ON THE STATE AGREEMENT PROBLEM FOR MULTIPLE NONLINEAR DYNAMICAL SYSTEMS

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Abstract: This paper studies the state agreement problem with the objective to ensure the asymptotic coincidence of all states of multiple nonlinear dynamical systems. The coupling structure of such systems is characterized in qualitative terms by means of a suitably defined directed graph. Under a suitable sub-tangentiality assumption on the vector fields of the systems, we obtain a necessary and sufficient graphical condition for their state agreement via nonsmooth analysis, with the invariance principle playing a central role. As applications, we study synchronization of coupled Kuramoto oscillators and synthesis of a rendezvous controller for a multi-agent system. *Copyright ©2005 IFAC*

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1. INTRODUCTION

Recent years have seen an increasingly broad range of studies on the state agreement problem, which goes back at least to (DeGroot, 1974). In this problem, the objective is to ensure the asymptotic coincidence of all or some states of multiple dynamical systems via distributed interactions. This is also referred to as the synchronization problem, see (Pogromsky *et al.*, 2002) and references therein.

In (Ando *et al.*, 1999), distributed algorithms were presented with the objective of getting a group of autonomous synchronous robots to congregate at a common location. These algorithms have been extended to various synchronous and asynchronous stop-and-go strategies in (Lin *et al.*, 2003), and (Cortes *et al.*, 2004). The work of (Jadbabaie *et al.*, 2003) on the agreement problem attracted the attention of many researchers. For example, represented by a linear continuous-time system, a group of agents were shown in (Lin *et al.*, 2004) to globally asymptotically converge to a single point under certain graphical conditions using the tool of the graph Laplacian. Meanwhile,

with this model, the problem of information consensus among multiple agents under fixed or dynamic topology was addressed by the same tool in (Beard and Stepanyan, 2003) and (Saber and Murray, 2003). Moreover, (Lin *et al.*, 2005) investigated the agreement problem of multiple unicycles, and Moreau studied the stability of linear continuous-time distributed consensus algorithms (Moreau, 2004) and nonlinear discrete-time distributed consensus algorithms (Moreau, 2005). In the latter work, assuming that the vector fields of the agents satisfy a certain convexity property, Moreau obtained necessary and sufficient graphical conditions for agreement with time-dependent graph topology. In a different context, multiple oscillators are coupled by some type of connection and the goal is synchronization (e.g., (Strogatz, 2000), (Jadbabaie *et al.*, 2004)).

As a natural extension to our previous works (Lin *et al.*, 2004) (Lin *et al.*, 2005), and inspired by the work of Moreau (Moreau, 2005), here we study the state agreement problem of *continuous time* nonlinear interconnected systems, which can describe a number of networks of coupled dynamical systems. The coupling structure of such systems

is usefully characterized in qualitative terms by means of a suitably defined *static* digraph, which is called the *interaction digraph* in the present paper. However, for such systems, the graph Laplacian alone cannot capture the nature of the system. So we would like, from the perspective of nonlinear dynamics, to understand how a dynamic network of interacting systems will behave collectively, given their individual dynamics and coupling architecture.

The state agreement problem turns out to be equivalent to asymptotic stability with respect to a specified set. Assuming that the systems' vector fields satisfy a certain sub-tangentiality condition, we prove that the interconnected system is globally asymptotically stable with respect to this set if and only if the interaction digraph is quasi strongly connected. Our technical approach relies on nonsmooth analysis involving the Dini derivative and with the invariance principle playing a central role, which is different from that for the linear cases in the literature that rely on algebraic graph theory; it is also different from the convexity argument in (Moreau, 2005).

Finally, as applications, we apply our result to the synchronization problem of the Kuramoto model of coupled nonlinear oscillators and to controller design to solve the rendezvous problem in multi-agent systems.

Several proofs are omitted due to pagelength requirements.

2. PRELIMINARIES

2.1 Convex Set and Cone

We introduce basic concepts, notations and some properties regarding convex set and cone.

The *convex hull* of $\mathcal{S} \subset \mathbb{R}^m$ is the smallest convex set containing $\mathcal{S}$. The convex hull of a finite set of points $x_1, \ldots, x_n \in \mathbb{R}^m$ is a *polytope* and is denoted by $\mathrm{co}\{x_1, \ldots, x_n\}$.

Consider any norm $\|\cdot\|$ in $\mathbb{R}^m$. For each nonempty subset $\mathcal{S}$ of $\mathbb{R}^m$ and each $x \in \mathbb{R}^m$, we denote the distance of x from $\mathcal{S}$ by $\|x\|_{\mathcal{S}} := \inf_{y \in \mathcal{S}} \|x - y\|$.

A set $\mathcal{K} \subset \mathbb{R}^m$ is a *cone* if $\lambda x \in \mathcal{K}$ when $x \in \mathcal{K}$ and $\lambda > 0$. Let $\mathcal{S} \subset \mathbb{R}^m$ be a closed convex set and $x \in \mathcal{S}$. The *tangent cone* (often referred to as *contingent cone*) to $\mathcal{S}$ at x is the set

$$\mathcal{T}(x, \mathcal{S}) = \left\{ y \in \mathbb{R}^m : \liminf_{\lambda \to 0} \frac{\|x + \lambda y\|_{\mathcal{S}}}{\lambda} = 0 \right\}$$

and the *normal cone* to $\mathcal{S}$ at x is

$$\mathcal{N}(x, \mathcal{S}) = \{ y^* \mid \langle y, y^* \rangle \leq 0, \ \forall y \in \mathcal{T}(x, \mathcal{S}) \}.$$

Note that if x is in the interior of $\mathcal{S}$, then $\mathcal{T}(x, \mathcal{S}) = \mathbb{R}^m$. Thus the set $\mathcal{T}(x, \mathcal{S})$ is non-trivial
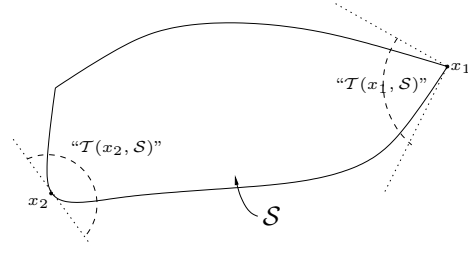

Fig. 1. Tangent cones are obtained by translation of "$\mathcal{T}(x_1, \mathcal{S})$" and "$\mathcal{T}(x_2, \mathcal{S})$" to the origin.

only on the boundary of $\mathcal{S}$. In particular, if $\mathcal{S}$ contains only one point, x, then $\mathcal{T}(x, \mathcal{S}) = \{0\}$. In geometric terms (see Fig. 1), the tangent cone for $x \in \partial S$ is a cone having center at the origin and which contains all the vectors whose directions point from x 'inside' (or they are 'tangent to') the set S.

Lemma 1. (Aubin, 1991) Let $\mathcal{S}_1, \mathcal{S}_2$ be closed convex sets in $\mathbb{R}^m$. If $x \in \mathcal{S}_1 \subset \mathcal{S}_2$, then $\mathcal{T}(x, \mathcal{S}_1) \subset \mathcal{T}(x, \mathcal{S}_2)$ and $\mathcal{N}(x, \mathcal{S}_2) \subset \mathcal{N}(x, \mathcal{S}_1)$.

2.2 Directed Graph

We review some selected notions in graph theory and present a property of a directed graph.

For a directed graph (digraph for short) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{v_1, \ldots, v_n\}$ is the set of nodes and $\mathcal{E}$ is the set of arcs, if there is a path in $\mathcal{G}$ from one node v_i to another node v_j, then v_j is said to be *reachable* from v_i, written $v_i \to v_j$. Note that every node of a digraph is reachable from itself.

A node v from which every node of the digraph is reachable is called a *centre* of the digraph.

A digraph is said to be *quasi strongly connected* (QSC) (called *arbitrated* in (Even, 1979)) if for every two nodes v_i and v_j there is a node v from which v_i and v_j are reachable.

A digraph is said to be *fully connected* if for every two nodes v_i and v_j there is an arc from v_i to v_j.

Lemma 2. (Berge and Ghouila-Houri, 1965) A digraph is QSC if and only if it has a centre.

2.3 Dini Derivative and Invariance Principle

The following is a brief introduction to the Dini derivative and LaSalle's invariance principle.

For the autonomous system

$$\dot{x} = f(x), \tag{1}$$

we assume only that $f : \mathcal{D} \longrightarrow \mathbb{R}^m$ is continuous, where $\mathcal{D}$ is an open subset of $\mathbb{R}^m$. With only continuity, uniqueness of solutions is not assured.

Let x_0 be a point of $\mathcal{D}$. The initial time will always be chosen equal to 0. A non-continuable solution with $x(0) = x_0$ will be written $x : (\alpha, \omega) \to \mathbb{R}^m$, where $\alpha \leq 0 \leq \omega$, and we shall write $J^+ = [0, \omega)$.

Let $V(x) : \mathcal{D} \to \mathbb{R}$ be locally Lipschitz. The upper Dini derivative of V along the trajectory of (1) is

$$D^+ V(x(t)) = \limsup_{\tau \to 0^+} \frac{V(x(t + \tau)) - V(x(t))}{\tau}.$$

Then we have the following property.

Lemma 3. (Danskin, 1966) Let $\mathcal{I}_0 = \{1, 2, \ldots, n\}$ and suppose for each $i \in \mathcal{I}_0$, $V_i(x) : \mathcal{D} \to \mathbb{R}$ is of class C^1. Let $V(x) = \max_{i \in \mathcal{I}_0} V_i(x)$. Then

(a) V is locally Lipschitz;
(b) $D^+ V(x(t)) = \max_{i \in \mathcal{I}(x(t))} \dot{V}_i(x(t))$, where $\mathcal{I}(x) = \{i \in \mathcal{I}_0 : V_i(x) = V(x)\}$.

The positive limit set of a solution $x(t)$ will be denoted $\Lambda^+(x_0)$. The following celebrated theorem is LaSalle's invariance principle.

Theorem 1. (Rouche et al.*, 1975)* Let x be a solution of (1) and $V : \mathcal{D} \to \mathbb{R}$ a locally Lipschitz function such that $D^+ V(x) \leq 0$ on $x(J^+)$. Then $\Lambda^+(x_0) \cap \mathcal{D}$ is contained in the union of all solutions that remain in $\mathcal{X} = \{x \in \mathcal{D} : D^+ V(x) = 0\}$ on their maximal intervals of definition.

3. THE STATE AGREEMENT PROBLEM

Consider a nonlinear interconnected large-scale system which is composed of n subsystems with the index set $\mathcal{I}_0 = \{1, 2, \ldots, n\}$ and is represented by the equations of the form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \ldots, x_n), \\ &\vdots \\ \dot{x}_n &= f_n(x_1, \ldots, x_n), \end{aligned} \tag{2}$$

where $x_i \in \mathbb{R}^m$, $i \in \mathcal{I}_0$, or in vector form

$$\dot{x} = f(x), \tag{3}$$

where $x \in \mathbb{R}^{mn}$. Associate to this system a digraph describing the coupling structure of the n subsystems.

Definition 1. An *interaction digraph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of

- a finite set $\mathcal{V}$ of n nodes, each node modeling a subsystem;
- an arc set $\mathcal{E}$ representing the links between subsystems. An arc from node j to node i indicates that subsystem j is a neighbor of subsystem i in the sense that f_i depends on x_j, i.e., there exist $x_j^1, x_j^2 \in \mathbb{R}^m$ such that

  $$f_i(x_1, \ldots, x_j^1, \ldots, x_n) \neq f_i(x_1, \ldots, x_j^2, \ldots, x_n).$$

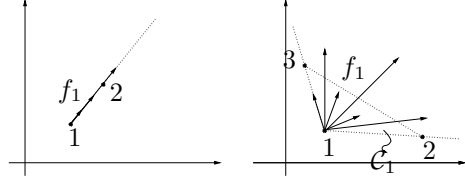 The set of neighbors of node i is denoted $\mathcal{N}_i$.

Fig. 2. Vector fields satisfying A2.

Let $\mathcal{C}_i = \mathrm{co}\{x_i, x_j : j \in \mathcal{N}_i\}$ be the polytope formed by the states of subsystem i and its neighbors. Now some properties of the vector field are assumed for each $i \in \mathcal{I}_0$:

A1. f_i is continuous on $\mathbb{R}^{mn}$;

A2. for all $x \in \mathbb{R}^{mn}$, $f_i(x) \in \mathcal{T}(x_i, \mathcal{C}_i)$, but $f_i(x) \neq 0$ if $\mathcal{C}_i$ is not a singleton and x_i is its vertex.

Fig. 2 illustrates two example situations of A2.

Next are some definitions of stability (asymptotical stability) with respect to a set.

Definition 2. Let $\Omega \subset \mathbb{R}^{nm}$ be a closed, invariant set for the system (2) (it is emphasized that Ω is not required to be bounded). Then with respect to Ω, the system (2) is called

(1) *stable* if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$\|x^0\|_\Omega \leq \delta \implies (\forall t \geq 0) \ \|x(t, x^0)\|_\Omega \leq \varepsilon;$$

(2) *globally asymptotically stable* (GAS) if it is stable and, $\forall x^0 \in \mathbb{R}^{mn}$, $\lim_{t \to \infty} \|x(t, x^0)\|_\Omega = 0$.

In what follows, let $\mathbf{1} = [1, \ldots, 1]^T \in \mathbb{R}^n$ and $\Omega = \{x \in \mathbb{R}^{mn} \mid x = \bar{x} \otimes \mathbf{1}, \bar{x} \in \mathbb{R}^m\}$. It can be easily checked that Ω is a closed invariant set for (2). Then state agreement is precisely asymptotic stability of the system (2) with respect to Ω.

The first result is stability of the interconnected system (2) without needing any property of the interaction digraph.

Theorem 2. The interconnected system (2) is stable with respect to Ω.

The second result shows the relevance of the coupling structure to global asymptotical stability with respect to Ω.

Theorem 3. The interconnected system (2) is GAS with respect to Ω if and only if the interaction digraph $\mathcal{G}$ is QSC.

Sketch of Proof: (Sufficiency) Consider an arbitrary $x^0 \in \mathbb{R}^{mn}$ and let $x(t)$ be a solution of (2) defined on the maximal interval $[0, \omega) \subseteq [0, \infty)$ with $x(0) = x^0$. Such a solution exists by Peano's Theorem.
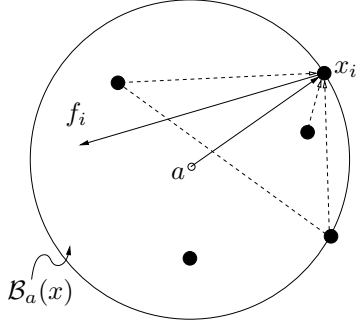
Fig. 3. Illustrations for $\mathcal{B}_a(x)$ and $f_i(x)$.

Fix a point $a \in \mathbb{R}^m$ and define

$$V_i^a(x) = \frac{1}{2}\|x_i - a\|^2, \quad V^a(x) = \max_{i \in \mathcal{I}_0} V_i^a(x).$$

Define $\mathcal{I}(x) = \{i \in \mathcal{I}_0 : V_i^a(x) = V^a(x)\}$, the set of indices where the maximum is reached. By Lemma 3, V^a is locally Lipschitz and its Dini derivative along the solution $x(t)$ is given by

$$D^+ V^a(x(t)) = \max_{i \in \mathcal{I}(x(t))} \dot{V}_i^a(x(t)).$$

Define $\mathcal{B}_a(x) = \{y \in \mathbb{R}^m : \|y - a\|^2 \leq 2V^a(x)\}$. Then $\mathcal{C}_i \subset \mathcal{B}_a(x)$ (see Fig. 3). It follows that, for each $i \in \mathcal{I}(x)$,

$$\dot{V}_i^a(x(t)) = L_f V_i^a(x) = (x_i - a)^T f_i(x) \leq 0$$

since $(x_i - a) \in \mathcal{N}(x_i, \mathcal{B}_a(x))$, the normal cone, and $f_i(x) \in \mathcal{T}(x_i, \mathcal{C}_i) \subset \mathcal{T}(x_i, \mathcal{B}_a(x))$ by assumption A2 and Lemma 1. Hence, $D^+ V^a(x(t)) \leq 0$. Thus $V^a(x(t)) \leq V^a(x(0))$ for all $t \in [0, \omega)$ and $x(t)$ is bounded. By properties of limit sets (Appendix III in (Rouche $et~al.$, 1975)), it follows that the positive limit set $\Lambda^+(x^0)$ is nonempty, compact, and connected. Furthermore, $\omega = \infty$ and $x(t) \to \Lambda^+(x^0)$ as $t \to \infty$.

Furthermore, by Theorem 1, $\Lambda^+(x^0) \subset \mathcal{M}$, where $\mathcal{M}$ is the union of all solutions that remain in $\mathcal{X}_a = \{x \in \mathbb{R}^{mn} : D^+ V^a(x) = 0\}$. This holds for any $a \in \mathbb{R}^m$. Choose any other two arbitrary points $b, c \in \mathbb{R}^m$. Then $\Lambda^+(x^0) \subset \mathcal{M}'$, too, where $\mathcal{M}'$ is the union of all solutions that remain in $\mathcal{X}_b \cap \mathcal{X}_c$.

Next, it can be shown that $\mathcal{M}' \subset \Omega$. Thus, the solution $x(t) \to \Omega$ as $t \to \infty$. Together with stability given in Theorem 2, global asymptotical stability follows.

(Necessity) To prove the contrapositive form, assume that $\mathcal{G}$ is not QSC, that is, there are two nodes i^* and j^* such that for any node k, either i^* or j^* is not reachable from k. Let $\mathcal{V}_1$ be the subset of nodes from which i^* is reachable and $\mathcal{V}_2$ be the subset of nodes from which j^* is reachable. Obviously, $\mathcal{V}_1$ and $\mathcal{V}_2$ are disjoint. Moreover, for each node $i \in \mathcal{V}_1$ (resp. $\mathcal{V}_2$), the set of neighbors of node i is a subset of $\mathcal{V}_1$ (resp. $\mathcal{V}_2$).

Choose any $z_1, z_2 \in \mathbb{R}^m$ such that $z_1 \neq z_2$, and pick initial conditions

$$x_i(0) = \begin{cases} z_1, & \forall~i \in \mathcal{V}_1, \\ z_2, & \forall~i \in \mathcal{V}_2. \end{cases}$$

Then by assumption A2,

$$x_i(t) = \begin{cases} z_1, & \forall~i \in \mathcal{V}_1, \\ z_2, & \forall~i \in \mathcal{V}_2, \end{cases} \quad \forall~t \geq 0.$$

This proves that the system is not GAS with respect to Ω. ∎

$Remark~1.$ For the interconnected system (2), suppose the condition in A2 does not hold for all $x \in \mathbb{R}^{mn}$, but instead it holds for all x in a set $\mathcal{D} \subset \mathbb{R}^{mn}$ which contains Ω and is positively invariant for the system (2). Then the interconnected system (2) is asymptotically stable with respect to Ω for all initial states in $\mathcal{D}$ if and only if the interaction digraph is QSC.

4. APPLICATIONS

In this section, we discuss two illustrative examples: synchronization of the Kuramoto model of coupled nonlinear oscillators; control synthesis for the rendezvous problem of multi-agent systems.

The Kuramoto model describes the dynamics of a set of n phase oscillators θ_i with natural frequencies ω_i. More details can be found in (Strogatz, 2000), (Jadbabaie $et~al.$, 2004). The time evolution of the i-th oscillator is given by

$$\dot{\theta}_i = \omega_i + K_i \sum_{j \in \mathcal{N}_i} \sin(\theta_j - \theta_i),$$

where $K_i > 0$ is the coupling strength and $\mathcal{N}_i$ is the set of neighbors. The coupling structure can be general so far, that is, $\mathcal{N}_i$ can be an arbitrary set of other nodes.

For identical coupled oscillators (i.e., $\omega_i = \omega, \forall i$), suppose initially $\max_{i,j} \|\theta_i(0) - \theta_j(0)\| < \pi$. Then we know $\max_{i,j} \|\theta_i(t) - \theta_j(t)\| < \pi$ for all $t \geq 0$. Applying the transformation $x_i = \theta_i - \omega t$ yields

$$\dot{x}_i = K_i \sum_{j \in \mathcal{N}_i} \sin(x_j - x_i).$$

Let $\mathcal{D} = \{x \in \mathbb{R}^n : \max_{i,j} \|x_i - x_j\| < \pi\}$. It turns out that $\mathcal{D}$ is positively invariant for the system above. Furthermore, it can be easily seen that assumption A1 is obviously satisfied and that the condition in A2 holds for all $x \in \mathcal{D}$. Thus, by Remark 1, if and only if the interaction digraph is QSC, the system above is asymptotically stable with respect to Ω for all initial states in $\mathcal{D}$, that is, there exists an $\bar{x} \in \mathbb{R}$ such that $x_i(t) \to \bar{x}$ for all i and therefore

$$\theta_i(t) \to \bar{x} + \omega t, \quad \dot{\theta}_i(t) \to \omega,$$

achieving synchronization of the oscillators. This is an extension of Theorem 1 in (Jadbabaie et
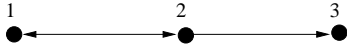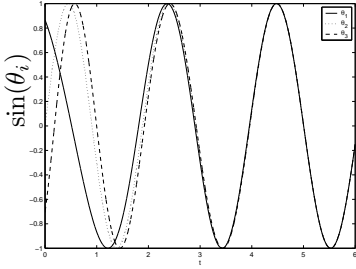
Fig. 4. The interaction digraph.

Fig. 5. Synchronization of coupled oscillators.

al., 2004), which assumes the graph is undirected and the initial state $\theta_i(0) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ for all i.

Fig. 5 shows simulation plots of three oscillators with the interaction digraph shown in Fig. 4. The natural frequency ω_i is set to be 3 for all i, K_1, K_2, K_3 are 1, 0.5, 2 respectively, and the initial conditions are $\theta_1(0) = \frac{2}{3}\pi$, $\theta_2(0) = 0$, $\theta_3(0) = -\frac{\pi}{4}$.

Next, we turn to control synthesis for the rendezvous problem of multi-agent systems in continuous time. Suppose there are n agents, each having the simple kinematic model of velocity control: $\dot{x}_i = u_i$, where $x_i \in \mathbb{R}^m$ is the position of agent i. Assume that, due to the limited field of view of its sensor, each agent can sense only the relative positions of its neighbor agents within radius r. Letting $\mathcal{N}_i(x)$ denote the set of neighbors of agent i, we thus have that $\{y_{ij} = x_j - x_i : \ j \in \mathcal{N}_i(x)\}$ is the information available to agent i.

The rendezvous problem is to design local distributed control laws

$$u_i = u_i \left(y_{ij} : j \in \mathcal{N}_i(x) \right)$$

such that

$$\lim_{t \to \infty} x_1(t) = \cdots = \lim_{t \to \infty} x_n(t) = \bar{x} \text{ for some } \bar{x} \in \mathbb{R}^m.$$

The controller above naturally induces an interaction digraph $\mathcal{G}(x)$, which depends on the state x and may be dynamic as the system evolves. Generally, if some agents are initialized so far away from the rest that they never acquire information from them, then the rendezvous problem can not be solved. Mathematically, this corresponds to the situation where $\mathcal{G}(x^0)$ is not QSC. So it is natural to assume that $\mathcal{G}(x^0)$ is QSC. Moreover, we wish the control law u_i to be devised such that the interaction digraph $\mathcal{G}(x(t))$ does not lose this property in the future, even though the control law may cause changes in $\mathcal{G}(x(t))$. Intuitively, u_i should make the maximum distance between agent i and its neighbor agents non-increasing.

Let $\mathcal{I}_i(x)$ denote the set of neighbor agents $j \in \mathcal{N}_i(x)$ that have maximum distance from agent i.

Proposition 1. Suppose $\mathcal{G}(x^0)$ is QSC. If, for all i, u_i satisfies the condition

$$(\forall x) \quad \max_{j \in \mathcal{I}_i(x)} (x_i - x_j)^T u_i \leq 0, \qquad (4)$$

then $\mathcal{G}(x(t))$ is QSC for all $t \geq 0$.

Proposition 2. Suppose $\mathcal{G}(x^0)$ is QSC. If u_i satisfies condition (4) and assumptions A1, A2, then the agents asymptotically rendezvous.

Proof: If $\mathcal{G}(x^0)$ is fully connected, then it is fixed for all time $t \geq 0$, since no arc will be dropped, by Proposition 1, and no arc can be added. Then the conclusion follows from Theorem 3.

If instead $\mathcal{G}(x^0)$ is not fully connected, then the interaction digraph $\mathcal{G}(x(t))$, $t \geq 0$ is dynamic and switches for a finite number of times. To prove this, suppose by contradiction that for all $t \geq 0$, $\mathcal{G}(x(t)) = \mathcal{G}(x^0)$. Then by Theorem 3, all the agents converge to a common position. So $\mathcal{G}(x(t))$ will become fully connected at some time t, which contradicts the assumption that $\mathcal{G}(x(t)) = \mathcal{G}(x^0)$ is not fully connected. Hence, there is a $t_1 \geq 0$ such that $\mathcal{G}(x(t_1))$ has more arcs than $\mathcal{G}(x^0)$ because no arcs will be dropped by Proposition 1. Repeating this argument a finite number of times eventually leads to the existence of t_i such that $\mathcal{G}(x(t_i))$ is fully connected, and thus, it is fixed after t_i. Then the conclusion follows from Theorem 3. ∎

The control law given next is based on the algorithm first proposed in (Ando *et al.*, 1999).

Proposition 3. A possible choice of u_i satisfying condition (4) and assumptions A1, A2 is $u_i = e(0, y_{ij} : j \in \mathcal{N}_i(x))$, the Euclidean center of the set $\mathcal{Z} = \{0, y_{ij}, j \in \mathcal{N}_i(x)\}$.

Proof: The Euclidean center of the set $\mathcal{Z}$ is the unique point w that minimizes the function $\max_{z \in \mathcal{Z}} \|w - z\|$. Furthermore, it can be easily shown that it lies in the polytope $\widetilde{\mathcal{C}}_i = \mathrm{co}\{0, y_{ij}, j \in \mathcal{N}_i(x)\}$ but not at its vertices if the polytope is not a singleton. Thus, by Maximum Theorem ((Sundaram, 1996)), the function $e(\cdot)$ is continuous and hence u_i satisfies assumption A1.

Next, $e(\cdot) \in \widetilde{\mathcal{C}}_i$ implies $e(\cdot) \in \mathcal{T}(0, \widetilde{\mathcal{C}}_i)$. Also, notice that $\mathcal{C}_i = \mathrm{co}\{x_i, x_j : j \in \mathcal{N}_i(x)\}$ is the translation of $\widetilde{\mathcal{C}}_i$ to the point x_i. Hence, $e(\cdot) \in \mathcal{T}(x_i, \mathcal{C}_i)$. In addition, if $\mathcal{C}_i$ is not a singleton and x_i is its vertex, this means $\widetilde{\mathcal{C}}_i$ is not a singleton and 0 is its vertex. Then by the fact that $e(\cdot)$ lies in $\widetilde{\mathcal{C}}_i$ but not at its vertices, it follows that $u_i = e(\cdot) \neq 0$. Thus u_i satisfies assumption A2.

Finally, u_i satisfies condition (4). This can be seen from geometry. We show the case when $m = 2$
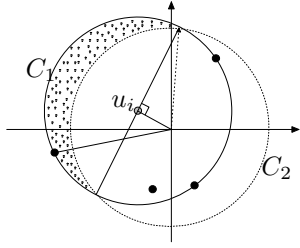
Fig. 6. The smallest enclosing circle.

and it is same when $m > 2$. If $u_i = 0$, then it trivially satisfies (4). If $u_i \neq 0$, then the picture is as in Fig. 6. The solid circle C_1 is the smallest circle enclosing the points 0 and $y_{ij}, j \in \mathcal{N}_i(x)$. The dotted circle C_2 is centered at the origin and goes through the intersection points between C_1 and its diameter, which is perpendicular to u_i. We know that if there are some y_{ij} in the closed shaded area, then one of them achieves the maximal distance from the origin. On the other hand, there is at least one $j \in \mathcal{N}_i(x)$ such that y_{ij} is in the closed semicircle of C_1, since otherwise it is not the smallest circle. Hence, y_{ij} lies in the closed shaded area if $j \in \mathcal{I}_i(x)$. Moreover, the angle between u_i and y_{ij} (y_{ij} is in the closed shaded area) is less than $\pi/2$. This implies that $\max_{j \in \mathcal{I}_i(x)} (x_i - x_j)^T u_i \leq 0$. ∎

5. CONCLUSION

In this paper, the state agreement problem for nonlinear continuous-time interconnected systems with static network architecture is studied. However, there are many situations where the network is ad hoc and time-varying. A future topic is the investigation of the state agreement problem for switched nonlinear interconnected systems.

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