Distributed Circular Formation Stabilization of Unicycles
Part I: Undirected Information Flow Graph

Mohamed I. El-Hawwary and Manfredi Maggiore

Abstract— We investigate the following problem: design a distributed control law making n kinematic unicycles converge to a common circle of prespecified radius, whose centre is stationary but dependent on the initial conditions, and traverse the circle in a desired direction. Moreover, the vehicles are required to converge to a formation on the circle, expressed by desired separations and ordering of the unicycles. We present a solution for the case when the information flow graph is undirected. In part II of this paper we generalize the solution to the case of arbitrary information flow graphs, and to the case of dynamic unicycles.

I. INTRODUCTION

Consider a system of n kinematic unicycles, with $n \geq 2$,

$$\begin{aligned} \dot{x}_1^i &= u_1^i \cos x_3^i \\ \dot{x}_2^i &= u_1^i \sin x_3^i \qquad i = 1, \ldots, n. \\ \dot{x}_3^i &= u_2^i \end{aligned} \qquad (1)$$

The state of unicycle i is $x^i = (x_1^i, x_2^i, x_3^i) \in \mathbb{R}^2 \times S^1$, where S^1 is the set $(\mathbb{R} \mod 2\pi)$ of real numbers modulo 2π, diffeomorphic to the unit circle. The state space of the overall system is $\mathcal{X} = (\mathbb{R}^2 \times S^1)^n$. Let $\chi = \mathrm{col}(x^1, \cdots, x^n)$ be the overall state, and let $x_3 = \mathrm{col}(x_3^1, \cdots, x_3^n)$. System (1) can be written in the driftless form $\dot{\chi} = g(\chi)u$, with

$$ g = \mathrm{blockdiag} \left\{ \begin{bmatrix} \cos x_3^1 & 0 \\ \sin x_3^1 & 0 \\ 0 & 1 \end{bmatrix}, \cdots, \begin{bmatrix} \cos x_3^n & 0 \\ \sin x_3^n & 0 \\ 0 & 1 \end{bmatrix} \right\}. $$

We will assume that each unicycle has access to its own absolute heading (this can be achieved with an on-board compass) and that it can exchange relative information with some other unicycles. As is customary in the multi-agent literature, the information flow shall be modelled by a directed graph $\mathcal{G}$. Each node of $\mathcal{G}$ represents a unicycle, and the edges of $\mathcal{G}$ represent which unicycles exchange information. Specifically, an edge from node i to node j means that unicycle i has access to its relative displacement and relative heading with respect to unicycle j. Let L denote the Laplacian of the digraph $\mathcal{G}$ of the n-unicycles. We will use the notation L^i for the i-th row of L, and we denote $L_{(2)} = L \otimes I_2$ where I_2 is the 2×2 identity matrix. Refer to [1] for an overview on algebraic graph theory and digraphs. In this paper, we assume that $\mathcal{G}$ is static, and it has a globally

reachable node, i.e. a node with arcs from every other node in the digraph. Equivalently, the graph has a spanning tree. A useful characterization of this property, used in the following sequel, is given in [2] as follows.

Lemma I.1 (Lemma 2, [2]). *The digraph $\mathcal{G}$ has a globally reachable node if and only if 0 is a simple eigenvalue of L.*

By this lemma, if a digraph with Laplacian L has a globally reachable node then $\ker L = \mathrm{span}\, \mathbf{1}$ where $\mathbf{1} = \mathrm{col}(1, \ldots, 1) \in \mathbb{R}^n$ and $\ker$ denotes the kernel. Note also that, by the Gershgorin circle Theorem [3], the eigenvalues of any Laplacian are either zero or have positive real part. Thus, if the digraph has a globally reachable node, then all the eigenvalues of L have positive real part except for one which is zero.

Circular Formation Control Problem (CFCP). Consider the n-unicycles in (1). For a given static information flow digraph $\mathcal{G}$ with a globally reachable node, design a distributed control law achieving the following objectives:

(i) *Circular path following.* For a suitable set of initial conditions, the unicycles should converge to a common circle of radius $r > 0$, whose centre is stationary but dependent on the initial condition, and traverse the circle in a desired direction (clockwise or counter-clockwise). The unicycles' forward speed should be bounded away from zero.

(ii) *Formation stabilization.* On the circle in part (i) of the problem, the n-unicycles are required to converge to a formation expressed by desired separations and ordering of the unicycles.

In Section III, we give a more precise formulation of CFCP as the problem of stabilizing a suitable subset of the overall state space $\mathcal{X}$. In this paper, we solve CFCP in the case where the information flow graph is undirected, which corresponds to the situation when the Laplacian L is symmetric. The solution, presented in Section V (see Proposition V.3), relies on recent results concerning the passivity-based stabilization of closed sets [4], [5]. These results are briefly reviewed in Section IV. In part II, we generalize the solution in two directions: we allow the information graph to be an arbitrary static directed graph with a globally reachable node, and we present the solution to CFCP for dynamic unicycles. Our control design for CFCP provides circular path following in the counter-clockwise direction, but can be easily modified to achieve clockwise path following.

Notation: In this paper we use the following notation. We will denote by S^n the n-torus, i.e., the Cartesian product $S^1 \times \cdots \times S^1$, n times. If A and B are two matrices or vectors,

$\text{col}(A, B)$ denotes the matrix $[A^\top \ B^\top]^\top$ where $\top$ denotes transpose, and $\text{blockdiag}(A, B)$ denotes the block-diagonal matrix with blocks A and B. If $a_1, \ldots, a_n$ are scalars, $\text{diag}(a_1, \ldots, a_n)$ is the diagonal matrix with diagonal entries a_i. By $\phi(t, x_0)$ we denote the solution of $\dot{x} = f(x)$ with initial condition x_0. Given an interval I of the real line and a set $S \in \mathcal{X}$, denote by $\phi(I, S)$ the set $\phi(I, S) := \{\phi(t, x_0) : t \in I, x_0 \in S\}$. We use $\| \cdot \|_S$ to denote the point-to-set distance to a set $S \subset \mathcal{X}$, $B_\alpha(x)$ an open ball of radius α centered at x, and $B_\alpha(S)$ the set of points with distance less than α to S. Denote by $\mathcal{N}(S)$ a generic open neighbourhood of S. We use the standard notation $L_f V$ to denote the Lie derivative of a C^1 function V along a vector field f on $\mathcal{X}$. For a function $f : \mathbb{R}^n \to \mathbb{R}^m$, $f^{-1}(0) = \{x : f(x) = 0\}$ denotes the zero level set of f. Finally, we denote by $A \otimes B$ the Kronecker product of two matrices A and B.

II. PREVIOUS RESULTS

The work in [6], [7] addresses the cyclic pursuit control problem where agent i has communication link with agent $i+1$. The authors obtain circular formations and show that the resulting relative equilibria are generalized regular polygons. The cyclic pursuit law in [6] has been studied in many other works, such as [8] and [9].

Another important research direction on formation stabilization is found in [10], where the authors investigate problems of synchronization for systems of particles modeled as unicycles. Potential functions are defined for various tasks and used to generate gradient control laws. Among other things, the authors stabilize the unicycles to a circle. The results are based on an all-to-all communication assumption. In [11] the authors extend the results in [10] to address different communication topologies. First, they provide a direct extension to the case of undirected time invariant communication topologies. Then they provide dynamic feedbacks to address the case where the communication topology is time varying and directed. The ideas used in [10] and [11] are incorporated in several other works, such as [12], [13].

The results above deal mainly with symmetric formations. In particular, the formations in [6], [7] are regular polygons. In [10] the authors show that general formations can be stabilized using phase potentials that are minimum at desired phase formations; the control design in [10] focuses on symmetric formations using specific potentials. In this paper we present controllers that stabilize arbitrary formations on the circle. Selecting the formation to be stabilized does not require extra design; the formation is simply encoded in a vector parametrizing our feedback controller.

As mentioned earlier, our results in part II solve the circular formation problem for general static graphs. All our controllers are time invariant static feedbacks. The information required by unicycle i is the relative displacement and relative heading with respect to neighbouring unicycles, and its own absolute heading. In [11], the authors allow for time varying and directed graphs by using dynamic feedbacks utilizing consensus filters that asymptotically reconstruct the averaged quantities required by the all-to-all stabilizing control law. The scheme in [11] requires extra communication, since particles must exchange relative estimated variables, in addition to relative displacement and relative heading.

III. CFCP AS A SET STABILIZATION PROBLEM

For $i \in \{1, \ldots, n\}$, define the function $c^i(x^i)$ as

$$c^i(x^i) = (x_1^i - r \sin x_3^i, x_2^i + r \cos x_3^i) \qquad (2)$$

For unicycle i, the point $c^i(x^i)$ lies at a distance r from (x_1^i, x_2^i), and the vector $\text{col}(x_1^i, x_2^i) - c^i(x^i)$ is orthogonal to the normalized velocity vector $(\cos x_3^i, \sin x_3^i)$ of unicycle i, see Figure 1. Therefore, the point $c^i(x^i)$ is the centre of the circle that the unicycle would follow in the counter-clockwise direction if its controls were chosen as $u_1^i = v$ and $u_2^i = v/r$. Using the functions c^i in (2), part (i) of the CFCP can be
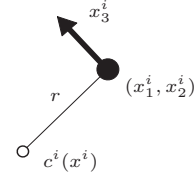

Fig. 1. The centre $c^i(x^i)$

stated as the stabilization of the set

$$\Gamma_1 = \{\chi : c^{i+1}(x^{i+1}) = c^i(x^i), i = 1, \cdots, n\} \qquad (3)$$

with the additional requirements that the linear velocities of the unicycles be bounded away from zero and that $c^i(x^i(t))$, $i = 1, \ldots, n$, tend to constant values. In the above, and in what follows, the indices $i \in \{1, \ldots, n\}$ are evaluated modulo n. For instance, $n + 1$ is identified with 1.

Remark. The function $c^i(x^i)$ gives a smooth map $\mathbb{R}^2 \times S^1 \to \mathbb{R}^2 \times S^1$, $(x_1^i, x_2^i, x_3^i) \mapsto (c^i(x^i), x_3^i)$ which is a diffeomorphism. Using this, instead of the dynamics (1), one can express the unicycle model as

$$\begin{aligned} \dot{c}_1^i &= (u_1^i - r u_2^i) \cos x_3^i \\ \dot{c}_2^i &= (u_1^i - r u_2^i) \sin x_3^i \\ \dot{x}_3^i &= u_2^i. \end{aligned} \qquad (4)$$

We now turn our attention to part (ii) of CFCP. Consider a formation where unicycle j travels on the circle at distance d from unicycle i, as shown in Figure 2. This formation
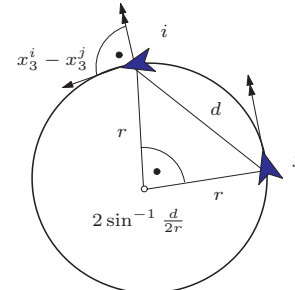

Fig. 2. Formation on the circle

constraint can be equivalently expressed as $x_3^i - x_3^j = 2\sin^{-1}\left(\frac{d}{2r}\right) \mod 2\pi$. In light of this observation, part (ii) of CFCP can be restated as the stabilization of the configuration on the circle where the unicycles headings differ by prespecified constant angles, or $x_3^i(t) = \bar{\alpha}(t) + \alpha_i \mod 2\pi$, $i = 1, \cdots, n$, for some differentiable function $\bar{\alpha}(t)$ and desired angles $\alpha_i \in [0, 2\pi)$ determine the ordering of the unicycles on the circle and their inter-distances. Part (ii) of CFCP can be restated as the stabilization of the set Γ_2 defined as

$$\Gamma_2 = \{\chi : L(x_3 - \alpha) = 0 \mod 2\pi\} \quad (5)$$

where $\alpha = \text{col}(\alpha_1, \cdots, \alpha_n)$ is the vector of desired angles specifying the formation. Notice, indeed, that since we assume that $\mathcal{G}$ has a globally reachable node, $\ker L = \text{span } \mathbf{1}$ and $\chi(t) \in \Gamma_2$ if and only if $x_3^i(t) = \bar{\alpha}(t) + \alpha_i \mod 2\pi$, $i = 1, \cdots, n$. Using the sets Γ_1 in (3) and Γ_2 in (5), CFCP can be restated as follows.

CFCP (equivalent statement). Consider the n-unicycles in (1). For a given information flow digraph $\mathcal{G}$ with a globally reachable node, and a desired formation specification expressed by a vector of angles $\alpha \in S^n$, design a distributed control law which asymptotically stabilizes[1] the set

$$\Gamma = \Gamma_1 \cap \Gamma_2$$
$$= \{\chi : L(x_3 - \alpha) = 0, c^{i+1}(x^{i+1}) = c^i(x^i), 1 \le i \le n\}, \quad (6)$$

where $c^i(x^i)$ is defined in (2). Additionally, the linear velocities u_1^i and angular velocities u_2^i of the unicycles should be bounded away from zero on Γ, and the unicycles should have a *common asymptotic centre of rotation,* by which it is meant that for all $\chi(0) \in \mathcal{X}$ there exists $\bar{c} \in \mathbb{R}^2$ such that $c(x^i(t)) \to \bar{c}$ as $t \to \infty$, $i = 1, \ldots, n$.

Note that Γ is closed but not compact since there are no restrictions on the centres of rotation $c^i(x^i)$.

IV. Passivity-based set stabilization

In this section we review recent results on the stabilization of closed sets by means of passivity-based feedback. We begin with some basic stability definitions concerning a smooth dynamical system $\Sigma : \dot{x} = f(x)$, $x \in \mathcal{X}$, and a closed set $\Gamma \subset \mathcal{X}$.

Definition IV.1 (Set stability and attractivity)**.** The set Γ is

1) *stable* for Σ if, for all $\varepsilon > 0$, there exists a neighbourhood $\mathcal{N}(\Gamma)$ such that $\phi(\mathbb{R}^+, \mathcal{N}(\Gamma)) \subset B_\varepsilon(\Gamma)$.
2) an *attractor* for Σ if there exists a neighbourhood $\mathcal{N}(\Gamma)$ such that, for all $x_0 \in \mathcal{N}(\Gamma)$, $\lim_{t \to \infty} \|\phi(t, x_0)\|_\Gamma = 0$. It is a *global attractor* if it is a attractor with $\mathcal{N}(\Gamma) = \mathcal{X}$.
3) *[globally] asymptotically stable* for Σ if it is stable and attractive [globally attractive] for Σ.

All stability notions in Definition IV.1 can be relativized to a subset of the state space as follows.

Definition IV.2 (Relative set stability and attractivity)**.** Let $\mathcal{O} \subset \mathcal{X}$ be such that $\mathcal{O} \cap \Gamma \ne \emptyset$. We say that Γ is

stable relative to $\mathcal{O}$ for Σ if, for any $\varepsilon > 0$, there exists a neighbourhood $\mathcal{N}(\Gamma)$ such that $\phi(\mathbb{R}^+, \mathcal{N}(\Gamma) \cap \mathcal{O}) \subset B_\varepsilon(\Gamma)$. Similarly, one modifies all other notions in Definition IV.1 by restricting initial conditions to lie in $\mathcal{O}$.

The next definition presents a notion of boundedness near a set.

Definition IV.3 (Local uniform boundedness)**.** Σ is *locally uniformly bounded (LUB) near* Γ if for each $x \in \Gamma$ there exist positive scalars λ and m such that $\phi(\mathbb{R}^+, B_\lambda(x)) \subset B_m(x)$.

Now we turn our attention to the control-affine system

$$\dot{x} = f(x) + \sum_{i=1}^m g_i(x)u_i := f(x) + g(x)u \quad (7)$$
$$y = h(x)$$

with state space $\mathcal{X} \subset \mathbb{R}^n$ that is either an open subset of $\mathbb{R}^n$ or a smooth submanifold. We assume that f and g_i, $i = 1, \ldots, m$, are smooth vector fields on $\mathcal{X}$, and $h : \mathcal{X} \to \mathbb{R}^m$ is a smooth mapping. Suppose that system (7) is passive with C^1 nonnegative storage function $V : \mathcal{X} \to \mathbb{R}$, i.e., [14],

$$(\forall x \in \mathcal{X}) \ L_f V(x) \le 0 \text{ and } L_g V(x) = h(x)^\top, \quad (8)$$

where $L_g V = [L_{g_1} V \ \cdots \ L_{g_m} V]$. We consider the class of passivity-based feedbacks defined as follows.

Definition IV.4 (Passivity-based feedback)**.** A smooth function $u = -\varphi(x)$, where $\varphi(x)$ is such that $\varphi(x) = 0$ whenever $h(x) = 0$, and $h(x)^\top \varphi(x) > 0$ whenever $h(x) \ne 0$, is called a *passivity-based feedback (PBF) with respect to the output* $h(x)$.

Now suppose that $\Gamma \subset \mathcal{X}$ is a closed set which is positively invariant for the open-loop system, i.e., such that, for all $x_0 \in \Gamma$, the solution of the open-loop system $\dot{x} = f(x)$ through x_0 remains in Γ for all positive times. The results in [4] answer this question: under what conditions does a passivity-based feedback with respect to $h(x)$ asymptotically stabilize the set Γ for the control system (7)? The answer to this question relies on the following notion of detectability. Let $\mathcal{O}$ denote the *maximal* open-loop invariant set contained in $h^{-1}(0)$, that is, the set with the property that if $\hat{\mathcal{O}}$ is any other open-loop invariant set contained in $h^{-1}(0)$, then $\hat{\mathcal{O}} \subset \mathcal{O}$.

Definition IV.5 (Γ-detectability)**.** System (7) is *locally Γ-detectable* if Γ is asymptotically stable relative to $\mathcal{O}$ for the open-loop system, and *Γ-detectable* if Γ is globally asymptotically stable relative to $\mathcal{O}$ for the open-loop system.

In [5], the following a procedure was introduced to design set stabilizing controllers.

Set stabilization procedure: Let Γ be a closed goal set that is controlled invariant for (7), i.e., there exists a smooth feedback $u^\star(x)$ such that Γ is a positively invariant set for the closed-loop system $\dot{x} = f(x) + g(x)u^\star(x)$.

1. *Candidate storage function and feedback transformation.*
   a) Find a candidate C^1 storage function $V : \mathcal{X} \to \mathbb{R}^+$ such that $\Gamma \subset V^{-1}(0)$ and $L_f V(x) \le 0$ for all $x \in \mathcal{X}$.

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b) Find, if possible, a locally Lipschitz matrix-valued function $\beta_1(x) : \mathcal{X} \to \mathbb{R}^{m \times k}$, for some $k \in \{1, \ldots, m-1\}$, such that $\beta_1(x)$ has full rank k and $L_g V(x)\beta_1(x) = 0_{1 \times k}$ for all $x \in \mathcal{X}$.

c) Let $\beta_2(x) : \mathcal{X} \to \mathbb{R}^{m \times m-k}$ be any locally Lipschitz function such that $[\beta_1(x) \ \ \beta_2(x)]$ is nonsingular for all $x \in \mathcal{X}$, and define the feedback transformation

$$u = \beta_1(x)\bar{u} + \beta_2(x)\tilde{u}, \tag{9}$$

where $\bar{u} \in \mathbb{R}^k$ and $\tilde{u} \in \mathbb{R}^{m-k}$ are new control inputs. Define an output function $h(x) := L_{g\beta_2}V(x)^\top$.

2. Γ-*detectability enforcement.* Find, if possible, a feedback $\bar{u}(x)$ such that Γ is (globally) asymptotically stable relative to $\mathcal{O}$ for the system $\dot{x} = [f(x) + g(x)\beta_1(x)\bar{u}(x)]|_\mathcal{O}$, where $\mathcal{O}$ is the maximal subset of $h^{-1}(0)$ invariant under the vector field $f + g\beta_1\bar{u}$.

3. *Passivity-based stabilization.* Pick any PBF $\tilde{u}(x)$, and let $u(x) = \beta_1(x)\bar{u}(x) + \beta_2(x)\tilde{u}(x)$, where $\bar{u}(x)$ is the feedback chosen in step 2.

Proposition IV.6 (Set stabilizing procedure)**.** *The feedback $u(x)$ designed according to the procedure above has the following properties:*

(a) If Γ is compact, then $u(x)$ asymptotically stabilizes it.

(b) If Γ is closed and unbounded, then $u(x)$ asymptotically stabilizes it provided that the closed-loop system is LUB near Γ.

(c) In both cases above, if all trajectories of the closed-loop system are bounded, and the Γ-detectability property enforced in step 2 of the procedure is global, then the stabilization of Γ is global as well.

In the rest of this paper we apply this procedure to solve CFCP.

V. SOLUTION OF CFCP FOR UNDIRECTED GRAPHS

In this section we use the passivity-based set stabilization procedure outlined before to solve CFCP when the information flow digraph is undirected.

Step 1: Candidate storage function.

Let $c(\chi) = \text{col}(c^1(x^1), \cdots, c^n(x^n)) \in \mathbb{R}^{2n}$ with $c^i(x^i)$ defined in (2). Consider the following candidate storage function

$$V(\chi) = \frac{1}{2}c(\chi)^\top L_{(2)}\,c(\chi) \tag{10}$$

Since L is symmetric, $L_{(2)}$ is positive semidefinite. Also, since the information digraph has a globally reachable node, from Lemma I.1 we have that $L_{(2)}$ has 2 eigenvalues at 0 with geometric multiplicity 2, and thus

$$\ker L_{(2)} = \text{Image}\left\{\text{col}(I_2, \cdots, I_2)\right\}, \tag{11}$$

from which it follows that $V^{-1}(0)$ is the set where all the centres of rotation coincide, i.e., $V^{-1}(0) = \Gamma_1$. Based on the observation that any feedback of the form $(u_1^i, u_2^i) = (\bar{u}^i(\chi), \bar{u}^i(\chi)/r)$, $i = 1, \cdots, n$, keeps the centres of rotation,

and hence V, constant along solutions of the closed-loop system, we choose the feedback transformation

$$u^i = \begin{bmatrix} u_1^i \\ u_2^i \end{bmatrix} = \beta_1^i\,\bar{u}^i + \beta_2^i\,\tilde{u}^i = \begin{bmatrix} 1 \\ 1/r \end{bmatrix}\bar{u}^i + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\tilde{u}^i, \ 1 \le i \le n.$$

Setting, for $i = 1, \ldots, n$, $u_1^i = \bar{u}^i$, $u_2^i = \bar{u}^i/r + \tilde{u}^i$, we obtain the feedback transformation

$$u = \beta_1\,\bar{u} + \beta_2\,\tilde{u}, \tag{12}$$

with $\beta_1 = \text{blockdiag}\{[1 \ \ 1/r]^\top, \cdots, [1 \ \ 1/r]^\top\}$, $\beta_2 = \text{blockdiag}\{[0 \ \ 1]^\top, \cdots, [0 \ \ 1]^\top\}$. The above feedback transformation has the property that $L_g V(\chi)\beta_1 = 0_{1 \times n}$. Moreover, $L_f V = 0$ because $f = 0$. Therefore, for any feedback $\bar{u}(\chi)$, the system with input $\tilde{u}$ and output $y = h(\chi) := L_{g\beta_2}V(\chi)^\top$ is passive. The output y is given as follows

$$y = h(\chi) = -r\mathcal{R}(x_3)L_{(2)}\,c(\chi) \tag{13}$$

where

$$\mathcal{R}(x_3) = \text{blockdiag}\left\{\left[\cos x_3^1 \ \ \ \sin x_3^1\right], \cdots, \left[\cos x_3^n \ \ \ \sin x_3^n\right]\right\}$$

Step 2: Γ-detectability enforcement.

Lemma V.1. *Let $\bar{u}(\chi)$ be any feedback which is bounded away from zero component-wise, i.e., for some $\varepsilon > 0$, $\inf_\chi \bar{u}^i(\chi) \ge \varepsilon > 0$ for $i = 1, \cdots, n$. Then, the maximal subset of $h^{-1}(0)$ invariant under the vector field $f + g\beta_1\bar{u}$ is Γ_1, i.e., $\mathcal{O} = \Gamma_1$.*

Proof: As observed earlier, if $\tilde{u} = 0$ and $\inf_\chi \bar{u} > \varepsilon > 0$ component-wise, then each unicycle moves along a circle of radius r, and so the vector $L_{(2)}\,c$, in the output function (13), is constant. Suppose that, for some solution $\chi(t)$ of the system with $\tilde{u} = 0$, $h(\chi(t)) \equiv 0$. Then, either $L_{(2)}\,c(\chi(t)) = 0$ which, because of (11), is only possible when all the centres coincide, i.e., when $\chi(t) \in \Gamma_1$, or, for some i, the constant vector $L_{(2)}c$ is perpendicular to the vector $[0 \ 0 \ \cdots \ \cos x_3^i(t) \ \sin x_3^i(t) \ \cdots \ 0 \ 0]^\top$, for $i = 1, \cdots, n$ and $t \in \mathbb{R}$, implying that $x_3^i(t)$ is constant. However, by assumption the unicycles move along n circles with nonzero linear velocity vectors, and therefore the angle $x_3^i(t)$ is not constant. ∎

As mentioned earlier, the functions $c^i(x^i)$ in (2) remain constant along the solutions of (1) with feedback transformation (12) and $\tilde{u} = 0$. When $\tilde{u} = 0$, the restriction of the vector field $f + g\beta_1\bar{u}$ to $\mathcal{O} = \Gamma_1$ is

$$\dot{x}_1^i = \bar{u}^i \cos x_3^i, \ \ \dot{x}_2^i = \bar{u}^i \sin x_3^i, \ \ \dot{x}_3^i = \frac{1}{r}\bar{u}^i \tag{14}$$

Using the model (4), the dynamics above takes the form

$$\dot{c}_1^i = 0, \ \ \dot{c}_2^i = 0, \ \ \dot{x}_3^i = \frac{1}{r}\bar{u}^i \tag{15}$$

i.e., the dynamics of the unicycles are entirely described by those of their angular velocities $\dot{x}_3^i$. Under the assumption of Lemma V.1, the goal set Γ can be expressed as

$$\Gamma = \{\chi \in \mathcal{O} : L(x_3 - \alpha) = 0\}, \tag{16}$$

so we need to design $\bar{u}$ to stabilize the set $\{x_3 : L(x_3 - \alpha) = 0\}$. In designing the stabilizer, we must take into account

the fact that $x_3^i \in S^1$, so the stabilization must be performed modulo 2π. To fulfill the assumption of Lemma V.1, we also need $\bar{u}^i$ to be bounded away from zero. There are many ways to obtain these objectives. We base our design on the following candidate Lyapunov function

$$W(x_3) = \sum_{i=1}^{n}[1 - \cos(L^i(x_3 - \alpha))] \quad (17)$$

where L^i is the i-th row of the Laplacian L. Note that $W \geq 0$ and $W = 0$ if and only if $L^i(x_3 - \alpha) = 0 \mod 2\pi$, for $i = 1, \cdots, n$. Thus $W^{-1}(0)$ is precisely the set we wish to stabilize. The derivative of W along (14) is

$$\dot{W} = \sum_{i=1}^{n} \sin(L^i(x_3 - \alpha))L^i \bar{u}/r = \frac{1}{r}S(x_3)^\top L \bar{u}, \quad (18)$$

where

$$S(x_3) = \begin{bmatrix} \sin(L^1(x_3 - \alpha)) \\ \vdots \\ \sin(L^n(x_3 - \alpha)) \end{bmatrix}.$$

Lemma V.2. *The feedback*

$$\bar{u}^i = v - v_1 \sin(L^i(x_3 - \alpha)), \ i = 1, \ldots, n, \quad (19)$$

where and $v > v_1 > 0$ are design constants, is bounded away from zero component-wise and makes the set Γ asymptotically stable relative to Γ_1 for system (1) after feedback transformation (12) and $\tilde{u} = 0$, thus enforcing local Γ-detectability of the system with input $\tilde{u}$ and output $y = h(\chi)$ in (13).

Proof: By Lemma V.1, the maximal subset of $h^{-1}(0)$ invariant under the vector field $f + g\beta_1\bar{u}$ is $\mathcal{O} = \Gamma_1$. Referring to the system restriction on $\mathcal{O}$ in (15), to prove the Lemma it suffices to show that the set $W^{-1}(0)$ is asymptotically stable for the system $\dot{x}_3^i = \bar{u}^i/r$, $i = 1, \cdots, n$, with $\bar{u}^i$ given in (19). By substituting the control (19) into the derivative (18) we get $\dot{W} = -v_1 S(x_3)^\top L S(x_3)$. The matrix L is positive semidefinite with one eigenvalue at zero and so W is nonincreasing along solutions, proving that $W^{-1}(0)$ is stable. As for its attractivity, since $(x_3^1, \ldots, x_3^n) \in S^n$ is compact, we can apply the LaSalle invariance principle and conclude that, for all initial conditions, $S(x_3^1(t), \ldots, x_3^n(t)) \to \ker L = \operatorname{span} \mathbf{1}$. Therefore, there exists a C^1 real-valued function $s(t)$ such that $\sin(L^i(x_3 - \alpha)) \to s(t)$ for all i. Let $\Omega = \{x_3 : W(x_3) < 1 - \min\{\cos(2\pi/n), 0\}\}$. The set Ω is positively invariant. Moreover, since for each $x_3 \in \Omega$ and each $i \in \{1, \ldots, n\}$, $1 - \cos(L^i(x_3 - \alpha)) \leq W(x_3) < 1 - \min\left\{\cos\frac{2\pi}{n}, 0\right\}$, we have $\cos(L^i(x_3 - \alpha)) > \min\{\cos(2\pi/n), 0\}$, so that

$$|L^i(x_3 - \alpha)| < \min\{2\pi/n, \pi/2\} \text{ modulo } 2\pi. \quad (20)$$

Now let $x_3(0)$ be an arbitrary initial condition in Ω. Since for all $i \in \{1, \ldots, n\}$, $|L^i(x_3 - \alpha)| < \pi/2$ we can invert the $\sin$ function and deduce that $(\forall i \in \{1, \ldots, n\})$ $L^i(x_3 - \alpha) \to \arcsin s(t) \mod 2\pi$, or $L(x_3 - \alpha) \to \mathbf{1}\arcsin s(t) \mod 2\pi$. Since $\ker L = \ker L^\top = \operatorname{span} \mathbf{1}$, we have $\mathbf{1}^\top L(x_3 - \alpha) = 0$, and therefore it must

be that $\mathbf{1}^\top \mathbf{1} \arcsin s(t) = 0 \mod 2\pi$, or $n \arcsin s(t) = 0 \mod 2\pi$. In other words, $\arcsin s(t) \in \{2\pi k/n + 2\pi l, \ k, l \in \mathbb{N}\}$. But since $|L^i(x_3 - \alpha)| < \min\{2\pi/n, \pi/2\} \mod 2\pi$, it must be that $\arcsin s(t) = 0 \mod 2\pi$, proving that $W^{-1}(0)$ is attractive, and hence asymptotically stable. ∎

Step 3: Passivity-based stabilization.

We are now ready to solve CFCP in the case of undirected information flow graph.

Proposition V.3 (Solution of CFCP for undirected information flow graph)**.** *Assume that the information flow graph is undirected and has a globally reachable node. For any $v > v_1 > 0$, there exists $K^\star > 0$ such that for all $K \in (0, K^\star)$ the feedback*

$$u_1^i = v - v_1 \sin(L^i(x_3 - \alpha))$$
$$u_2^i = \frac{u_1^i}{r} - Kh_i(\chi), \qquad i = 1, \cdots, n \quad (21)$$

where $h(\chi)$ is given in (13), solves CFCP and renders the goal set Γ in (6) asymptotically stable, and Γ_1 in (3) globally asymptotically stable for the closed-loop system.

The proof is omitted due to space limitations.

Remark. Note that unicycle i needs to compute

$$h_i(\chi) = [0 \ 0 \ \cdots \ -r\cos x_3^i \ -r\sin x_3^i \ \cdots \ 0 \ 0]L_{(2)}c(\chi)$$

and $L^i(x_3 - \alpha)$. In order to perform this computation, the unicycle needs to sense its relative displacement and orientation with respect to its neighbours in the information flow graph, as well as its absolute orientation x_3^i. Therefore, feedback (21) is a distributed control law respecting the information flow graph.

Simulations

We consider 6 unicycles exchanging information in a cyclic manner: unicycle i exchanges information with unicycles $i + 1$ and $i - 1$. The Laplacian of the information flow digraph is

$$L = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

We present simulation results for the following two formations.

A. The unicycles converge to a circular formation whereby they are uniformly distributed on the circle in a counter-clockwise cyclic order $\{1, 2, 3, 4, 5, 6\}$. To achieve this formation the vector α is set as: $\alpha = \begin{bmatrix} 0 & \frac{2\pi}{6} & \frac{4\pi}{6} & \frac{6\pi}{6} & \frac{8\pi}{6} & \frac{10\pi}{6} \end{bmatrix}^\top$.

B. The unicycles converge to a circular formation whereby they are uniformly distributed on half of the circle in a counter-clockwise cyclic order $\{1, 2, 3, 4, 5, 6\}$. To achieve this formation the vector α is set as: $\alpha = \begin{bmatrix} 0 & \frac{2\pi}{10} & \frac{4\pi}{10} & \frac{6\pi}{10} & \frac{8\pi}{10} & \frac{10\pi}{10} \end{bmatrix}^\top$.

Figures 3 and 4 show the simulations results for cases A and B using feedback (21) with the following parameters: $r = 1$, $v = 1$, $v_1 = 0.9$, and $K = 1$. Empirically, we observed that by either increasing or decreasing K beyond 1 the convergence of the centres of rotation degrades.
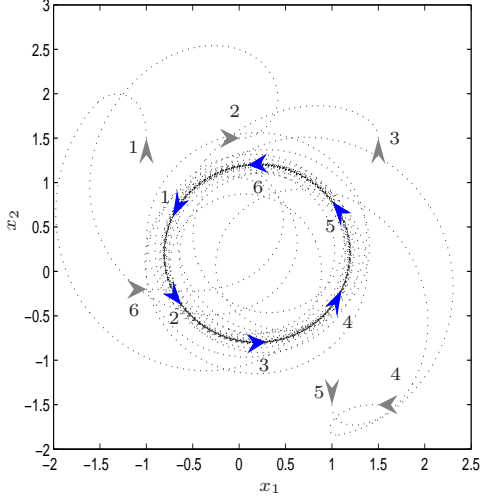

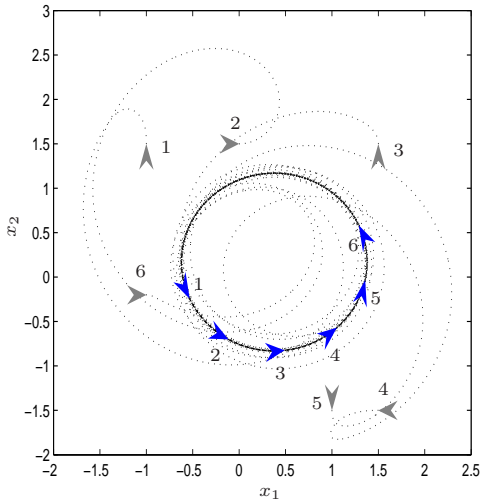
Fig. 3. CFCP Simulation - A

Fig. 4. CFCP Simulation - B

Global solution of CFCP

The passivity-based design in Section V took into account the fact that $x_3^i \in S^1$ and so the stabilization was performed modulo 2π. This was accomplished by using the function W in (17), which is 2π-periodic with respect to x_3^i, $i = 1, \ldots, n$. Moreover, the centres $c^i(x^i)$, upon which the output (13) depends, are 2π-periodic with respect to x_3^j, $j = 1, \ldots, n$. Motivated by the fact that several results in literature, including the work in [6], do not account for the fact that $x_3 \in S^1$, in this section we present a variation of the controller solving

CFCP in Proposition V.3 which assumes that $x_i^3 \in \mathbb{R}$, rather than S^1, but globally asymptotically stabilizes the goal set Γ, hence solving CFCP globally.

Proposition V.4 (Global solution of CFCP for undirected information flow graph). *Assume that the information flow graph is undirected and has a globally reachable node. Let $v > 0$, and let[2] $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be defined as $\varphi(y) = \phi(y)y$, where $\phi : \mathbb{R}^n \to (0, +\infty)$ is a locally Lipschitz map such that $\sup_{\mathbb{R}^n} \|\varphi\| < v$. Then, there exists $K^\star > 0$ such that, for all $K \in (0, K^\star)$, the feedback*

$$
\begin{aligned}
u_1^i &= v - \varphi_i(L(x_3 - \alpha)) \\
u_2^i &= \frac{u_1^i}{r} - K h_i(\chi), \qquad i = 1, \cdots, n,
\end{aligned}
\tag{22}
$$

where $h(\chi)$ is defined in (13) and φ_i denotes the i-th component of φ, globally asymptotically stabilizes Γ in (6) and solves CFCP globally when the state space is taken to be $\mathcal{X} = \mathbb{R}^{3n}$.

To prove this result one cannot use the same method as that of Proposition V.3 because $x_3^i(t)$ is no longer a solution on the compact set S^1.

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[2] One possible choice of function φ is $\varphi(y) = v_1 \min\{1, 1/\|y\|\}y$, with $0 < v_1 < v$.