A Hybrid Orbital Stabilizer With Guaranteed Basin of Attraction for Mechanical Systems With Underactuation Degree One*

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***Abstract*— We enhance a recently proposed hybrid controller that asymptotically stabilizes a class of closed orbits (so-called oscillations) for mechanical control systems with underactuation degree one. The controller in question enforces virtual holonomic constraint (VHCs) within a parametric family and instantiates new VHCs at certain events so as to asymptotically stabilize the target orbit. In this paper we propose a novel feedback mechanism in the jump map of the hybrid controller ensuring the asymptotic stabilization of oscillations with guaranteed basin of attraction. We demonstrate the new controller with a four degrees-of-freedom robot mimicking a child on a swing. For this robot, we are able to stabilize a large range of oscillations with guaranteed basin of attraction.**

I. Introduction

This paper investigates the orbital stabilization problem for a class of underactuated mechanical systems with n degrees-of-freedom (DOFs) and $n-1$ actuators. The problem is to asymptotically stabilize a type of closed orbit called an *oscillation* which corresponds to a repetitive motion in which some phase variable oscillates periodically without performing full rotations. Examples of such repetitive motions abound in nature and robotics. The motion of a child pumping a swing to achieve a certain desired amplitude of oscillation, or a gibbon swinging its arms to move from branch to branch at constant speed are examples of asymptotically stable oscillations.

In recent work [11], we proposed a hybrid orbital stabilizer for oscillations that relies on virtual holonomic constraints (VHCs). As is well-documented (e.g., [16], [1], [15]), VHCs are an ideal tool to induce closed orbits in underactuated mechanical systems. The challenge though is that once a VHC is enforced, there are no control DOFs left that can be used to asymptotically stabilize a target orbit. The idea used in [11] to circumvent this problem was partially inspired by the papers [10], [3] and Chapter 7 of the book [17], and it involves embedding the nominal VHC that creates the target closed orbit in a parametric family of VHCs. When certain events occur, a hybrid supervisor updates the VHC parameters instantiating a new VHC in the family. This process induces an orbital stabilization mechanism.

Our work in [11] ensures *local* asymptotic stabilization of a target oscillation. This paper proposes an enhancement ensuring asymptotic stabilization with *guaranteed* basin of attraction. The main result giving precise theoretical guarantees is found in Theorem 4.2. As an application of our theory, we consider a four DOFs robot mimicking a child on a swing. For this robot, we are able to stabilize a wide range of oscillations with a "large" basin of attraction.

There are several other approaches in the literature for orbital stabilization. In the context of VHCs, we mention the work in [16], [15] that relies on VHCs to identify a closed orbit and transverse linearization to stabilize it, as well as the work in [9] that uses dynamic VHCs for orbital stabilization. In [4], [5], the authors use VHCs and apply impulsive inputs to instantaneously change the velocities of the robot when its configuration crosses a certain threshold. Very important is also the extensive literature using VHCs to induce stable walking in bipedal robots (see, e.g., [18], [13], [17], [10], [3], [2]). In this context, the orbital stabilization mechanism if provided by the impulsive impacts of the swing foot with the ground. In a different context, [12] proposes using immersion and invariance for stabilization of closed orbits. In [14], the authors provide a constructive procedure building upon the work of [12]. In [19], the authors use energy shaping for passivity-based stabilization of closed orbits.

This paper is organized as follows. In Section II we succinctly review basic notions and terminology pertaining to VHCs. In Section III we review the hybrid controller proposed in [11]. Section IV presents the enhancement of the controller in [11] and the main theoretical result of this paper, whose proof is found in the Appendix. In Section V, the proposed controller is applied to stabilize oscillations for a four DOFs model mimicking a child on a swing. Finally, Section VI presents future perspectives for this work.

II. Preliminaries

We begin our development by reviewing virtual holonomic constraints and related concepts. We refer the reader to [7] for more details. Consider a mechanical system modelled as

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + \nabla P(q) = Bu, \tag{1}$$

where $q = (q_1, \ldots, q_n) \in \mathcal{Q}$ is the configuration vector and $u \in \mathbb{R}^{n-1}$ is the vector of control inputs, $D = D^\top$ is the mass matrix and $P : \mathcal{Q} \to \mathbb{R}$ is the potential energy. All quantities are assumed to be smooth. Each configuration variable q_i is either a real number (e.g., a displacement) or a

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real number modulo 2π (e.g., an angle). We assume B has full column rank $n-1$.

A **virtual holonomic constraint** (VHC) is a curve in the configuration manifold $\mathcal{Q}$ expressed as $h(q)=0$, where $h: \mathcal{Q} \to \mathbb{R}^{n-1}$ is smooth and such that the output function $y=h(q)$ yields vector relative degree $(2,\ldots,2)$ at each point $q \in h^{-1}(0)$. The idea in the VHC framework is to use the control inputs u to constrain the configuration q of the system to stay on this curve.

To achieve this goal, the control u needs to be designed to asymptotically stabilize the zero dynamics manifold associated with the output $y=h(q)$, given by

$$\Gamma = \{(q,\dot{q}) \in T\mathcal{Q} \mid h(q)=0, dh_q\dot{q}=0\}. \tag{2}$$

We call this zero dynamics manifold the **constraint manifold**. Under some mild assumptions (see [6, Proposition 12 and Remark 14]), Γ can be rendered asymptotically stable by means of a feedback linearization controller of the form

$$\begin{aligned} u =& \left(dh_q D^{-1}B\right)^{-1}\left[dh_q D^{-1}\left(C\dot{q}+\nabla_q P\right) - H\right.\\ &\left. -K_p h(q) - K_d dh_q\dot{q}\right]. \end{aligned} \tag{3}$$

where $H=[H_1 \ \ldots \ H_{n-1}]^\top$ and $H_i = \dot{q}^\top \operatorname{Hess}(h_i)\dot{q}$, and $K_p, K_d \in \mathbb{R}^{(n-1)\times(n-1)}$ are matrix gains chosen to stabilize the $n-1$ dimensional linear system $\ddot{y}+K_d\dot{y}+K_p y=0$.

A VHC can also be expressed in parametric form by a relation of the form $q=\sigma(\theta)$, where $\sigma:\Theta \to \mathcal{Q}$ and Θ is either $\mathbb{R}$ when the VHC is an open curve or[1] $[\mathbb{R}]_T$ for some $T>0$ when the VHC is a closed curve. By defining the *tangent map* $T\sigma: T\Theta \to T\mathcal{Q}$ as $T\sigma(\theta,\dot{\theta}) := (\sigma(\theta), \sigma'(\theta)\dot{\theta})$, a point $(q,\dot{q})$ in the constraint manifold Γ can be identified with a pair $(\theta,\dot{\theta}) \in T\Theta$ using the relation $(q,\dot{q}) = T\sigma(\theta,\dot{\theta})$.

When the constraint manifold Γ has been rendered asymptotically stable by a feedback u, the dynamics of the closed loop system, restricted to Γ are described by a second-order differential equation of the form

$$\ddot{\theta} = \Psi_1(\theta) + \Psi_2(\theta)\dot{\theta}^2, \tag{4}$$

where

$$\begin{aligned} \Psi_1(\theta) &= -\left.\frac{B^\perp \nabla_q P}{B^\perp D\sigma'(\theta)}\right|_{q=\sigma(\theta)} \\ \Psi_2(\theta) &= -\left.\frac{B^\perp D\sigma''(\theta) + B^\perp C\sigma'(\theta)}{B^\perp D\sigma'(\theta)}\right|_{\substack{q=\sigma(\theta)\\ \dot{q}=\sigma'(\theta)}}. \end{aligned} \tag{5}$$

We call this system the **constrained dynamics** and its meaning is this. Given any smooth feedback that asymptotically stabilizes Γ, the solutions of the closed-loop system on Γ have the form $(q(t),\dot{q}(t)) = T\sigma((\theta(t),\dot{\theta}(t)))$, where $(\theta(t),\dot{\theta}(t))$ is a solution of (4).

One of the advantages of the VHC framework is that, under suitable conditions, the constrained dynamics are also Euler-Lagrange, with Lagrangian $L(\theta,\dot{\theta}) = (1/2)M(\theta)\dot{\theta}^2 - V(\theta)$, where M and V are the **virtual mass** and **virtual potential** functions defined as

$$\begin{aligned} M(\theta) &= \exp\left(-2\int_0^\theta \Psi_2(\tau)d\tau\right) \\ V(\theta) &= -\int_0^\theta \Psi_1(\tau)M(\tau)d\tau, \end{aligned} \tag{6}$$

and the **virtual energy**

$$E(\theta,\dot{\theta}) = \frac{1}{2}M(\theta)\dot{\theta}^2 + V(\theta) \tag{7}$$

is an integral of motion of the constrained dynamics (see [7], [8], [16]). This fact is important because it implies that the constrained dynamics have an abundance of closed orbits, each one representing a repetitive motion of the mechanical system (see, e.g., [8]). It is for this reason that VHCs have proven useful for finding closed orbits (see, e.g., [16]).

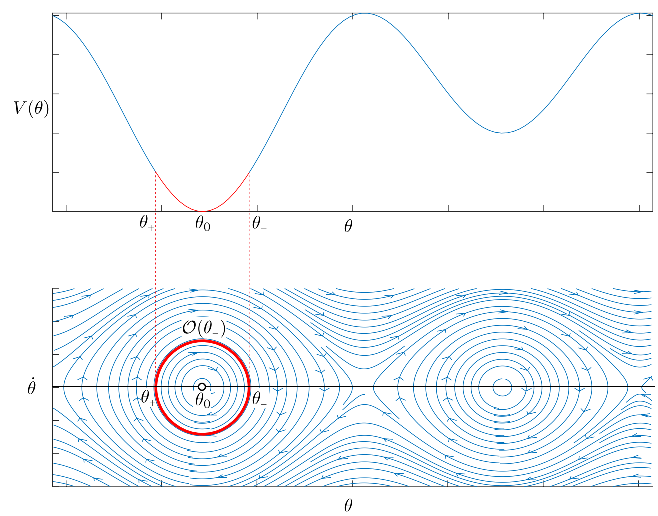

Fig. 1: An oscillation of the constrained dynamics (4).

When the virtual potential V has a local minimum at θ_0, the point $(\theta,\dot{\theta}) = (\theta_0, 0)$ is a stable equilibrium of the constrained dynamics, and the constrained dynamics will contain a continuum of closed orbits around this equilibrium corresponding to so-called **oscillations**. See Figure 1 for an illustration. Oscillations are stable but not asymptotically stable. The asymptotic stabilization of oscillations induced by a VHC is the focus of our recent work [11] and of this paper. As illustrated in Figure 1, an oscillation is characterized by two parameters, θ_- and θ_+ in Θ, the two unique points where the oscillation intersects the $\dot{\theta}$ axis; θ_- (resp., θ_+) corresponds to the point where $\dot{\theta}$ changes sign from positive to negative (resp., negative to positive). Since the parameter θ_- can be used to uniquely identify an oscillation, in what follows we will use the notation $\mathcal{O}(\theta_-)$ to denote an oscillation of (4), as done in Figure 1. Moreover, θ_- also uniquely identifies θ_+ and we will indicate this dependency using the notation $\theta_+ = \mathsf{B}(\theta_-)$.

[1] $[\mathbb{R}]_T$ denotes the set of real numbers modulo T.

III. Review of hybrid controller from [11]

Consider a VHC $h(q) = 0$ inducing a target oscillation $\mathcal{O}(\theta_-)$ that we wish to asymptotically stabilize. We refer to $h(q) = 0$ as the **nominal VHC**. The idea proposed in [11] to asymptotically stabilize the orbit is to embed this nominal VHC in a family $\{h^a(q) = 0\}_{a\in \mathsf{A}}$, parametrized by a parameter vector $a = (\mathsf{d}, \vartheta, \lambda)$. The hybrid controller enforces the VHC $h^a(q) = 0$ and keeps a constant during flow, updating it at certain events so as to asymptotically stabilize the target oscillation.

In the parameter vector $a = (\mathsf{d}, \vartheta, \lambda)$, the *toggle* variable $\mathsf{d} \in \{-1, 1\}$ keeps track of the direction of movement; the *memory* variable ϑ keeps track of the value of θ when the direction of movement changes; finally, the vector $\lambda \in \mathbb{R}^{n-1}$ shapes the geometry of the VHC. Whenever the system state crosses a chosen Poincaré section, the vector λ is updated according to a feedback law v to be designed.

Following [11], the control design begins with the definition of a smooth function $\theta^\star : \mathcal{W} \subset \mathcal{Q} \to \Theta$, where $\mathcal{W}$ is a neighborhood of $h^{-1}(0)$, such that for each $q \in h^{-1}(0)$, $\sigma(\theta^\star(q)) = q$. Intuitively, the function $\theta^\star$ maps a configuration $q \in \mathcal{W}$ to a curve parameter in Θ such that the point $\sigma(\theta^\star(q))$ is a projection of q onto $h^{-1}(0)$. When the VHC is expressed as a graph of a function of the form $\mathrm{col}(q_2 \ \dots \ q_n) = \phi(q_1)$, we can simply take $\theta^\star(q) = q_1$. More generally, Lemma 4.1 in [11] states that the set $\mathcal{W}$ and function $\theta^\star(q)$ are guaranteed to exist for any VHC.

Next, given a family of C^2 functions $\{\phi^a : \Theta \to \mathbb{R}^{n-1}\}_{a\in\mathsf{A}}$, we define the family of VHCs $h^a(q) = 0$ via $h^a(q) = h(q) - \phi^a(\theta^\star(q))$. Each VHC in this family has associated with it a constraint manifold Γ^a, a feedback u^a, and virtual mass, potential and energy function M^a, V^a, E^a using the expressions in (2), (3), (6) and (7) where we replace σ with σ^a, with σ^a computed as described in Section V of [11].

In [11, Defn. 5.2], precise conditions are given identifying a class $\mathcal{F}$ of functions ϕ^a ensuring that the **hybrid constraint manifold**

$$\bar{\Gamma} = \{(q, \dot{q}, a) \in \mathsf{C} \mid (q, \dot{q}) \in \Gamma^a\}, \tag{8}$$

is hybrid invariant. The set C in (8) is the flow set of the closed-loop system, reviewed below. One key requirement in the class $\mathcal{F}$ is that for each a of the form $a = (\mathsf{d}, \vartheta, 0)$, the function ϕ^a is identically zero, and the VHC $h^a(q) = 0$ reduces to the nominal VHC $h(q) = 0$.

The hybrid controller defined in [11] is given by

$$\mathcal{H}_{\text{osc}} : \begin{cases} (q, \dot{q}, a) \in \mathsf{C} & \dot{a} = 0 \\ (q, \dot{q}, a) \in \mathsf{D} & a^+ = G_{\text{osc}}(q, \dot{q}, a) \\ & u = u^a(q, \dot{q}), \end{cases} \tag{9}$$

where the flow and jump sets C, D are given by

$$\begin{aligned} \mathsf{C} =& \{(q, \dot{q}, a) \mid (q, \dot{q}) \in T\mathcal{W},\ \mathsf{d}\,\dot{\theta}^\star(q, \dot{q}) \geq 0\} \cup \\ & \{(q, \dot{q}, a) \mid (q, \dot{q}) \in T\mathcal{W},\ |\theta^\star(q) - \vartheta| \leq \delta\} \\ \mathsf{D} =& \{(q, \dot{q}, a) \mid (q, \dot{q}) \in T\mathcal{W},\ \mathsf{d}\,\dot{\theta}^\star(q, \dot{q}) \leq 0\} \cap \\ & \{(q, \dot{q}, a) \mid (q, \dot{q}) \in T\mathcal{W},\ |\theta^\star(q) - \vartheta| \geq \delta\} \end{aligned} \tag{10}$$

and the jump map G_{osc} is given by

$$G_{\text{osc}} : \begin{cases} \mathsf{d}^+ = -\mathsf{d} \\ \vartheta^+ = \theta^\star(q) \\ \lambda^+ = \begin{cases} \mathrm{sat}_{\mathsf{B}_r}(v) & \mathsf{d} = 1 \\ \lambda & \mathsf{d} = -1, \end{cases} \end{cases} \tag{11}$$

where

$$\mathrm{sat}_{\mathsf{B}_r}(\lambda) = \begin{cases} \lambda & \|\lambda\| \leq r \\ r\frac{\lambda}{\|\lambda\|} & \|\lambda\| \geq r. \end{cases} \tag{12}$$

The constants $r, \delta > 0$ are parameters used in the definition of the class $\mathcal{F}$ of function families, and v is the feedback updating λ that we mentioned earlier. This feedback is presented in Theorem 4.2 below.

We refer the reader to Section VII of [11] for a detailed description of the controller above. Here, we highlight the two main control mechanisms in $\mathcal{H}_{\text{osc}}$: the continuous feedback u^a enforcing a VHC $h^a(q) = 0$ in the family, and the discrete feedback v updating the VHC parameters a at jumps. Each time a is updated, a new VHC is instantiated. Roughly, jumps occur when the closed-loop system changes direction of motion, as perceived by the time derivative of $\theta^\star(q)$. There is also a mechanism (implemented using the variable $\delta > 0$) preventing multiple consecutive jumps in the presence of measurement noise.

The closed-loop system formed by system (1) and hybrid controller $\mathcal{H}_{\text{osc}}$ in (9), (11) has state $(q, \dot{q}, a)$. Following [11], we lift the orbit $\mathcal{O}(\theta_-)$ to this augmented state space by defining a **lifted orbit** $\mathcal{O}_{\text{lift}}(\theta_-)$ as

$$\begin{aligned} \mathcal{O}_{\text{lift}}(\theta_-) = \{(q, \dot{q}, a) \in \mathsf{C} \mid & (\theta^\star(q), \dot{\theta}^\star(q, \dot{q})) \in \mathcal{O}(\theta_-), \\ & \mu_{\theta_-}(a) = 0\}, \end{aligned} \tag{13}$$

where

$$\mu_{\theta_-}(a) = \begin{cases} (\vartheta - \theta_-, \lambda) & \mathsf{d} = 1 \\ (\vartheta - \theta_+, \lambda) & \mathsf{d} = -1. \end{cases}$$

The condition $\mu_{\theta_-}(a) = 0$ means that $\lambda = 0$, i.e., that the VHC $h^a(q) = 0$ reduces to the nominal VHC $h(q) = 0$, and that the memory variable ϑ is either θ_- or θ_+, depending on the direction of movement encoded in the toggle variable d. The projection of $\mathcal{O}_{\text{lift}}(\theta_-)$ onto the set $T\mathcal{Q}$ via the map $(q, \dot{q}, a) \mapsto (q, \dot{q})$ is precisely the target orbit $\mathcal{O}(\theta_-)$ expressed in $(q, \dot{q})$ coordinates, i.e., the orbit $T\sigma(\mathcal{O}(\theta_-))$.

A Poincaré analysis was used in [11] to show that stability of this lifted orbit $\mathcal{O}_{\text{lift}}(\theta_-)$ is equivalent to stability of the origin of the scalar discrete-time LTI system

$$z(k+1) = z(k) + b^{\theta_-}/(\Psi_1(\theta_-)M(\theta_-))\, v, \tag{14}$$

where $b^{\theta_-} \in \mathbb{R}^{1\times(n-1)}$ is a constant vector given by

$$\begin{aligned} b^{\theta_-} =& \frac{\partial}{\partial\lambda}\Big|_{\lambda=0}\Big(V^{a_-(\lambda)}(\theta_+) - V^{a_-(\lambda)}(\theta_-) \\ & + V^{a_+(\lambda)}(\theta_-) - V^{a_+(\lambda)}(\theta_+)\Big), \end{aligned} \tag{15}$$

where $a_-(\lambda) = (-1, \theta_-, \lambda)$, and $a_+(\lambda) = (1, \mathsf{B}(\theta_-), \lambda)$.

The main result in [11, Theorem 6.1], states that if $b^{\theta_-} \neq 0$, and if $v^{\theta_-}(z)$ is a feedback asymptotically stabilizing the origin of (14), setting $v = v^{\theta_-}(\theta^\star(q) - \theta_-)$ in the jump map G_{osc} in (11) ensures that the lifted orbit $\mathcal{O}_{\text{lift}}(\theta_-)$ will be asymptotically stable for the resulting closed-loop system formed by (1) and (9). The superscript θ_- in b^{θ_-} and v^{θ_-} is used to indicate their dependency on the chosen oscillation.

IV. Main Result

The main result in [11] is purely local. We now propose an enhancement ensuring a guaranteed basin of attraction for a target oscillation $\mathcal{O}(\bar{\theta}_-)$. We replace the parameter θ_- used in the design of the controller in [11], with a new feedback $\eta(\theta)$ given by

$$\eta(\theta) = \bar{\theta}_- + \epsilon \, \mathrm{sign}(\theta - \bar{\theta}_-) \left\lfloor \left| \frac{|\theta - \bar{\theta}_-|}{\epsilon} - 1 \right| \right\rfloor, \qquad (16)$$

where $\epsilon > 0$ is a new design parameter. We will show that, by choosing ϵ small enough and setting the feedback v appropriately, we can not only stabilize the lifted orbit $\mathcal{O}_{\text{lift}}(\bar{\theta}_-)$, but also provide a guaranteed basin of attraction.

The idea we use here is simple: if the plant state is too far from the desired orbit $\mathcal{O}_{\text{lift}}(\bar{\theta}_-)$ (outside the basin of attraction obtained with the original controller), then instead of stabilizing $\mathcal{O}_{\text{lift}}(\bar{\theta}_-)$, we change the feedback v to stabilize an intermediate orbit between the current plant state and $\mathcal{O}_{\text{lift}}(\bar{\theta}_-)$. If the chosen orbit is close enough to the plant state and is closer to the desired orbit than the current plant state, then the plant state will eventually become closer to the desired orbit. We can adjust the orbit being stabilized every time the system state reaches the jump set to make it progressively closer to the desired orbit until the plant state enters the basin of attraction of the original controller.

The feedback $\eta(\theta)$ in (16) implements this idea by breaking the set Θ into segments of length ϵ. If $|\theta - \bar{\theta}_-| < 2\epsilon$, then $\eta(\theta) = \bar{\theta}_-$. If $\theta \in [\bar{\theta}_- + k\epsilon, \bar{\theta}_- + k\epsilon + \epsilon)$, with k an integer greater than 1, we have $\eta(\theta) = \bar{\theta}_- + (k-1)\epsilon$. This is illustrated in Figure 2.

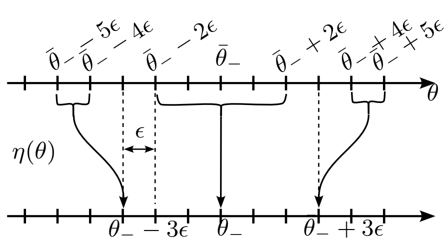

Fig. 2: Illustration of the map η.

Before stating the main result, we need the following definition:

Definition 4.1 (Controllability interval D): Let D to be an open interval of Θ such that for each $\theta_- \in \mathsf{D}$, the following properties hold:

(i) the orbit $\mathcal{O}(\theta_-)$ of (4) is an oscillation;
(ii) $|\theta_- - \mathsf{B}(\theta_-)| > \delta$;
(iii) the vector $b^{\theta_-} \in \mathbb{R}^{1\times(n-1)}$ in (15) is nonzero. △

Theorem 4.2: Consider the underactuated mechanical system (1) and a regular VHC $h(q) = 0$. Fix parameters $\delta, r > 0$, pick a family of functions $\{\phi^a\}_{a\in\mathsf{A}}$ in class $\mathcal{F}$ as in [11, Defn. 5.2], and define the hybrid controller $\mathcal{H}_{\text{osc}}$ in (9), (11). Suppose the controllability interval D in Definition 4.1 is non-empty, and let $v^\eta : \Theta \to \mathbb{R}^{n-1}$ be a feedback that is C^1 with respect to (θ, η) and that, for each $\eta \in \mathsf{D}$, stabilizes the origin of the scalar discrete-time LTI system

$$z(k+1) = z(k) + b^\eta/(\Psi_1(\eta)M(\eta))\, v^\eta. \qquad (17)$$

Then, for each target oscillation $\mathcal{O}(\bar{\theta}_-)$, with $\bar{\theta}_- \in \mathsf{D}$ and for each compact interval $I \subset \mathsf{D}$ containing $\bar{\theta}_-$, there exists $\epsilon^\star > 0$ such that, for each $\epsilon \in (0, \epsilon^\star)$, using the feedback $\eta(\theta)$ defined in (16) and setting $v = v^{\eta(\theta^\star(q))}(\theta^\star(q) - \eta(\theta^\star(q)))$ in the jump map G_{osc} in (11), the resulting closed-loop system enjoys the following properties:

(a) the hybrid constraint manifold $\bar{\Gamma}$ in (8) is forward invariant;
(b) the lifted orbit $\mathcal{O}_{\text{lift}}(\bar{\theta}_-)$ is asymptotically stable and there are no Zeno solutions in a neighborhood of $\mathcal{O}_{\text{lift}}(\bar{\theta}_-)$;
(c) the basin of attraction of $\mathcal{O}_{\text{lift}}(\bar{\theta}_-)$ includes the set $\mathcal{O}_{\text{lift}}(I)$.

The proof is in the Appendix.

Remark 4.3: The guaranteed basin of attraction in Theorem 4.2 includes all initial conditions in which the plant state is initialized on the constraint manifold Γ anywhere on $T\sigma(\mathcal{O}(I))$, and where the controller state a is initialized with $\lambda = 0$ (i.e., the controller first instantiates the nominal VHC), and with d, λ set consistently with the initial direction of movement. Note that the set $T\sigma(\mathcal{O}(I))$ is an annular subset of the nominal constraint manifold. An example of such a set can be seen in Figure 4 for the child on the swing example presented in Section V. △

Remark 4.4: One possible choice for the feedback v^η in the construction above is simply $v^\eta(z) = -\frac{\Psi_1(\eta)M(\eta)}{b^\eta(b^\eta)^\top}(b^\eta)^\top z$. This choice of feedback places the eigenvalue of system (17) at zero making the origin exponentially stable, satisfying the conditions on Theorem 4.2. More generally, if $L^\eta \in \mathbb{R}^{n-1}$ is a vector such that $b^\eta L^\eta \neq 0$ for all $\eta \in \mathsf{D}$, then we may set $v^\eta(z) = -\frac{\Psi_1(\eta)M(\eta)}{b^\eta L^\eta} L^\eta z$. And further, instead of placing the eigenvalue at 0 one may place it anywhere within the interval $(-1, 1)$. △

V. Application: Child on a Swing

We now use this proposed controller to stabilize an oscillation for the 4 degrees-of-freedom robot in Figure 3 modelling a child on a swing. A MATLAB script computing the differential equations modelling this robot can be found on `https://github.com/manfredimaggiore/CDC25`. For simplicity, our model assumes that the links are rigid massless rods with point-masses at their ends, but the results with distributed masses would be entirely analogous.

The nominal VHC used for this example is given by

$$h(q) = \begin{bmatrix} q_2 - \frac{\pi}{2} \\ q_3 \\ q_4 - 1.2\pi \end{bmatrix}, \qquad (18)$$

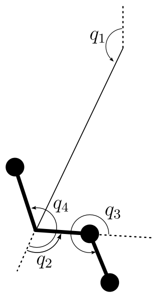

Fig. 3: Child on a swing, modelled as a 4 degrees-of-freedom robot.

and corresponds to the child swinging with their legs fully extended forward and perpendicular to the swing rod, while the torso leans behind the swing rod. One can easily see that the output $y = h(q)$ yields vector relative degree $(2, 2, 2)$ everywhere[2].

This VHC can be expressed as a graph of a function of q_1, and therefore a natural parametrization is $\sigma(\theta) = [\theta \;\; \pi/2 \;\; 0 \;\; 1.2\pi]^\top$. As such, the map $\theta^\star$ readily be chosen as $\theta^\star(q) = q_1$.

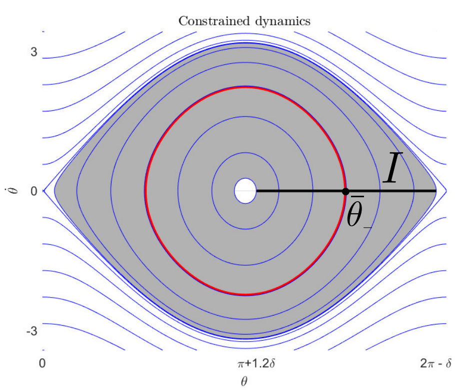

Fig. 4: Constrained dynamics of the nominal VHC. The orbit highlighted in red is the orbit we want to stabilize. The black segment on the θ axis corresponds to the set I and the shaded annular region is an illustration of the guaranteed basin of attraction.

The phase portrait of the constrained dynamics is shown in Figure 4. It is easily verified that the constrained dynamics are Euler-Lagrange. This is intuitively obvious because when $h(q) = 0$, the swing reduces to a pendulum.

We use the class $\mathcal{F}$ family of functions introduced in [11] of the form

$$\phi^{(1,\vartheta,\lambda)}(\theta) := \phi(\theta - \vartheta, \lambda) \tag{19}$$

$$\phi^{(-1,\vartheta,\lambda)}(\theta) := \lambda - \phi^{(1,\vartheta,\lambda)}(\theta), \tag{20}$$

[2]Indeed, since $dh_q = B^\top$ for this example, the decoupling matrix $dh_q D^{-1}(q)B$ is given by $B^\top D^{-1}(q)B$, and since D^{-1} is positive definite and B has full rank 3, the decoupling matrix is invertible.

where we take $\delta = \pi/20, r = 2$ and

$$\phi(\theta, \lambda) := \begin{cases} 0 & \text{if } |\theta| > \delta \\ \lambda\,(\theta-\delta)^2(\theta+\delta)^2/\delta^4 & \text{if } |\theta| \le \delta. \end{cases} \tag{21}$$

Following Remark 4.4, we take $L^\eta = \begin{bmatrix} 0 & 2 & 1 \end{bmatrix}^\top$ (independent of η) and for each η in the interval $[\pi + 1.2\delta, 2\pi - \delta]$ we compute $b^\eta L^\eta$. The result is shown in Figure 5, where we can see that for each η in this interval we have $b^\eta L^\eta \neq 0$. Furthermore, we can see from the phase portrait of the

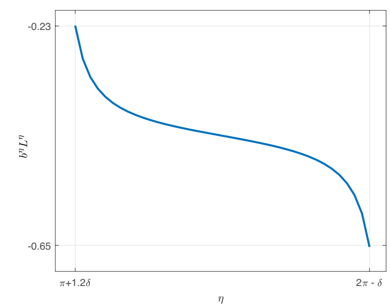

Fig. 5: The function $b^\eta L^\eta$ for different values of η in the interval $[\pi + 1.2\delta, 2\pi - \delta]$.

constrained dynamics that for all $\eta \in [\pi + 1.2\delta, 2\pi - \delta]$ we have that the associated orbit $\mathcal{O}(\eta)$ is an oscillation satisfying $|\theta_- - \theta_+| > \delta$. As such, this interval is contained in the controllability interval D from Definition 4.1 and we can take the set I in Theorem 4.2 to be the entire interval $[\pi + 1.2\delta, 2\pi - 2\delta]$. We choose $\bar\theta_- = \frac{3}{2}\pi$ and design the feedback η in (16) to stabilize the orbit $\mathcal{O}(\bar\theta_-)$.

Theorem 4.2 states that there exists $\epsilon > 0$ small enough in the construction of the feedback η such that the set $\mathcal{O}_{\text{lift}}(I)$ is contained in the basin of attraction of the target orbit. Here we take $\epsilon = \pi/20$. To verify that this gives us the desired basin of attraction, we run a sequence of simulations with initial conditions of the form $q(0) = \sigma(\pi + \zeta\delta)$, $\dot q(0) = 0, \lambda = 0, \mathsf{d} = -1, \vartheta = \frac{1}{2}\pi$, where we vary ζ from 1.2 to 19.

The results are shown in Figure 6. We see that the value of $\theta^\star(q)$ at consecutive intersections converges to the desired value $\bar\theta_-$, and simultaneously, the VHC parameter λ also converges to zero. These two facts combined imply that the solution converges to the desired orbit on the nominal constraint manifold.

VI. Conclusion

We have presented an improvement of the hybrid orbit stabilizer developed in [11], allowing one to asymptotically stabilize a closed orbit with a guaranteed basin of attraction. A possible topic of future research is the extension of the theory of [11] and this paper to handle mechanical systems with impulsive impacts. A primary area of application of such an extension would be the motion control of bipedal robots.

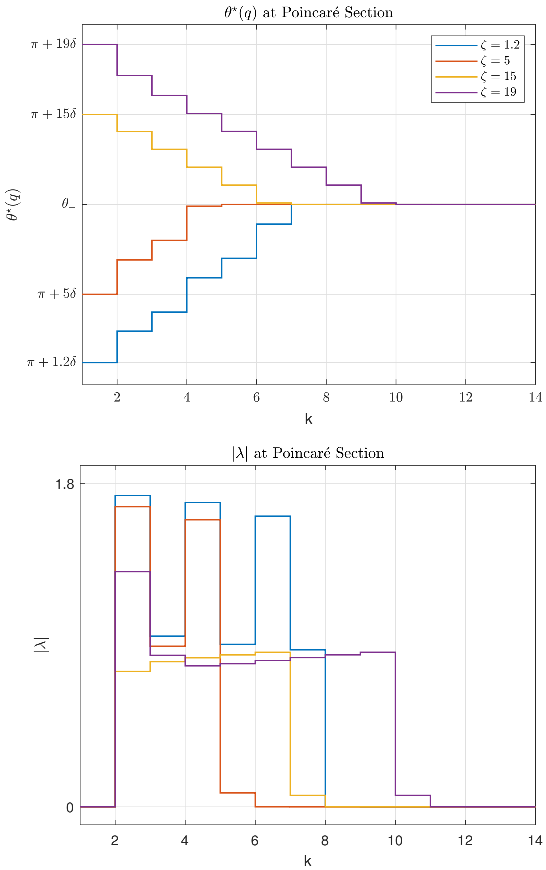

Fig. 6: Simulation results for different initial conditions. On the top, the value of $\theta^\star(q)$ at consecutive intersections of the solution with the Poincaré Section $\mathsf{P}_{\mathrm{osc}}$. On the bottom, the magnitude of the VHC parameter λ at these intersections.

Appendix
Proof of Theorem 4.2

Properties (a) and (b) of Theorem 4.2 are a direct consequence of Theorem 6.1 of [11]. The only property of Theorem 4.2 that still needs to be proved is property (c) regarding the guaranteed basin of attraction.

In [11], a reduced order system with state $x := (\theta, \dot{\theta}, a)$ with $a = (\mathsf{d}, \vartheta, \lambda)$ was derived to model the dynamics of the closed loop system restricted to the hybrid constraint manifold $\bar{\Gamma}$. For this system, a Poincaré section was defined as

$$\mathsf{P}_{\mathrm{osc}} := \{(\theta, \dot{\theta}, a) \mid \mathsf{d} = 1, \dot{\theta} = 0\}. \tag{22}$$

For the given target orbit $\mathcal{O}_{\mathrm{lift}}(\bar{\theta}_-)$, it was shown that the intersection of the orbit with this Poincaré section is given by the point $\bar{x} := (\bar{\theta}_-, 0, (1, \bar{\theta}_+, 0))$.

The associated Poincaré return map $\mathsf{g}_{\mathrm{osc}}$ was shown to have the form

$$\mathsf{g}_{\mathrm{osc}}(x) = \begin{bmatrix} \theta_2(\theta, v) \\ 0 \\ (1, \theta_1(\theta, v), \lambda_1(\theta, v)) \end{bmatrix}. \tag{23}$$

It was argued in [11] that, since the map $\mathsf{g}_{\mathrm{osc}}$ depends only on the state θ and the input v, stability of the equilibrium $\bar{x}$ for the Poincaré system $x^+ = \mathsf{g}_{\mathrm{osc}}(x)$ is equivalent to stability of the equilibrium $\bar{\theta}_-$ for the θ component of $\mathsf{g}_{\mathrm{osc}}$, given by

$$\theta^+ = \theta_2(\theta, v). \tag{24}$$

Similarly, one can argue that the basin of attraction of $\bar{x}$ for $\mathsf{g}_{\mathrm{osc}}$ can be entirely characterized by the θ component of the system in the following sense: if a set $\mathcal{D}_1$ is in the basin of attraction of $\bar{\theta}_-$ for system (24), then the basin of attraction of $\bar{x}$ for $\mathsf{g}_{\mathrm{osc}}$ includes the set

$$\mathcal{D}_2 = \{(\theta, 0, (1, \vartheta, \lambda)) \mid \theta \in \mathcal{D}_1\}. \tag{25}$$

It was shown in [11] that the linearization of (24) at the equilibrium $\theta = \eta$ is given by (17) where $z = \theta - \eta$. In the statement of Theorem 4.2, v was chosen to be the feedback $v = v^{\eta(\theta)}(\theta - \eta(\theta))$. By assumption, the feedback $v^\eta(\theta - \eta)$ is chosen to be such that, for each $\eta \in I$, the origin of (17) with control $v = v^\eta(\theta - \eta)$ is exponentially stable.

Letting $\Sigma(\theta, \eta) := \theta_2(\theta, v^\eta(\theta - \eta))$, the above means that, for each $\eta \in I$, the equilibrium $\theta = \eta$ is exponentially stable for the discrete time system

$$\theta^+ = \Sigma(\theta, \eta). \tag{26}$$

Equivalently, this means that for each $\eta \in I$ we have

$$\left| \left(\frac{\partial \Sigma}{\partial \theta} \right)_{(\eta,\eta)} \right| < 1. \tag{27}$$

The following Lemma will be useful what follows:

Lemma 1.1: Consider a parameterized scalar discrete-time system in the form

$$\theta^+ = \Sigma(\theta, \eta) \tag{28}$$

where η is a constant parameter in a compact interval I. Assume that the function Σ is continuously differentiable and that for each $\eta \in I$ we have $\Sigma(\eta, \eta) = \eta$ and

$$\left| \left(\frac{\partial \Sigma}{\partial \theta} \right)_{(\eta,\eta)} \right| < 1, \tag{29}$$

meaning that η is an exponentially stable equilibrium of (28). Then there exists $\epsilon^\star > 0$ such that for each $\epsilon \in (0, \epsilon^\star)$ and each $\eta \in I$ the interval $(\eta - 2\epsilon, \eta + 2\epsilon)$ is positively invariant and contained in the basin of attraction of η for system (28).

Proof: Consider the coordinate transformation $z = \theta - \eta$. In this coordinate system, system (28) is given by

$$z^+ = g(z, \eta) := \Sigma(z + \eta, \eta) - \eta. \tag{30}$$

Consider the Lyapunov function $V(z) = z^2$. For system (30) we have that

$$\Delta V = g(z, \eta)^2 - z^2 \tag{31}$$

From the Mean Value Theorem we know that, for each $z \in I$, there exists $\tilde{z}$ between 0 and z such that

$$\frac{g(z, \eta) - g(0, \eta)}{z - 0} = \left. \frac{\partial g}{\partial z} \right|_{(\tilde{z}, \eta)} \tag{32}$$

Since $g(0,\eta)=\Sigma(\eta,\eta)-\eta=0$ we have that

$$g(z,\eta)=z\left.\frac{\partial g}{\partial z}\right|_{(\tilde{z},\eta)} \tag{33}$$

and therefore

$$\Delta V=\left(\left.\frac{\partial g}{\partial z}\right|_{(\tilde{z},\eta)}^{2}-1\right)z^2 \tag{34}$$

for some $\tilde{z}$ between 0 and z. Since z^2 is always positive, we will focus on the coefficient

$$\alpha(\tilde{z},\eta)=\left(\left.\frac{\partial g}{\partial z}\right|_{(\tilde{z},\eta)}^{2}-1\right). \tag{35}$$

Consider the set $\alpha^{-1}(\mathbb{R}^-)$, i,e. the set of all values of $\tilde{z},\eta$ such that $\alpha(\tilde{z},\eta)<0$. From the definition of g in (30) we have that

$$\left.\frac{\partial g}{\partial z}\right|_{(\tilde{z},\eta)}=\left.\frac{\partial\Sigma}{\partial\theta}\right|_{(\tilde{z}+\eta,\eta)}. \tag{36}$$

Since Σ is assumed to be continuously differentiable, then α is continuous and the set $\alpha^{-1}(\mathbb{R}^-)$ is open. Furthermore, assumption (29) implies that $\alpha(0,\eta)<0$ and therefore the set $\alpha^{-1}(\mathbb{R}^-)$ contains the slice $0\times I$. Since the set I is assumed to be compact, we can use the Tube Lemma to assert that there exists $\epsilon^*>0$ such that

$$(-2\epsilon^\star,2\epsilon^\star)\times I\subset\alpha^{-1}(\mathbb{R}^-) \tag{37}$$

Since $\tilde{z}$ in (34) is between 0 and z then $z\in(-2\epsilon^\star,2\epsilon^\star)$ implies that $\tilde{z}\in(-2\epsilon^\star,2\epsilon^\star)$. Therefore expression (37) means that, for any $\eta\in I$ and $z\in(-2\epsilon^\star,2\epsilon^\star)$ we have $\Delta V<0$ and for any $\epsilon\in(0,\epsilon^\star)$ the interval $(-2\epsilon,2\epsilon)$ is forward invariant and contained in the basin of attraction of the origin for system (30). In θ-coordinates this means that the interval $(\eta-2\epsilon,\eta+2\epsilon)$ is positively invariant and contained in the basin of attraction of η for system (28). ■

The proof of Theorem 4.2 is split into three parts. First, we show that for a sufficiently small $\epsilon^\star$ the set

$$\bar{I}=\{(\theta,0,(1,\vartheta,\lambda))\in\mathsf{P}_{\text{osc}}\mid\theta\in I,|\theta-\vartheta|>\delta\} \tag{38}$$

is contained in the domain of the Poincaré map. Then, we show that the set I is forward invariant and contained in the basin of attraction of $\bar{\theta}_-$ for system (39). Finally, we show that these two facts combined imply that the set $\mathcal{O}_{\text{lift}}(I)$ is contained in the basin of attraction of $\mathcal{O}_{\text{lift}}(\bar{\theta}_-)$.

To show that the set $\bar{I}$ in (38) is in the domain of the Poincaré map g_{osc} we start by noting that when $v=0$ the plant state stays on the nominal constraint manifold. Since the interval I is a subset of D, properties (i) and (ii) from Definition 4.1, imply that, when $v=0$, all orbits starting from the set $\bar{I}$ are oscillations with $|\theta_- - \theta_+|>\delta$. As such, when $v=0$ then the entire set $\bar{I}$ is in the domain of the Poincaré map g_{osc}. Furthermore, since I is in the interior of D, continuity of solutions implies that for v close enough to 0, the entire set $\bar{I}$ is in the domain of g_{osc}.

We note that, for any $\theta_-\in I$, we have that $|\theta_- -\eta(\theta_-)|\le 2\epsilon$. Furthermore, from the definition of v^η in Theorem 4.2 we have that $v^\eta(0)=0$. Since v^η is assumed to be continuous with respect to (θ,η) we have that for any $\rho>0$ we can find $\epsilon>0$ such that $|v^\eta(\theta-\eta(\theta))|<\rho$. This means that by choosing ϵ small enough we can guarantee that the entire set $\bar{I}$ is in the domain of g_{osc}.

Next, we want to show that the set I is forward invariant and contained in the basin of attraction of $\bar{\theta}_-$ for system (39). Recall that the θ component of the Poincaré map g_{osc} when the control input is chosen to be $v=v^{\eta(\theta)}(\theta-\eta(\theta))$ is given by

$$\theta^+=\Sigma(\theta,\eta(\theta)). \tag{39}$$

We have seen that system (26) satisfies all the conditions of Lemma 1.1. Let $\epsilon^\star$ by given by Lemma 1.1 and let the parameter ϵ used in the definition of $\eta(\theta)$ in (16) be such that $\epsilon\in(0,\epsilon^\star)$.

We start by noting that, for θ in the interval $I_2=(\bar{\theta}_- - 2\epsilon,\bar{\theta}_- +2\epsilon)$, we have $\eta(\theta)=\bar{\theta}_-$. Since $\eta(\theta)$ is constant in this region, Lemma 1.1 tells us that the interval I_2 is positively invariant and contained within the basin of attraction of $\bar{\theta}_-$ for system (39).

Now assume that the interval $I_k=(\bar{\theta}_- - k\epsilon,\bar{\theta}_- + k\epsilon)\cap I$ is positively invariant and contained within the basin of attraction of $\bar{\theta}_-$ for system (39). We will show that this implies that the interval $I_{k+1}=(\bar{\theta}_- -(k+1)\epsilon,\bar{\theta}_- +(k+1)\epsilon)\cap I$ is also positively invariant and contained within the basin of attraction of $\bar{\theta}_-$ for system (39).

To do that, we write the interval I_{k+1} as $I_{k+1}^-\cup I_k\cup I_{k+1}^+$, where $I_{k+1}^-=(\bar{\theta}_- -(k+1)\epsilon,\bar{\theta}_- - k\epsilon]\cap I$ and $I_{k+1}^+=[\bar{\theta}_- + k\epsilon,\bar{\theta}_- +(k+1)\epsilon)\cap I$. The interval I_k is already assumed to be positively invariant and contained in the basin of attraction of $\bar{\theta}_-$ for system (39), so we only need to show that solutions starting on I_{k+1}^- and I_{k+1}^+do not leave the interval I_{k+1} and eventually enter the interval I_k.

We will focus on the case of $\theta\in I_{k+1}^+$ since the argument for the remaining case is symmetric. If $I_{k+1}^+=\emptyset$, then $I_k\cup I_{k+1}=I_k$ which is positively invariant and contained in the basin of attraction by assumption. If $I_{k+1}^+\neq\emptyset$, then for all $\theta\in I_{k+1}^+$ we have $\eta(\theta)=\bar{\theta}_- +(k-1)\epsilon$. Since $\eta(\theta)$ is constant in this region then system (39) coincides with system (28) with $\eta=\bar{\theta}_- +(k-1)\epsilon$, and Lemma 1.1 tells us that, for any $\tilde{\epsilon}\in(0,\epsilon^\star)$ the interval $(\eta-2\tilde{\epsilon},\eta+2\tilde{\epsilon})$ is positively invariant and contained in the basin of attraction of $\eta=\bar{\theta}_- +(k-1)\epsilon$ for system (28).

If I_{k+1}^+ is in the interior of I, we choose $\tilde{\epsilon}=\epsilon$. If I_{k+1}^+ contains points in the boundary of I we choose $\tilde{\epsilon}$ to be the larges value in $(0,\epsilon^\star)$ such that that $(\eta-2\tilde{\epsilon},\eta+2\tilde{\epsilon})$ is contained in I. In either case, positive invariance of the interval $(\eta-2\tilde{\epsilon},\eta+2\tilde{\epsilon})$ combined with positive invariance of I_k implies that the interval $I_k\cup I_{k+1}^+$ is positively invariant.

The fact that the interval $(\eta-2\tilde{\epsilon},\eta+2\tilde{\epsilon})$ is in the basin of attraction of $\eta=\bar{\theta}_- +(k-1)\epsilon$ for system (28) means that, for any solution of system (28) with initial condition in I_{k+1}^+, there exists a time $K>0$ such that after K updates the state will enter the interval $(\eta-\epsilon,\eta+\epsilon)$. Since $(\eta-\epsilon,\eta+\epsilon)\subset I_k$ and I_k is assumed to be in the basin of attraction of $\bar{\theta}_-$ for system (39), this means that I_{k+1}^+ is also in the basin of attraction of $\bar{\theta}_-$ for system (39).

Using a symmetric argument for I_{k-1}^{-} we obtain that the interval I_{k+1} is positively invariant and contained in the basin of attraction of $\bar{\theta}_-$ for system (39). By induction we have that the interval I is positively invariant and contained in the basin of attraction of $\bar{\theta}_-$ for system (39). A visual representation of the intervals of interest can be seen in Figure 7.

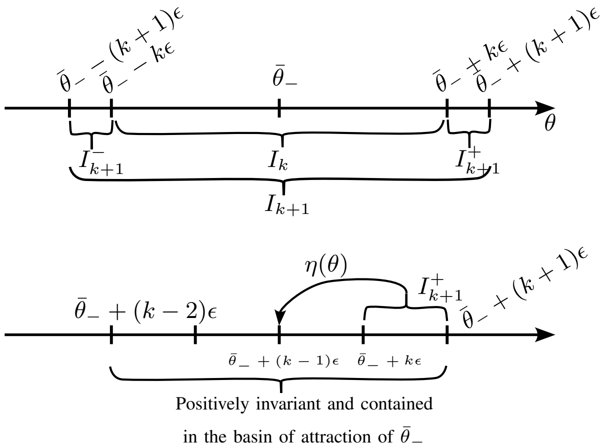

Fig. 7: Illustration of the relevant intervals used in part 1 of the proof of Theorem 4.2.

We have shown that there exists $\epsilon^\star > 0$ such that, if we choose the parameter ϵ in the construction of $\eta(\theta)$ satisfying $\epsilon \in (0, \epsilon^\star)$, then the set I is in the basin of attraction of $\bar{\theta}_-$ for the Poincaré system (39). We will now show that $\epsilon \in (0, \epsilon^\star)$ also implies that the set $\mathcal{O}_{\text{lift}}(I)$ is contained in the basin of attraction of $\mathcal{O}_{\text{lift}}(\bar{\theta}_-)$ for the full closed-loop system using the proposed controller $\mathcal{H}_{\text{osc}}$. By part (b) of the theorem, the orbit $\mathcal{O}_{\text{lift}}(\bar{\theta}_-)$ is asymptotically stable so it suffices to show that each solution initialized in $\mathcal{O}_{\text{lift}}(I)$ enters any neighborhood of $\mathcal{O}_{\text{lift}}(\bar{\theta}_-)$ in finite time.

We start by noting that any solutions starting on $\mathcal{O}_{\text{lift}}(I)$ will remain on the constraint manifold Γ of the nominal VHC until they reach the Poincaré Section P_{osc} in (22). Since all orbits in the set $\mathcal{O}(I)$ are oscillations that intersect the set $\{(\theta, \dot{\theta}) : \dot{\theta} = 0\}$ in the region where $\theta \in I$, we have that solutions starting on $\mathcal{D}$ must intersect P_{osc} for the first time at a point where $\theta^\star(q) \in I$.

To see why this is the case, consider any initial condition in $\mathcal{O}_{\text{lift}}(I)$. The solution from this initial condition starts on the flow set and on the constraint manifold of the nominal VHC, and it will flow until it reaches the jump set. Recall that all initial conditions in $\mathcal{O}_{\text{lift}}(I)$ have $(\theta^\star(q), \dot{\theta}^\star(q, \dot{q})) \in \mathcal{O}(I)$ and correspond to oscillations with amplitude greater than δ. In the case where the initial condition has $\dot{\theta}^\star(q, \dot{q}) > 0$, $\mathsf{d} = 1$ and $\vartheta = \theta_+$, the associated solution will reach the jump set at a point where $\theta \in I$ and $\dot{\theta}^\star(q, \dot{q}) = 0$. In the case, where the initial condition has $\dot{\theta}^\star(q, \dot{q}) < 0$, $\mathsf{d} = -1$ and $\vartheta = \bar{\theta}_-$, the first jump of the associated solution will update ϑ and d but will keep $\lambda = 0$, meaning that the solution will remain in the nominal constraint manifold until it jumps again, this time at a point where $\theta \in I$ and $\dot{\theta}^\star(q, \dot{q}) = 0$.

In either case, we note that, since the interval I is contained in the basin of attraction of $\bar{\theta}_-$ for system (39), then for any initial condition in $\mathcal{O}_{\text{lift}}(I)$ and any $\zeta > 0$ there exists an integer $J > 0$ such that after J jumps, the intersection of the solution starting with the Poincaré Section P_{osc} will happen at a point where $|\theta^\star(q) - \bar{\theta}_-| < \zeta$. Since $\zeta > 0$ is arbitrary, the solution is guaranteed to enter any neighborhood of $\mathcal{O}_{\text{lift}}(\bar{\theta}_-)$ in finite time. As discussed earlier, part (b) of the theorem ensures that the solution in question is contained in the basin of attraction of the orbit $\mathcal{O}_{\text{lift}}(\bar{\theta}_+)$. ∎

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