

Path Following for the PVTOL: A Set Stabilization Approach

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Abstract—This article proposes a path following controller that regulates the center of mass of the planar vertical take-off and landing aircraft (PVTOL) to the unit circle and makes the aircraft traverse the circle in a desired direction. A static feedback controller is designed using the ideas of transverse feedback linearization, finite time stabilization and virtual constraints. No time parameterization is given to the desired motion on the unit circle. Instead, our approach relies on the nested stabilization of two sets on which the dynamics of the PVTOL exhibit desirable behavior.

I. INTRODUCTION

Ever since Hauser, Sastry and Meyer’s 1992 paper [1] pointed out the interesting control problems associated with the vertical/short takeoff and landing aircraft, this control system has become a benchmark for controller design. The bulk of existing research can be partitioned into two main categories : set-point stabilization [2], [3], [4], [5] and design of tracking/path following controllers [6], [7], [8], [9], [10], [11], [12]. This paper falls into the latter category and deals with the path following problem.

It is well-known that when the center of mass of the PVTOL is used as an output, the so-called nominal output, the system is non-minimum phase, i.e. the resulting zero dynamics are not asymptotically stable. This fact presents a challenge to controller design. In particular if one is not careful in designing a tracking controller, the PVTOL will begin to rotate about its longitudinal axis uncontrollably making several rotations as the vehicle executes a trajectory. This is obviously undesirable.

In [6] the authors overcome this problem by using alternative outputs which render the system differentially flat or linear. The flat output is the Huygens center of oscillation, see also [5]. The resulting time-varying control law does not solve the tracking problem for the center of mass of the aircraft.

The fact that the PVTOL has non-minimum phase zero dynamics associated with its center of mass suggests that path following controllers may be more appropriate than

tracking controllers. In the tracking approach, the path to be followed is parameterized by time. For the PVTOL this parameterization is not always suitable. Moving along the path too quickly can lead to the undesirable rotations mentioned earlier. Path following controllers do not have an a priori parameterization of the curve to be followed. In fact, there may not even be any reference points at all.

In [9] the authors use non-causal nonlinear system inversion to design tracking controllers and then convert the resulting tracking controller into path following controllers using a popular projection technique introduced in [13]. The authors demonstrate their technique on a circular path. The authors in [11] consider general C^2 paths and investigate the problem of finding a trajectory and an open-loop control such that the center of mass of the PVTOL is on the path and the roll angle is bounded. A desired (“quasi-static”) trajectory for the roll dynamics is defined. A nonlinear optimal control problem is then solved to *numerically* compute a bounded roll trajectory that approximates the quasi-static one. An interesting feature of this approach is that it doesn’t seem to require the center of mass of the PVTOL to move “sufficiently” slowly along the path. On the other hand, it is not clear, from a theoretical viewpoint, whether the approach avoids complete revolutions of the PVTOL about its longitudinal axis. In [12], the authors show that there exists a constant κ such that for every curve γ with $\|\ddot{\gamma}\|_\infty \leq \kappa$, it is possible to find suitable initial conditions for the roll angle and velocity such that the center of mass exactly tracks γ and the aircraft does not overturn. As a consequence they show that any curve can be exactly tracked after a suitable reparameterization. This corresponds to making the center of mass of the PVTOL traverse the curve “sufficiently” slowly.

In this paper we design a path following controller to drive the center of mass of the PVTOL to the unit circle and make it traverse the circle in a desired direction. Rather than parametrizing the path, we take a nested set stabilization approach to solve the problem. We first stabilize a four-dimensional controlled-invariant submanifold of the state space, Γ_1^* , on which the center of mass of the PVTOL is constrained to lie on the circle. We further stabilize a two dimensional submanifold $\Gamma_2^* \subset \Gamma_1^*$ that corresponds to a “virtual constraint” involving the roll angle and the position of the PVTOL on the unit circle: at a given point on the circle, the PVTOL should have a certain roll angle. Our approach guarantees the key feature of *output invariance* of the path, by which we mean that if the PVTOL’s center of mass starts on the unit circle with initial velocity tangent to the circle, the PVTOL’s center of mass will remain on the unit circle for all future time. This is desirable because once

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the aircraft is on the path with the correct velocity vector, the path is followed exactly, which is essential to avoid collisions with the ground.

II. PATH FOLLOWING PROBLEM

We study the well-known, simplified, model of a V/STOL aircraft in planar vertical takeoff and landing (PVTOL) mode as introduced in [1]

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -u_1 \sin x_5 + \epsilon u_2 \cos x_5 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -g + u_1 \cos x_5 + \epsilon u_2 \sin x_5 \\
\dot{x}_5 &= x_6 \\
\dot{x}_6 &= \mu u_2 \\
y &= h(x) = \text{col}(x_1, x_3),
\end{aligned} \tag{1}$$

which we concisely represent as $\dot{x} = f(x) + g(x)u$, $y = h(x)$. The state space of (1) is $M := \mathbb{R}^4 \times S^1 \times \mathbb{R}$, the states $(x_1, x_3) \in \mathbb{R}^2$ represent the coordinates of the center of mass of the aircraft in the vertical plane and $x_5 \in S^1$ is the roll angle. The parameters μ and ϵ are positive constants and $g > 0$ is the acceleration due to gravity.

We are interested in designing a path following feedback control law $u(x)$ for (1) and the unit circle. Let $\lambda : M \rightarrow \mathbb{R}$, $x \mapsto (x_1^2 + x_3^2 - 1)/2$ and let $\Gamma := \{x \in M : \lambda(x) = 0\}$. The problem of driving y to the unit circle can be cast as an output stabilization problem [14] for the system

$$\begin{aligned}
\dot{x} &= f(x) + g(x)u \\
\dot{y} &= \lambda(x).
\end{aligned} \tag{2}$$

It is well-known that the output of the PVTOL can be made to follow any twice continuously differentiable path. It is not clear, however, if the path can be followed while maintaining boundedness of the internal dynamics especially for *arbitrary* initial conditions in an open subset of M . The control objectives of this paper are to design a static feedback $u : U \subset M \rightarrow \mathbb{R}^2$ such that for all initial conditions in an open set U the closed loop dynamics of (1) enjoy the following properties:

- G1** $\tilde{y}(t) \rightarrow 0$ as $t \rightarrow \infty$.
- G2** The path is invariant for the nominal output of the aircraft, as defined in the introduction.
- G3** $y(t)$ traverses the entire set $\{(x_1, x_3) : x_1^2 + x_3^2 = 1\}$ in a desired direction.
- G4** $\exists \bar{t} : |x_5(t)| < \pi$, $\forall t \geq \bar{t}$, that is for $t > \bar{t}$ the aircraft does not experience full rotations about its longitudinal axis.

III. TRANSVERSE FEEDBACK LINEARIZATION

In order to accomplish the objectives listed above, we will use transverse feedback linearization for multi-input systems [15]. In this problem, transverse feedback linearization simply amounts to input-output linearization for system (2). First, we note that the output $\lambda(x)$ of (2) yields a well-defined

relative degree of 2 on $M \setminus \{x : x_1 = x_3 = 0\}$. For, the decoupling matrix

$$L_g L_f \lambda(x) = \begin{bmatrix} (x_3 \cos x_5 - x_1 \sin x_5) & \epsilon(x_1 \cos x_5 + x_3 \sin x_5) \end{bmatrix}$$

has full rank on $M \setminus \{x : x_1 = x_3 = 0\}$. Therefore, in particular, the zero dynamics manifold of (2) (the largest controlled invariant manifold contained in Γ) is

$$\begin{aligned}
\Gamma_1^* &:= \{x \in M : \lambda(x) = L_f \lambda(x) = 0\} \\
&= \{x \in M : (x_1^2 + x_3^2 - 1)/2 = (x_1 x_2 + x_3 x_4) = 0\}.
\end{aligned}$$

From the physical point of view, the set Γ_1^* is the collection of all motions of the PVTOL aircraft that can be made to lie entirely in Γ by appropriate feedback. We refer to this manifold as the *path following manifold* [16]. Solving the output stabilization problem for (2) amounts to stabilizing the set Γ_1^* . We do so by feedback linearizing the dynamics transversal to Γ_1^* . System (2) is non-square because it has one output and two inputs. Hence we first apply a regular feedback transformation $u \mapsto \beta(x)v$ that decomposes the inputs into ‘transversal’ and ‘tangential’ groups. Consider the matrix-valued function $\beta : M \rightarrow \mathbb{R}^{2 \times 2}$ defined as follows

$$\beta(x) := \frac{1}{x_1^2 + x_3^2} \begin{bmatrix} x_3 \cos x_5 - x_1 \sin x_5 & \epsilon(x_1 \cos x_5 + x_3 \sin x_5) \\ (1/\epsilon)(x_1 \cos x_5 + x_3 \sin x_5) & -x_3 \cos x_5 + x_1 \sin x_5 \end{bmatrix}.$$

It is easily seen that $\beta(x)$ is nonsingular on $M \setminus \{x_1 = x_3 = 0\}$. Moreover, the first column of $\beta(x)$ is the right inverse of the decoupling matrix $L_g L_f \lambda(x)$, and the second column of $\beta(x)$ is a basis for the kernel of $L_g L_f \lambda(x)$. Now consider the regular feedback transformation $u = \beta(x)v$, where $v = \text{col}(v_1, v_2)$, and the map $\Xi : M \setminus \{x : x_1 = x_3 = 0\} \rightarrow S^1 \times \mathbb{R} \times S^1 \times \mathbb{R}^3$, $x \mapsto (z, \xi) = (z_1, z_2, z_3, z_4, \xi_1, \xi_2)$, defined as

$$\begin{aligned}
z_1 &:= x_5 \\
z_2 &:= x_6 \\
z_3 &:= \arg(x_1 + ix_3) \\
z_4 &:= x_1 x_4 - x_2 x_3 \\
\xi_1 &:= (x_1^2 + x_3^2 - 1)/2 \\
\xi_2 &:= x_1 x_2 + x_3 x_4.
\end{aligned} \tag{3}$$

This transformation is easily seen to be a diffeomorphism onto its image. Moreover, in transformed coordinates the set Γ_1^* is given by $\Xi(\Gamma_1^*) = \{(z, \xi) : \xi = 0\}$. Consider next, the regular feedback transformation,

$$v_1 = g x_3 - x_2^2 - x_4^2 + v^{\text{tr}}, \quad v_2 = \frac{1}{\epsilon}(x_1 g + v^{\text{tr}}).$$

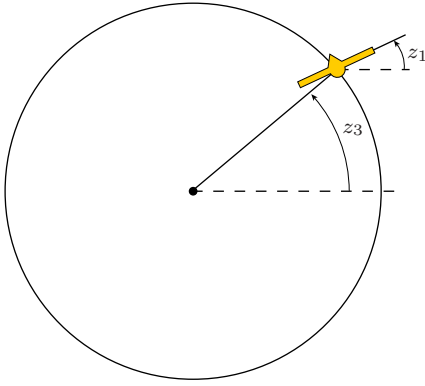


Fig. 1. Physical interpretation of the states $z_1 \in S^1$ (roll angle) and $z_3 \in S^1$ (PVTOL's position on the unit circle) when $\xi = 0$.

In (z, ξ) coordinates, using the two feedback transformations defined above, system (2) assumes the *normal form*

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= \frac{\mu}{\epsilon} \left[g \sin z_1 \right. \\ &\quad - \frac{1}{(1 + 2\xi_1)^{3/2}} \cos(z_1 - z_3)(z_4^2 + \xi_2^2) \\ &\quad \left. + \frac{1}{\sqrt{1 + 2\xi_1}} (\cos(z_1 - z_3)v^{\text{th}} + \sin(z_1 - z_3)v^{\text{ll}}) \right] \quad (4) \\ \dot{z}_3 &= \frac{z_4}{1 + 2\xi_1} \\ \dot{z}_4 &= v^{\text{ll}} \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= v^{\text{th}}, \end{aligned}$$

with output $\tilde{y} = \xi_1$.

The singularity set $\{x : x_1 = x_3 = 0\}$ is mapped by Ξ to the point $\xi_1 = -1/2$. Hence, (4) is valid on $\{(z, \xi) : \xi_1 > -1/2\}$. This normal form is particularly insightful as pertains to the path following problem. First, stabilizing the ξ -subsystem to zero corresponds to driving the center of mass (x_1, x_3) of the aircraft to the unit circle and making its velocity vector (x_2, x_4) tangent to it. For this reason, we refer to the ξ -subsystem as the *transversal* subsystem. When $\xi = 0$, the z -subsystem models the dynamics on the path following manifold. Specifically, z_1 and z_2 represent the roll angle and velocity, while z_3 and z_4 determine the position of the center of mass on the unit circle, and its angular velocity; see Figure 1. Therefore, physically, when $\xi = 0$ the z subsystem describes the motion of the PVTOL as it slides on the circle. We refer to the z -subsystem as the *tangential* subsystem. The normal form (4) illustrates another type of decomposition: the transversal subsystem is only driven by the control input v^{th} . On the other hand, when $\xi(t) \equiv 0$ (and thus $v^{\text{th}}(t) \equiv 0$), the tangential subsystem is only driven by v^{ll} . Accordingly, we refer to v^{th} and v^{ll} as the transversal and tangential control inputs, respectively.

IV. TRANSVERSAL CONTROL DESIGN

To simplify the subsequent stability analysis, we define a continuous feedback yielding stabilization to Γ_1^* (and hence

meeting goals **G1** and **G2**) in finite time. To this end, we follow the finite-time stabilization theory developed in [17], [18].

$$v^{\text{th}}(\xi) = -\frac{1}{k_1} \left(\text{sgn}(k_1 \xi_2) |k_1 \xi_2|^{\frac{1}{2}} + \text{sgn}(\phi(\xi)) |\phi(\xi)|^{\frac{1}{3}} \right) \quad (5)$$

where $k_1 > 0$ is a design parameter, and

$$\phi(\xi) = k_1 \xi_1 + \frac{2}{3} \text{sgn}(k_1 \xi_2) |k_1 \xi_2|^{\frac{3}{2}}.$$

A control law analogous to $v^{\text{th}}(\xi)$ is used in [19] and is based on the controllers introduced in [17].

Proposition 4.1: The control law (5) globally stabilizes ξ to 0. Furthermore, there exists a continuous function $T_\xi(\zeta, k_1), \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow [0, +\infty)$, with the following properties:

- (i) For all $\|\xi(0)\| < \zeta$, $\xi(t) = 0$ for all $t \geq T_\xi(\zeta, k_1)$.
- (ii) $T_\xi(\zeta, k_1) \rightarrow 0$ as $k_1 \rightarrow 0^+$, and $T_\xi(0, k_1) = 0$.

The existence and properties of T_ξ follow from the proof of Proposition 1 in [17], and utilizes [18, Theorem 4.2].

V. TANGENTIAL CONTROL DESIGN

In this section we design a tangential controller v^{ll} meeting design goals **G3** and **G4**. In light of the results of the previous section, for all t greater than $T_\xi(\|\xi(0)\|, k_1)$, we have $\xi(t) = 0$ and therefore also $v^{\text{th}}(t) = 0$. Therefore, for sufficiently large t , the tangential dynamics of the PVTOL restricted to the path following manifold are

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= \frac{\mu}{\epsilon} \left(g \sin z_1 - z_4^2 \cos(z_1 - z_3) + \sin(z_1 - z_3)v^{\text{ll}} \right) \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= v^{\text{ll}} \end{aligned} \quad (6)$$

with $(z_1, z_2, z_3, z_4) \in S^1 \times \mathbb{R} \times S^1 \times \mathbb{R}$.

A. Virtual constraint: the roll dynamics manifold

In order to meet design goals **G3** and **G4**, we will impose a ‘‘virtual constraint’’ $z_1 = f(z_3)$, where $f : S^1 \rightarrow S^1$ is a function to be determined that satisfies the following constraints

- 1) f is smooth (at least C^2).
- 2) $f(z_3 + 2\pi) = f(z_3)$.
- 3) $|f| \leq \pi$ so that while traversing the unit circle, the PVTOL does not perform complete revolutions along its longitudinal axis.

In order for $z_1 = f(z_3)$ to be a feasible constraint for (6) we need

$$\begin{aligned} \dot{z}_1 &= z_2 = f'(z_3)z_4 \\ \ddot{z}_1 &= \dot{z}_2 = f''(z_3)z_4^2 + f'(z_3)v^{\text{ll}} \end{aligned}$$

which holds if and only if

$$\begin{aligned} \frac{\mu}{\epsilon} \left(g \sin z_1 - z_4^2 \cos(z_1 - z_3) + \sin(z_1 - z_3)v^{\text{ll}} \right) \Big|_{z_1=f(z_3)} \\ = f''(z_3)z_4^2 + f'(z_3)v^{\text{ll}}, \end{aligned}$$

or,

$$v^{\parallel} = \frac{\mu}{\epsilon} \frac{\left(g \sin z_1 - z_4^2 \cos(z_1 - z_3) - \frac{\epsilon}{\mu} f''(z_3) z_4^2 \right)}{f'(z_3) - \frac{\mu}{\epsilon} \sin(z_1 - z_3)}.$$

If the denominator of v^{\parallel} is never zero when $z_1 = f(z_3)$ and $z_2 = f'(z_3)z_4$ then the set Γ_2^* defined by

$$\Xi(\Gamma_2^*) := \{(z, \xi) : z_1 = f(z_3), z_2 = f'(z_3)z_4, \xi = 0\} \quad (7)$$

is a controlled invariant subset of the path following manifold Γ_1^* . We refer to Γ_2^* as the *roll dynamics manifold*. Therefore, the problem now becomes: find a function $f(z_3)$ satisfying properties (1)-(3) above and such that $f'(z_3) - \frac{\mu}{\epsilon} \sin(f(z_1) - z_3)$ is never zero. We take the simplest approach and impose that $f'(z_3) - \frac{\mu}{\epsilon} \sin(f(z_1) - z_3)$ be equal to a non-zero constant,

$$f' = \frac{\mu}{\epsilon} \sin(f(z_3) - z_3) - \delta, \quad \delta > 0. \quad (8)$$

Equation (8) is an ODE of the form

$$x' = \frac{\mu}{\epsilon} \sin(x - t) - \delta$$

whose solution can be found explicitly as¹

$$x = t + 2 \arctan \left(\frac{1}{1 + \delta} \left(\lambda \tan \left(K - \frac{\lambda}{2} t \right) + \frac{\mu}{\epsilon} \right) \right),$$

with $\lambda = \sqrt{(1 + \delta)^2 - (\mu/\epsilon)^2}$ and $K \in \mathbb{R}$ a constant. This solution satisfies property (2) (i.e., it is 2π -periodic) if $\tan(K - \lambda t/2)$ is 2π -periodic. This is true if $\lambda/2$ is an integer multiple of $1/2$. We therefore impose that $\lambda = 1$ and solve for δ to obtain

$$\delta = -1 + \sqrt{1 + \left(\frac{\mu}{\epsilon} \right)^2}. \quad (9)$$

In summary we have obtained that the function

$$f(z_3) = z_3 + 2 \arctan \left(\frac{1}{1 + \delta} \left(\tan(K - z_3/2) + \frac{\mu}{\epsilon} \right) \right) \quad (10)$$

with δ as in (9) satisfies properties (1) and (2). To check if (3) is satisfied, note that the salient properties of the function (10) are

(i) The maximum excursion of f is

$$\begin{aligned} \Delta f &= \max_{z_3 \in S^1} (f(z_3)) - \min_{z_3 \in S^1} (f(z_3)) \\ &= 4 \arcsin \left(\frac{\epsilon}{\mu} \delta \right). \end{aligned}$$

(ii) The extrema are at $(z_3^*)_{1,2} = 2K - 2 \arctan(\phi_{1,2} - \frac{\mu}{\epsilon})$ where $\phi_{1,2}$ are the solutions to

$$\frac{\epsilon}{\mu} \delta y^2 - 2(1 + \delta)y + \frac{\epsilon}{\mu} \delta (1 + \delta)^2 = 0.$$

The solutions are both real and positive. We let $\phi_1 < \phi_2$.

¹Note that, when the argument of $\tan(\cdot)$ is $\pm\pi/2$ modulo 2π , the function $x(t)$ has a jump of 2π . However, $x(t)$ is taken to be in S^1 , and so $x(t)$ is actually continuous, and in fact smooth.

(iii) The values of f at the extrema are

$$f_{1,2}^* = (z_3^*)_{1,2} + 2 \arctan \left(\frac{1}{1 + \delta} \phi_{1,2} \right)$$

and f_1^* is a maximum, f_2^* is a minimum.

We select the parameter K so that $f_1^* = -f_2^*$. Physically this means that as the PVTOL moves around the circle, the maximum tilt of the aircraft to the left and to the right is the same. Using properties (i)-(iii) above we find

$$K = \arctan \left(\phi_2 - \frac{\mu}{\epsilon} \right) - \arctan \left(\frac{\phi_2}{1 + \delta} \right) - \arcsin \left(\frac{\epsilon \delta}{\mu} \right). \quad (11)$$

B. Stabilization of the roll dynamics manifold Γ_2^*

The foregoing analysis shows that the set $\Gamma_2^* \subset \Gamma_1^*$ is controlled-invariant and, on it, design goal **G4** is met. Therefore, the next step is to choose the tangential control such that the set Γ_2^* is stabilized in finite time. To this end, let $e_1 := z_1 - f(z_3) \in S^1$, $e_2 := z_2 - f'(z_3)z_4 \in \mathbb{R}$, and $e = (e_1, e_3)$. Then,

$$\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= \frac{\mu}{\epsilon} (g \sin(e_1 + f(z_3)) - z_4^2 \cos(e_1 + f(z_3) - z_3) - \\ &\quad \frac{\epsilon}{\mu} f''(z_3) z_4^2) - \left(f'(z_3) - \frac{\mu}{\epsilon} \sin(f(z_3) - z_3 + e_1) \right) v^{\parallel}. \end{aligned} \quad (12)$$

Therefore we choose as our tangential control law

$$v^{\parallel} = \frac{\frac{\mu}{\epsilon} \left(g \sin(z_1) - z_4^2 \cos(z_1 - z_3) - \frac{\epsilon}{\mu} f''(z_3) z_4^2 \right) - \varphi(e)}{f'(z_3) - \frac{\mu}{\epsilon} \sin(z_1 - z_3)} \quad (13)$$

with f given by (10), δ given by (9), K given (11) and $\varphi(e)$ a function yet to be determined. Note that, on Γ_2^* , the denominator of v^{\parallel} is given by $f'(z_3) - \mu/\epsilon \sin(z_1 - f(z_3))$ which, by construction, is equal to $-\delta < 0$. Hence, v^{\parallel} in (13) is well-defined on a neighborhood of Γ_2^* . It is useful to estimate the size of this neighborhood. To this end, note that a sufficient condition for the denominator of (13) to be bounded away from zero is $|1 - \cos e_1| + |\sin(e_1)| < \epsilon \delta / \mu = -\epsilon / \mu + \sqrt{(\epsilon / \mu)^2 + 1}$. After application of the control law (13) the dynamics (12) become

$$\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= \varphi(e). \end{aligned} \quad (14)$$

Just as we did for the transversal controller, we again define a continuous control for $\varphi(e)$ yielding stabilization of Γ_2^* in finite time based on the control law for the rotational double integrator from [17]

$$\begin{aligned} \varphi(e) &= -\frac{1}{k_2} \left(\operatorname{sgn}(k_2 e_2) |k_2 e_2|^{\frac{1}{2}} \right. \\ &\quad \left. + k_2^{\frac{1}{2}} \operatorname{sgn}(\sin(\phi(e))) |\sin(\phi(e))|^{\frac{1}{3}} \right), \end{aligned} \quad (15)$$

where $k_2 > 0$ is a design parameter and

$$\phi(e) = e_1 + \frac{2}{3} k_2^{\frac{1}{2}} \operatorname{sgn}(e_2) |e_2|^{\frac{3}{2}}.$$

The equilibrium $e = (0 \bmod 2\pi, 0)$ in the closed-loop double integrator in (14) is *not* globally finite-time stable², for the continuous control law (15) introduces a saddle point at $e = (\pi \bmod 2\pi, 0)$. However, the equilibrium $e = (0 \bmod 2\pi, 0)$ of (14) is almost globally finite-time stable, as for any initial condition on $S := S^1 \times \mathbb{R} \setminus \{e : \phi(e) = \pi \bmod 2\pi\}$ the solution converges to the equilibrium in question in finite time. Finally, there exists a continuous “settling time” function $T_e(\zeta, k_2), \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow [0, +\infty)$, with the following properties:

- (i) For all $e(0) \in \Gamma_2^*$ such that $\|e(0)\| < \zeta$, $e(t) = 0$ for all $t \geq T_e(\zeta, k_2)$.
- (ii) $T_e(0, k_2) = 0$.

Consider the closed-loop error subsystem (12) with v^\parallel as in (13) and $\varphi(\cdot)$ as in (15). We have obtained that all solutions $(e_1(t), e_2(t))$ contained in the set $\{e \in S^1 \times \mathbb{R} : |1 - \cos e_1| + |\sin e_1| < -\epsilon/\mu + \sqrt{(\epsilon/\mu)^2 + 1}, \phi(e) \neq \pi\}$ converge to the origin in finite time. In particular, since the origin $e = (0 \bmod 2\pi, 0)$ is stable, and since the set $\{\phi(e) = \pi\}$ is disjoint from the origin, the above implies that for all $e(0)$ in a sufficiently small neighborhood of the origin, the solution $e(t)$ is well-defined and converges to zero in finite time. Now we go back to the tangential subsystem in (6) with control law v^\parallel as above, and draw the implications of this result. Suppose that the PVTOL is initialized on the circle with velocity tangent to it, so that the PVTOL remains on the path and its motion is described by the tangential subsystem. If the initial roll angle z_1 and roll velocity z_2 of the aircraft are not too far from their desired values $f(z_3)$ and $f'(z_3)z_4$, then as the PVTOL slides on the circle, its roll angle and velocity converge in finite time to the desired values or, what is the same, $x(t) \rightarrow \Gamma_2^*$ in finite time. After this time, the motion of the PVTOL is entirely described by the restriction of the closed-loop tangential dynamics to Γ_2^* , and therefore goal **G4** is met. What remains to be ascertained is whether goal **G3** is met, i.e., whether the PVTOL traverses the circle.

C. Motion on the roll dynamics manifold

In this section we analyze the restriction to Γ_2^* of the closed-loop tangential subsystem in (6), with v^\parallel as in (13), and $\varphi(\cdot)$ as in (15). We begin by noticing that Γ_2^* is a two-dimensional closed, embedded submanifold of Γ_1^* . This follows directly from the definition of Γ_2^* in (7), since Γ_2^* is the graph of the map $F : (z_3, z_4) \mapsto (f(z_3), f'(z_3)z_4)$. Moreover, the motion on Γ_2^* is described by

$$\begin{aligned} \dot{z}_3 &= z_4 \\ \dot{z}_4 &= v^\parallel \Big|_{\Gamma_2^*} = \\ &= \frac{\mu}{\epsilon \delta} \left(-g \sin(f(z_3)) + z_4^2 \cos(f(z_3) - z_3) + \frac{\epsilon}{\mu} f''(z_3) z_4^2 \right). \end{aligned} \quad (16)$$

Hence, the dynamics of the closed-loop tangential subsystem on Γ_2^* are completely characterized by those of the system

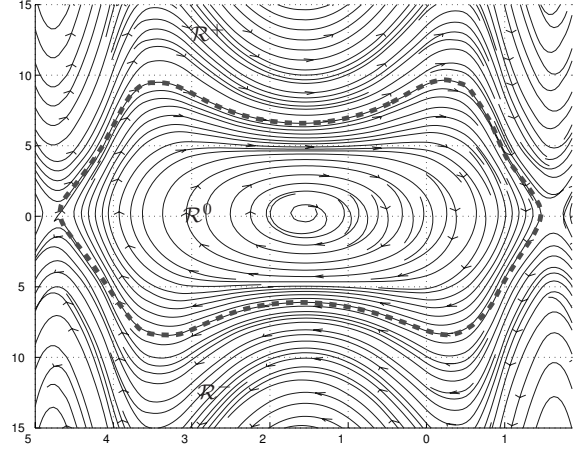


Fig. 2. Phase portrait of the PVTOL constrained to the set Γ_1^* with $\mu/\epsilon = 1$ and K given by (11). Note that $z_3 \in S^1$ so the points $z_3 = n\pi, n \in \mathbb{Z}$ are identified.

in (16), with state space $S^1 \times \mathbb{R}$. Figure 2 shows the phase portrait of system (16) when³ $\mu/\epsilon = 1$ and K is calculated according to (11). The equilibria on this cylinder are given by $(\bar{z}_3, 0)$, where \bar{z}_3 are the values of z_3 such that $f(z_3)$ in (10) is either $0 \bmod 2\pi$ or $\pi \bmod 2\pi$. For the value of K calculated by (11) these values are exactly $\bar{z}_3 = \{\pi/2, -\pi/2\} \bmod 2\pi$ regardless of the value of μ/ϵ . The eigenvalues of the Jacobian of (16) at $(\bar{z}_3, 0)$ are given by

$$\lambda = \pm \sqrt{-\frac{g\mu}{\epsilon\delta} f'(\bar{z}_3)} = \pm \sqrt{\frac{g\mu}{\epsilon}} \sqrt{\frac{\mu}{\epsilon\delta} \sin(\bar{z}_3) + 1}.$$

Hence we obtain that the equilibrium $(\pi/2 \bmod 2\pi, 0)$ is a saddle while the equilibrium at $(-\pi/2 \bmod 2\pi, 0)$ is a center, see Figure 2.

Next note that, based purely on the phase portrait in Figure 2, there are two homoclinic orbits departing from the saddle point $(\pi/2 \bmod 2\pi, 0)$. One of these corresponds to positive z_4 or counterclockwise motion along the unit circle, while the other one corresponds to negative z_4 , or clockwise motion on the unit circle. These two homoclinic orbits divide the cylinder into three components. Call \mathcal{R}^0 the region of the cylinder enclosed by the homoclinic orbits. Let \mathcal{R}^+ denote the component of the cylinder corresponding to $\eta_4 > 0$, and \mathcal{R}^- the component corresponding to $\eta_4 < 0$; see Figure 2. For our purposes, the key to these distinctions is that as long as the state remains in \mathcal{R}^+ or \mathcal{R}^- the z_3 dynamics make full rotations and hence the PVTOL is able to traverse the entire unit circle. Region \mathcal{R}^0 is to be avoided because in it the motion on Γ_2^* corresponds to the PVTOL not traversing the entire unit circle in output space.

Let us summarize what we have obtained so far. If properly initialized, the PVTOL reaches the set Γ_2^* at time $T = \max(T_\xi(|\xi(0)|, k_1), T_e(|e(0)|, k_2))$. If at time T we have $(z_3(T), z_4(T)) \in \mathcal{R}^-$, then the PVTOL traverses the circle in the clockwise direction. If $(z_3(T), z_4(T)) \in \mathcal{R}^+$,

²This is hardly surprising, though, as it is well-known that *no continuous control law* can globally stabilize an equilibrium on the cylinder.

³The discussion in this section deals specifically with the case $\mu/\epsilon = 1$ but the qualitative observations remain unchanged for other values of μ/ϵ .

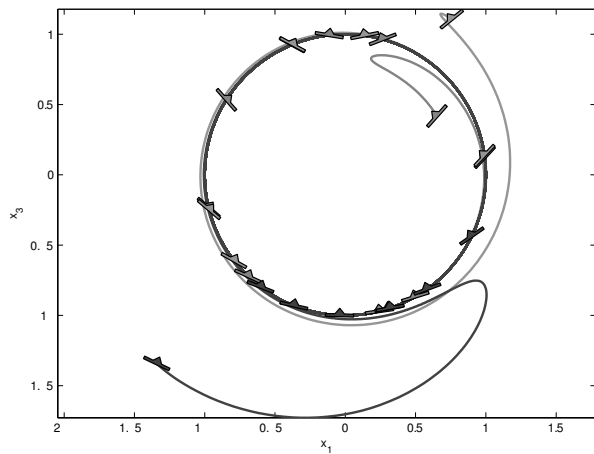


Fig. 3. Three different simulations of the PVTOL approaching and then traversing the unit circle with control gains $k_1 = 0.05$ (transversal) and $k_2 = 0.1$ (tangential).

then the PVTOL traverses the circle in the counterclockwise direction. In the next section, we use this fact to assert the existence of a region of the state space of the PVTOL such for all initial conditions in this region, goals **G1-G4** are met.

VI. STABILITY ANALYSIS

The following notation will be useful for the subsequent development. Given $(z_3, z_4) \in S^1 \times \mathbb{R}$ let $\|(z_3, z_4)\|_{\mathcal{R}^0} := \inf_{p \in \mathcal{R}^0} \|(z_3, z_4) - p\|$. Note that if $\|(z_3, z_4)\|_{\mathcal{R}^0} \neq 0$ then (z_3, z_4) belongs to one of the regions \mathcal{R}^+ or \mathcal{R}^- . Let $\pi(z)$ be the projection map $z \mapsto (z_3, z_4)$.

Proposition 6.1: Consider system (1) with transversal controller v^{\perp} in (5), tangential controller v^{\parallel} in (13), (15), variables z and ξ defined in (3), and variables $e_1 = z_1 - f(z_3)$, $e_2 = z_2 - f'(z_3)z_4$. For all $\varepsilon_2 > \varepsilon_1 > 0$, there exist $\zeta > 0$, such that for all initial conditions such that

$$\begin{aligned} \varepsilon_1 &< \|(z_3(0), z_4(0))\|_{\mathcal{R}^0} < \varepsilon_2 \\ \|e(0)\| &< \zeta, \|\xi(0)\| < \zeta \end{aligned} \quad (17)$$

there exists $T > 0$ yielding $\xi([T, +\infty)) = 0$, $e([T, +\infty)) = 0$, and, for all $t \geq 0$,

$$\|(z_3(t), z_4(t))\|_{\mathcal{R}^0} > 0.$$

Due to space limitations, we omit this straightforward proof.

VII. SIMULATIONS

Figure 3 shows simulation results for the PVTOL (1) following a unit circle for various initial conditions. The simulation parameters are $\mu = \epsilon = 1$, $g = 9.81$. The control parameters are $k_1 = 0.05$, $k_2 = 0.1$.

The salient feature of these simulations is that the circle is stabilized without following any particular reference point. Instead, a set is stabilized with the tangential control v^{\parallel} controlling the motion on the set. This causes the resulting motion on the path to depend on the region, \mathcal{R}^+ or \mathcal{R}^- , that the initial conditions $(z_3(0), z_4(0))$ start in.

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