to the origin. In view of the change of coordinates $\eta = z - Mx$, $\eta = 0$ and $x = 0$ imply $z = 0$, thus concluding the first part of the proof.

We now show that (15) is a strict Lyapunov function. Firstly, recall that detectability of (F, L) guarantees the existence of a scalar $\mu \in (0,1)$, a gain $\Gamma \in \mathbb{R}^{n_z \times n_u}$ and a positive definite matrix $Q \in \mathbb{R}^{n_z \times n_z}$ satisfying

$$(F - \Gamma L)^\top Q (F - \Gamma L) \preceq \mu Q, \qquad \underline{q} I \preceq Q \preceq \bar{q} I, \tag{24}$$

for some $0 < \underline{q} \leq \bar{q}$. Secondly, notice that in η coordinates the candidate strict Lyapunov function (15) reads

$$W(x, \eta) := V(x, \eta) + \varepsilon \eta^\top Q \eta. \tag{25}$$

In view of (22), it follows that

$$\begin{aligned} W(x^+, \eta^+) \leq V(x,\eta) - (1-\rho) x^\top P x - u^\top (2\mathbf{R} + N) u \\ + \varepsilon (F\eta - \Psi u)^\top Q (F\eta - \Psi u). \end{aligned} \tag{26}$$

Considering the final $\varepsilon-$scaled quadratic term in this inequality, by adding and subtracting ΓL and exploiting (24), we find that

$$\begin{aligned} &(F\eta - \Psi u)^\top Q (F\eta - \Psi u) \leq \mu\, \eta^\top Q \eta \\ &+2\eta^\top (F - \Gamma L)^\top Q (\Gamma L \eta - \Psi u) + (\Gamma L \eta - \Psi u)^\top Q (\Gamma L \eta - \Psi u). \end{aligned}$$

Recalling Young's inequality $2a^\top b \preceq c a^\top a + c^{-1} b^\top b$ valid for any vectors a, b and any positive scalar $c > 0$, by selecting $a = \eta (F - \Gamma L) Q^{\frac{1}{2}}$ and $b = Q^{\frac{1}{2}} (\Gamma L \eta - \Psi u)$ and $c = (1-\mu)(2|F - \Gamma L|^2)^{-1}$, the second term of the previous inequality can be bounded as follows

$$\begin{aligned} &2\eta^\top (F - \Gamma L)^\top Q (\Gamma L \eta - \Psi u) \\ &\qquad \leq \tfrac{1-\mu}{2} \eta^\top Q \eta + \nu_0 (\Gamma L \eta - \Psi u)^\top Q (\Gamma L \eta - \Psi u), \end{aligned}$$

where $\nu_0 = 2|F - \Gamma L|^2 (1-\mu)^{-1}$. On the other term, we also have

$$\begin{aligned} |\Gamma L \eta - \Psi u| &\leq |\Gamma L \eta| + |\Psi u| \leq |\Gamma (u - Kx)| + |\Psi u| \\ &\leq (|\Gamma| + |\Psi|)|u| + |K||x|. \end{aligned}$$

Combining all the inequalities above we further obtain

$$(F\eta - \Psi u)^\top Q (F\eta - \Psi u) \leq \tfrac{1+\mu}{2} \eta^\top Q \eta + \nu_1 |u|^2 + \nu_2 |x|^2 \tag{27}$$

where $\nu_1 = (|\Gamma| + |\Psi|)\nu_3$, $\nu_2 = |K|\nu_3$, $\nu_3 = (1 + \nu_0)|Q|$. Hence, combining (26) with (27) yields

$$\begin{aligned} \Delta W \leq &- (1 - \rho - \varepsilon \nu_2) x^\top P x - (r - \nu_1 \varepsilon)|u|^2 \\ &- \varepsilon (1 - \tfrac{1+\mu}{2}) \eta^\top Q \eta. \end{aligned} \tag{28}$$

where $r > 0$ is such that $2\mathbf{R} + N \succeq rI$. Then, by selecting $0 < \varepsilon < \min(\frac{r}{\nu_1}, \frac{1-\rho}{\nu_2})$, ΔW in (28) is strictly negative, concluding the proof. □

3.2 Inverse Optimality of the Forwarding Design

We now show that (13) is an optimal solution for an LQR problem (2) over the extended system (1), and we provide the associated cost in the original coordinates. To this aim, we partition the state cost matrix in (2) as follows

$$\mathbf{X} := \begin{pmatrix} \mathbf{X}_{xx} & \mathbf{X}_{xz}^\top \\ \mathbf{X}_{xz} & \mathbf{X}_{zz} \end{pmatrix} \tag{29}$$

where $\mathbf{X}_{xx} \succ 0, \mathbf{X}_{zz} \succeq 0$, and $\mathbf{X}_{xz} \in \mathbb{R}^{n_z \times n_x}$. The result of this section is tightly related to the findings of (Ahmed-Ali et al., 1999; Monaco and Normand-Cyrot, 2015). In particular, we present the explicit formulation of the optimal control problem related to forwarding-based control in linear discrete-time systems. In the context of nonlinear control, this result can be of particular interest for guaranteeing that learning-based control approaches are locally stabilizing, e.g., (Zoboli et al., 2021; Minami et al., 2023). We also note that this connection implies that our forwarding-based designs enjoy certain guaranteed gain and phase robustness margins; see, e.g., Jinyoung Lee and Shim (2012) or (Haddad and Chellaboina, 2008, Chp. 14.7) for details.

Theorem 2. *Let Assumptions 1 and 2 hold and let* $\mathbf{R} \succ 0$. *Moreover, let* $Q_x \succ 0$, $Q_z \succeq 0$ *be solutions of* $A^\top P A = P - Q_x$ *and* $F^\top F = \mathrm{I} - Q_z$, *respectively, with* $P \succ 0$ *as in* (11). *Finally, let* M *be defined as in* (3). *Then, the control law* (14) *with gains* (13) *is the optimal solution to the minimization problem* (2) *subject to dynamics* (1), *with*

$$\begin{aligned} \mathbf{X}_{xx} &= Q_x + M^\top Q_z M \\ &\quad + \begin{pmatrix} B^\top P A \\ B^\top M^\top F M \end{pmatrix}^\top \begin{pmatrix} Y & -Y \\ -Y & Y \end{pmatrix} \begin{pmatrix} B^\top P A \\ B^\top M^\top F M \end{pmatrix}, \\ \mathbf{X}_{zz} &= Q_z + F^\top M B Y B^\top M^\top F, \\ \mathbf{X}_{xz} &= F^\top M B Y (B^\top P A - B^\top M^\top F M) - Q_z M, \end{aligned} \tag{30}$$

where $Y = Y^\top := (\mathbf{R} + B^\top (P + M^\top M) B)^{-1} \succ 0$.

Proof. Before starting the proof, we remark that by following notation (16) we have $Y = (\mathbf{R} + N)^{-1}$ and, consequently, $K = -Y B^\top P A$ and $L = Y \Psi^\top F$. As shown in the proof of Theorem 1, $V(\zeta) = \zeta^\top \mathbf{P} \zeta$ with $\mathbf{P}$ as in (12) is a Lyapunov function for the closed-loop system (6),(7). By LQR theory (e.g., (Bertsekas, 2017, Section 3.1)), the Lyapunov function is also the optimal value function for a suitably defined LQR problem. Hence, our aim is to show that the matrix $\mathbf{P}$ is the solution of a DARE (10) for system (6) and that (7) is the optimal controller arising from such a DARE.

We start by studying the DARE for (6). By defining the state weight in (x, η)-coordinates as

$$\mathbf{Q} := \begin{pmatrix} \mathbf{Q}_{xx} & \mathbf{Q}_{x\eta}^\top \\ \mathbf{Q}_{x\eta} & \mathbf{Q}_{\eta\eta} \end{pmatrix}$$

and by selecting $\mathbf{P}$ as a solution, DARE (10) for the cascade (6) reads

$$\begin{pmatrix} A^\top (P - P B Y B^\top P) A & -A^\top P B Y \Psi^\top F \\ -F^\top \Psi Y B^\top P A & F^\top (\mathrm{I} - \Psi Y \Psi^\top) F \end{pmatrix} = \widetilde{\mathbf{P}} \tag{31}$$

where

$$\widetilde{\mathbf{P}} := \begin{pmatrix} P - \mathbf{Q}_{xx} & -\mathbf{Q}_{x\eta}^\top \\ -\mathbf{Q}_{x\eta} & \mathrm{I} - \mathbf{Q}_{\eta\eta} \end{pmatrix}.$$

Since $A^\top P A = P - Q_x$ and $F^\top F = \mathrm{I} - Q_z$, equality (31) can be rewritten as

$$\begin{pmatrix} T_{xx} & T_{x\eta}^\top \\ T_{x\eta} & T_{\eta\eta} \end{pmatrix} = 0 \tag{32}$$

with

$$\begin{aligned} T_{xx} &= \mathbf{Q}_{xx} - (Q_x + A^\top P B Y B^\top P A) \\ T_{\eta\eta} &= \mathbf{Q}_{\eta\eta} - (Q_z + F^\top \Psi Y \Psi^\top F) \\ T_{x\eta} &= \mathbf{Q}_{x\eta} - F^\top \Psi Y B^\top P A \end{aligned}$$

that is solved by the choice

$$\begin{aligned} \mathbf{Q}_{xx} &= Q_x + A^\top P B Y B^\top P A \\ \mathbf{Q}_{\eta\eta} &= Q_z + F^\top \Psi Y \Psi^\top F, \\ \mathbf{Q}_{x\eta} &= F^\top \Psi Y B^\top P A. \end{aligned} \tag{33}$$

Note that $\mathbf{Q} \succeq 0$ since it can be rewritten as

$$\mathbf{Q} = \begin{pmatrix} Q_x & 0 \\ 0 & Q_z \end{pmatrix} + \begin{pmatrix} K^\top & L^\top \end{pmatrix} \begin{pmatrix} Y^{-1} & -Y^{-1} \\ -Y^{-1} & Y^{-1} \end{pmatrix} \begin{pmatrix} K \\ L \end{pmatrix},$$

where we used the definitions of K, L in (13) and the fact that $Y = Y Y^{-1} Y$. We now show that under the selection (30) the above equality is satisfied, namely, that all terms $T_{xx}, T_{x\eta}, T_{\eta\eta}$ are identically zero. Consider the weight matrix $\mathbf{X}$ in (29) in the original (x, z)-coordinates. By definition of the coordinate change $\eta = z - Mx$ corresponding to $\zeta = \mathbf{T}\xi$ with

$$\mathbf{T} = \begin{pmatrix} \mathrm{I} & 0 \\ -M & \mathrm{I} \end{pmatrix},$$

one can rewrite $\zeta^\top \mathbf{Q}\zeta = \xi^\top \mathbf{T}^\top \mathbf{Q}\mathbf{T}\xi = \xi^\top \mathbf{X}\xi$, leading to the selection of the state weighting matrix $\mathbf{X}$ in (2) as

$$\mathbf{X} = \mathbf{T}^\top \mathbf{Q}\mathbf{T}. \tag{34}$$

Accordingly, exploiting (33) and making computations block-component wise yield

$$\begin{aligned}
&\mathbf{X}_{xx} = \mathbf{Q}_{xx} - (M^\top \mathbf{Q}_{x\eta} + \mathbf{Q}_{x\eta}^\top M) + M^\top \mathbf{Q}_{\eta\eta} M,\\
&\mathbf{X}_{zz} = \mathbf{Q}_{\eta\eta},\\
&\mathbf{X}_{xz} = \mathbf{Q}_{x\eta} - \mathbf{Q}_{\eta\eta} M,
\end{aligned}$$

that correspond to (30). Moreover, since M is solution to the Sylvester equation (5), the change of coordinates is always well-defined and $\mathbf{Q} \succeq 0$ implies $\mathbf{X} \succeq 0$. Then, $\mathbf{P}$ is solution to the DARE (31) for the extended system.

We now show that (13) is its related optimal gain. Since $\mathbf{P}$ is solution to the DARE (31), the optimal controller gain is (9). By the definitions of $\mathbf{A}, \mathbf{B}$ in (6) and the ones of Y and $\mathbf{P}$, we obtain

$$\mathbf{K} = -Y\left(B^\top PA \;\; -\Psi^\top F\right) = (K \;\; L),$$

with K, L as in (13). Consequently, the control law (7) is the optimal solution to the minimization problem (2) under (30), and this concludes the proof. □

Remark 4. Notice that, if F in (1) is neutrally stable, $Q_z = 0$. Then, while the weight on the x state can be independently controlled via Q_x, the cost on the z state is regulated solely by the input weight matrix $\mathbf{R}$ entering in Y. This strong interconnection is due to the cascade structure, which inevitably intertwines the behavior of z to the one of u and x.

4. FORWARDING FOR INPUT SATURATED SYSTEMS

Since the first subsystem in the feedforward form (1) is assumed to be stable, one may question the role of the feedback term Kx in (7) aimed at stabilizing its dynamics. Indeed, this term can be dispensed with, and in this section we simplify the resulting controller. As the framework of stable or prestabilized autonomous dynamics naturally fits the study of global stabilization under saturated input (see, e.g., (Tarbouriech et al., 2011, Section 1.6.2)), we propose the abovementioned alternative designs in the context of saturated actuation. This analysis also provides simple hints on the potential of our forwarding designs for nonlinear dynamics.

We consider a linear time-invariant discrete-time system with input saturation in the feedforward form

$$\begin{aligned}
x^+ &= Ax + B\,\mathrm{sat}(u),\\
z^+ &= Fz + Gx,
\end{aligned} \tag{35}$$

where $(x, z) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$ is the state, $u \in \mathbb{R}^{n_u}$ is the control input, and $\mathrm{sat} : \mathbb{R}^{n_u} \to \mathbb{R}^{n_u}$ is a centralized saturation function defined as

$$\mathrm{sat}(s) := \begin{cases} \dfrac{s}{|s|}\min\{r, |s|\}, & s \neq 0,\\ 0 & s = 0, \end{cases}$$

for any desired saturation level $r > 0$. Note that with such a definition, for any the following inequalities hold

$$s^\top R\,\mathrm{sat}(s) = s^\top Rs \min\left\{\frac{r}{|s|}, 1\right\} \geq 0 \tag{36}$$

for any $R \succeq 0$ and any $s \in \mathbb{R}^{n_u}$.

We first propose a result relaxing the necessity of a stabilizing feedback gain K in (7). Furthermore, we will show that global asymptotically stability of the origin can be obtained. Note that since F is only marginally stable, classical LMI-based conditions such as (Tarbouriech et al., 2011, Proposition 3.26) or Lin et al. (1996) cannot be directly applied. Moreover, this structure allows for the derivation of small-gain controllers requiring measurements of one subsystem's state only, as shown in the last part of this section.

Theorem 3. *Let Assumptions 1 and 2 hold. Then, for all* $\mathbf{R} \succ 0$ *the origin of system* (35) *in closed-loop with*

$$u = L(z - Mx) \tag{37}$$

with L *as in* (13)*, is globally asymptotically stable and locally exponentially stable.*

Proof. In this proof, we exploit the definition of Ψ in (16) to improve readability. Under the coordinate transformation $\eta = z - Mx$, the closed-loop system is

$$\begin{aligned}
x^+ &= Ax + B\,\mathrm{sat}(L\eta)\\
\eta^+ &= F\eta - \Psi\,\mathrm{sat}(L\eta).
\end{aligned} \tag{38}$$

Since A is Schur-Cohn stable and given the feedforward form of the closed-loop dynamics, stability of (38) is proven by showing exponential stability of the η dynamics. To this end, let $V_\eta(\eta) = \eta^\top \eta$ and let

$$\overline{\mathbf{R}} := \mathbf{R} + B^\top PB. \tag{39}$$

This notation highlights the relation to LQR design in the η-dynamics, since the controller gain L in (13) reads

$$L = (\overline{\mathbf{R}} + \Psi^\top \Psi)^{-1}\Psi^\top F. \tag{40}$$

Then, the Lyapunov candidate one-step increment reads

$$\begin{aligned}
\Delta V_\eta &= (F\eta - \Psi\,\mathrm{sat}(u))^\top (F\eta - \Psi\,\mathrm{sat}(u)) - \eta^\top \eta\\
&= \eta^\top (F^\top F - \mathrm{I})\eta - \mathrm{sat}(u)^\top \big(2\Psi^\top F\eta - \Psi^\top \Psi\,\mathrm{sat}(u)\big).
\end{aligned}$$

By the definition of L in (40), we have $(\overline{\mathbf{R}} + \Psi^\top \Psi)u = \Psi^\top F\eta$. Then, substituting it into ΔV_η and using the fact that $F^\top F \preceq \mathrm{I}$ yields

$$\begin{aligned}
\Delta V_\eta &\leq -2\,\mathrm{sat}(u)^\top (\overline{\mathbf{R}} + \Psi^\top \Psi)u + \mathrm{sat}(u)^\top \Psi^\top \Psi\,\mathrm{sat}(u).\\
&\leq -2\,\mathrm{sat}(u)^\top \overline{\mathbf{R}}u - \mathrm{sat}(u)^\top \Psi^\top \Psi(2u - \mathrm{sat}(u))\\
&\leq -2\,\mathrm{sat}(u)^\top \overline{\mathbf{R}}u < 0
\end{aligned} \tag{41}$$

for any $u \neq 0$, where in the last step we used inequality (36), the fact that $\overline{\mathbf{R}}$ is positive definite, and the fact that $\mathrm{sat}(u)^\top \Psi^\top \Psi(2u - \mathrm{sat}(u)) > 0$ for any $u \neq 0$, which can be verified again using the property (36). The proof is concluded identically to the proof of Theorem 1 with LaSalle's like arguments and noting that, since the saturation is locally linear, the closed-loop dynamics is linear around the origin, thus proving local exponential stability. □

While Theorem 3 exploits open-loop stability of the x system, the design (37) still requires a state-feedback term which depends on x. We can further simplify the design by looking for a feedback depending solely on the variable z. To this end, we can exploit further Assumption 2, namely, we can leverage the fact that F in (1) is at worst neutrally stable. This allows for the design of a "small-gain" controller, whose aim is to push the eigenvalues of F into the interior of the unit disc without disrupting the stability of A.

Theorem 4. *Let Assumptions 1 and 2 hold. Then, there exists $\mathbf{R} \succ 0$ and $\varepsilon^\star > 0$ such that, for all $\varepsilon \in (0, \varepsilon^\star)$, the origin of system* (35) *in closed-loop with*

$$u = \varepsilon L z \tag{42}$$

with L defined as in (13)*, is globally asymptotically stable and locally exponentially stable.*

Proof. In this proof, we exploit the definition of Ψ and N in (16) to improve readability. Consider the system in the (x, η)-coordinates in (38). Straightforward computations show that the Lyapunov function

$$V(x, \eta) = x^\top P x + \eta^\top \eta$$

satisfies

$$\begin{aligned}\Delta V(x, \eta) =& x^\top \left(A^\top P A - P\right) x + \eta^\top \left(F^\top F - I\right) \eta \\ &+ \operatorname{sat}(u)^\top \left[2 \left(B^\top P A x - \Psi^\top F \eta\right) + N \operatorname{sat}(u)\right].\end{aligned}$$

By definition of $L = (\mathbf{R} + N)^{-1} \Psi^\top F$ and (42) one gets

$$u = \varepsilon L (\eta + M x) \implies \Psi^\top F \eta = \tfrac{1}{\varepsilon} (\mathbf{R} + N) u - \Psi^\top F M x.$$

Substituting the equality above into $\Delta V(x, \eta)$ yields

$$\begin{aligned}\Delta V(x, \eta) =& x^\top \left(A^\top P A - P\right) x + \eta^\top \left(F^\top F - I\right) \\ &+ 2 \operatorname{sat}(u)^\top \left(B^\top P A + \Psi^\top F M\right) x \\ &- \tfrac{2}{\varepsilon} \operatorname{sat}(u)^\top (\mathbf{R} + N) u + \operatorname{sat}(u)^\top N \operatorname{sat}(u).\end{aligned} \tag{43}$$

By Young's inequality and Assumption 1, we obtain

$$2 \operatorname{sat}(u)^\top B^\top P A x \leq \gamma_1 \rho x^\top P x + \frac{1}{\gamma_1} \operatorname{sat}(u)^\top B^\top P B \operatorname{sat}(u), \tag{44a}$$

and recalling that $F^\top F = I$,

$$2 \operatorname{sat}(u)^\top \Psi^\top F M x \leq \gamma_2 x^\top M^\top M x + \frac{1}{\gamma_2} \operatorname{sat}(u)^\top \Psi^\top \Psi \operatorname{sat}(u) \tag{44b}$$

for two arbitrary positive scalars γ_1, γ_2. With the selection $\gamma_1 = \frac{1-\rho}{2\rho}$ and $\gamma_2 = \frac{(1-\rho)\underline{p}}{4|M^\top M|}$ where $\underline{p}$ is the smallest eigenvalue of P, we obtain

$$\begin{aligned}1 - (1 + \gamma_1)\rho &= \frac{1 - \rho}{2} \\ \gamma_2 x^\top M^\top M x &\leq \frac{1 - \rho}{4} x^\top P x.\end{aligned} \tag{45}$$

Combining (43), (44) and (45) and exploiting again Assumption 2, we have

$$\begin{aligned}\Delta V(x, \eta) \leq& -\frac{1 - \rho}{4} x^\top P x - \frac{2}{\varepsilon} \operatorname{sat}(u)^\top (\mathbf{R} + N) u \\ &+ \operatorname{sat}(u)^\top \left(N + \frac{1}{\gamma_1} B^\top P B + \frac{1}{\gamma_2} \Psi^\top \Psi\right) \operatorname{sat}(u) \\ \leq& -\frac{1 - \rho}{4} x^\top P x - \frac{2}{\varepsilon} \operatorname{sat}(u)^\top (\mathbf{R} + N) u \\ &+ (1 + \gamma) \operatorname{sat}(u)^\top N \operatorname{sat}(u).\end{aligned}$$

with $\frac{1}{\gamma} := \min\{\gamma_1, \gamma_2\}$. Selecting any $\varepsilon \in (0, \frac{2}{1+\gamma})$ and using (36) ensures

$$-\frac{2}{\varepsilon} \operatorname{sat}(u)^\top N u + (1 + \gamma) \operatorname{sat}(u)^\top N \operatorname{sat}(u) \leq 0$$

for any u, resulting in

$$\Delta V(x, \eta) \leq -\frac{1 - \rho}{4} x^\top P x - \frac{2}{\varepsilon} \operatorname{sat}(u)^\top \mathbf{R} u.$$

The proof is concluded identically to the proof of Theorem 3. □

5. CONTROL OF SYSTEMS IN THE PRESENCE OF SATURATED DELAYED INPUTS

A direct application of the results of the previous section is given by the stabilization problem of systems in the presence of actuator saturations and input delays. In particular, consider a system of the form

$$z_{k+1} = \mathbf{F} z_k + \mathbf{G} \operatorname{sat}(u_{k-\tau}) \tag{46}$$

where $z \in \mathbb{R}^{n_z}$, $u \in \mathbb{R}$, $\tau \in \mathbb{N}$ is the delay and $\mathbf{F}$ is an invertible matrix satisfying $\mathbf{F}^\top S \mathbf{F} \preceq S$. Recalling that a unitary discrete-time delay corresponds to a simple integrator, it is readily seen that system (46) can be put in the form (35) in which

$$A = \begin{pmatrix} 0_{(\tau-1)\times 1} & \mathrm{I}_{(\tau-1)\times(\tau-1)} \\ 0 & 0_{1\times(\tau-1)} \end{pmatrix}, \quad B = \begin{pmatrix} 0_{(\tau-1)\times 1} \\ 1 \end{pmatrix},$$

and $F = S^{\frac{1}{2}} \mathbf{F} S^{-\frac{1}{2}}$, $G = S^{\frac{1}{2}} \mathbf{G} \begin{pmatrix} 1 & 0_{1\times(\tau-1)} \end{pmatrix}$. The matrix A having all eigenvalues in 0 is Schur-Cohn stable and the conditions of Theorem 3 are all satisfied. We remark that the resulting controller is different from the one presented in (Yang et al., 2021) based on a *nested saturations* approach. As a simple numerical simulation to show the performances of our design, we consider (46) with

$$\mathbf{F} = \begin{pmatrix} -4.441 & 5.8834 & 2.1624 \\ -3.2257 & 4.1251 & 2.0109 \\ 0.3078 & -0.0531 & 0.3159 \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \tau = 4,$$

which satisfies $\mathbf{F}^\top S \mathbf{F} \preceq S$ with

$$S = \begin{pmatrix} 1.826 & -2.2572 & -0.2095 \\ -2.2572 & 2.8888 & 0.0967 \\ -0.2095 & 0.0967 & 0.4733 \end{pmatrix}.$$

Then, the transformed matrices F, G become

$$F = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} -0.1449 & 0 & 0 & 0 \\ -0.0218 & 0 & 0 & 0 \\ 0.6722 & 0 & 0 & 0 \end{pmatrix}$$

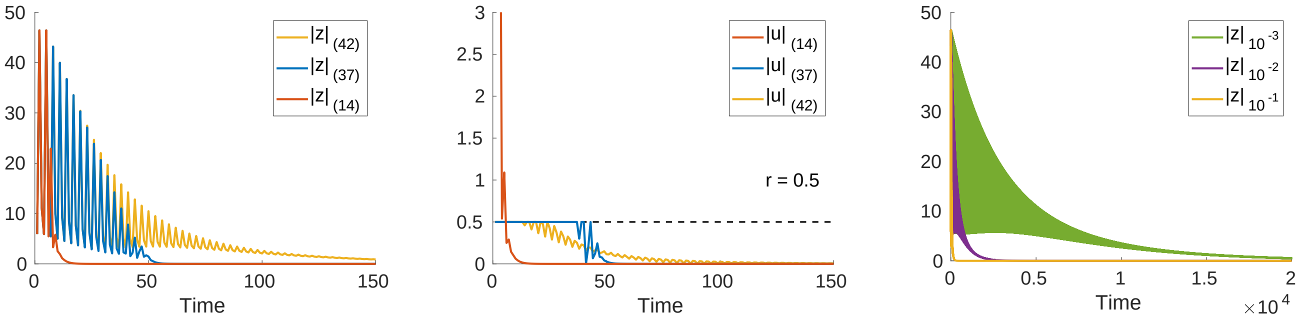
and the Sylvester equation (3) is solved with

$$M = \begin{pmatrix} -0.6722 & 0.0218 & 0.1449 & -0.6722 \\ 0.1449 & -0.6722 & 0.0218 & 0.1449 \\ 0.0218 & 0.1449 & -0.6722 & 0.0218 \end{pmatrix}.$$

We select $\mathbf{R} = 0.01$ and a saturation threshold $r = 0.5$. We present the simulation results under the not saturated control law (14), the saturated law (37) and the saturated controller (42) for $\varepsilon \in \{0.1, 0.01, 0.001\}$. All trajectories are initiated from the same random initial condition. A comparison between the three types of control laws with $\varepsilon = 0.1$ in (42) is presented in Figure 1a, showcasing the different rates of convergence. Figure 1b depicts the norms of the control inputs for the three scenarios, highlighting the difference in control efforts. Finally, Figure 1c presents the norm trajectories under the saturated control law (42) with three different value of ε, whose choice significantly impacts the convergence rate.

6. CONCLUSIONS AND PERSPECTIVES

In this article we revised the forwarding approach for the problem of stabilization of linear systems in feedforward



(a) (b) (c)

Fig. 1. **(a)** Trajectories of the norm of z under control laws (14) (not saturated), (37) (saturated) and (42) (saturated, $\varepsilon = 0.1$). All trajectories start from the same initial condition. **(b)** Input norm for the trajectories in (a). **(c)** Trajectories of the norm of z under the saturated control law (42) for $\varepsilon_1 = 0.1, \varepsilon_2 = 0.01, \varepsilon_3 = 0.001$.

(or cascade) form. Based on a preliminary change of coordinates, it is possible to decouple the systems and transform the problem into a simultaneous stabilization one. This allows easy derivation of various types of control laws. Moreover, these controllers can be directly employed in the stabilization of systems subject to saturation and delayed inputs.

Although the proposed results are conceptually similar to existing results in forwarding stabilization, we believe that this new viewpoint will improve understanding of stabilization of feedforward nonlinear systems. In particular, we aim to develop new feedback laws capable of making closed-loop feedforward nonlinear systems incrementally stable (Fromion and Scorletti, 2002; Tran et al., 2016) or convergent (Pavlov and van de Wouw, 2008; Tran et al., 2018). These results would benefit a wide range of applications, such as repetitive control (Aarnoudse et al., 2023), output regulation (Pavlov and van de Wouw, 2011) of nonlinear systems, and nonlinear constrained convex optimization problems (Häberle et al., 2020; Kelly and Simpson-Porco, 2024).

Appendix A. TECHNICAL LEMMAS

The following lemma presents a result on the detectability properties of control gains arising from Riccati-based designs in discrete time. Similar findings have been proposed in (Yang et al., 2021, Lemma 3) in the context of observability.

Lemma 1. *Let F be invertible and (F, G) be a detectable pair. Then, for any $Y \succ 0$, the pair (F, YGF) is detectable.*

Proof. Due to detectability assumption, there exist $W \succ 0$ such that

$$F^\top WF - W - G^\top G \preceq 0, \tag{A.1}$$

see, e.g., (Hespanha, 2018, Theorem 16.6). By letting $L = YGF$, the above inequality and invertibility of F imply

$$F^\top WF - W - F^{-\top}L^\top Y^{-2}LF^{-1} \preceq 0,$$

By left and right multiplication of $F^\top$ and F respectively, we obtain

$$F^\top(F^\top WF)F - F^\top WF - L^\top Y^{-2}L \preceq 0, \tag{A.2}$$

Moreover, since $Y \succ 0$, there exists $\underline{y}$ such that

$$\underline{y}\,\mathrm{I} \preceq Y^{-2}.$$

Then, by defining $\overline{W} := \underline{y}^{-1}F^\top WF \succ 0$, inequality (A.2) implies

$$F^\top \overline{W} F - \overline{W} - L^\top L \preceq 0,$$

thus showing detectability of the pair (F, L) and concluding the proof. □

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