

# Position Control for a Class of Vehicles in SE(3)

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**Abstract**—A hierarchical design framework is presented to control the position of a class of vehicles in SE(3) that are propelled by a thrust vector along a single body axis and incorporate some mechanism to induce torques about all body axes. A position control outer loop provides reference signals for an attitude control inner loop. The main result of this paper is a set of conditions under which a position controller designed for a point-mass system and an attitude controller can be combined to form a position controller that almost globally asymptotically stabilizes the vehicle to a desired position with desired heading. As opposed to the classical backstepping framework, the proposed approach is modular, in that position control and attitude control designs are completely separate. Thus, for instance, with the proposed technique one can employ any attitude controller from the vast literature on attitude stabilization, provided it enjoys a basic almost global stability property.

## I. INTRODUCTION

In this paper, we consider a class of vehicles in SE(3) that are propelled by a thrust vector along a single body axis and incorporate some mechanism to induce torques about all three body axes. Examples include vertical take-off and landing (VTOL) aircrafts such as helicopters as well as underwater and space vehicles. We investigate the problem of position control for such vehicles to a desired position in  $\mathbb{R}^3$ . These vehicles are underactuated since they can only achieve thrust in a single direction at a given time. That is, there are four degrees of freedom: one translational and three rotational. This makes the control problem more complex. In particular, to obtain a desired thrust direction, we must induce body torques that align the vehicle thrust vector to the desired axis. The work of Tayebi and collaborators [3], [4], [5], [6] adopts a two-stage design strategy to solve the position control problem. An outer translational control loop assigns a desired thrust vector, treating the vehicle as a point mass. An inner attitude control loop then applies a torque input that aligns the thrust vector to the desired axis, while simultaneously controlling the vehicle's heading. This latter stage is designed using the backstepping technique. A similar approach is found in [7].

In some papers, attitude is parameterized with Euler-angles and the control yields only local results. For instance, in [8] a model predictive controller is used for the translational control stage whereas a robust nonlinear  $\mathcal{H}^\infty$  controller is used for the attitude control stage. In [9], a sliding mode

controller is used for both stages where neural networks are used for disturbance rejection. In [10], the approach has three control stages. The first stage uses the thrust input and yawing torque to control the vehicle elevation and yaw angle, respectively. In the second stage, the pitching torque is used to control  $y$ -position and pitch angle. In this stage, a nested saturation control is used to bound the pitching torque. The third stage is similar to the second, where the rolling torque is used to control  $x$ -position and roll angle.

To avoid singularities associated with Euler-angles, other approaches parameterize attitude using global representations. In some literature, this is done with rotation matrices and the control yields almost-global results. In [7], a simple linear proportional-derivative controller is chosen for the outer translational control loop. In [11], rather than using the two-stage approach, the authors develop a position controller which is evolved through a series of simpler controllers (i.e., from thrust direction control to velocity control to position control). In [4] and [6], the attitude parameterization is done with quaternions, and the controller is designed without measurements in linear and angular velocity, respectively. The resulting control produces a global result. However, quaternions suffer from an unwinding issue related to attitude control [12].

The goal of this paper is to develop a hierarchical control design framework for position stabilization that provides benefits over the backstepping approach. Like [3], [4], [5], [6], [7], we use a two-stage approach. However, rather than relying on specific position control and attitude control designs, we show that *any* outer position control stage belonging to a suitable class can be combined with *any* inner attitude control stage in a suitable class in such a way that the resulting hierarchical controller stabilizes the desired position almost-globally. This result has useful consequences. First, by decoupling position control from attitude control, the complexity of the control design process is significantly reduced and the final control is intuitive and structured. Second, the proposed hierarchical design is modular in that one can replace either one of the control stages without having to redesign the remaining stage. As a result, one can leverage the rich literature on attitude control to systematically generate position controllers for thrust-propelled vehicles in SE(3). Finally, the modularity of our approach allows one to easily change the control specification for the outer control stage. For instance, one may swap the position stabilizer with a path following controller.

In this paper, we will provide a solution to the position control problem for the rotation matrix parameterization. Identical results can be formulated using quaternions, but

This research was supported by the National Sciences and Engineering Research Council of Canada.

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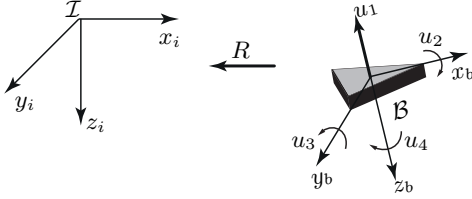


Fig. 1: Vehicle class under consideration.

they are not included for space limitations. Our results rely on the so-called reduction theorem for asymptotic stability of sets by P. Seibert and J.S. Florio in [13]. Some of the ideas presented here were explored in the context of co-axial helicopters in [14].

*Notation.* We let  $v \cdot w$  denote the Euclidean inner product between vectors  $v$  and  $w \in \mathbb{R}^3$  and the vector  $e_i$  represents the  $i$ -th Euclidean axis in  $\mathbb{R}^3$ . Let  $\mathbf{S}(x)$  be the skew-symmetric representation of the vector  $x$ , so that  $\mathbf{S}(x)y = x \times y$  for all  $x, y \in \mathbb{R}^3$ , and  $\mathbf{S}^{-1}(s)$  be its inverse. If  $\|\cdot\|$  is a vector norm and  $\Gamma$  is a closed subset of a manifold  $\mathcal{X}$  a metric space, we denote by  $\|x\|_\Gamma$  the point-to-set distance of  $x \in \mathcal{X}$  to  $\Gamma$ , both  $x$  and  $\Gamma$  being viewed as subsets of  $\mathcal{X}$ . If  $\epsilon > 0$ , we let  $B_\epsilon(\Gamma) = \{x \in \mathcal{X} : \|x\|_\Gamma < \epsilon\}$ . By  $\mathcal{N}(\Gamma)$  we denote a generic neighbourhood of  $\Gamma$  in  $\mathcal{X}$ . Finally, if  $A$  and  $B$  are two sets, we denote by  $A \setminus B$  the set-theoretic difference of  $A$  and  $B$ .

## II. MODELING

Consider the vehicle depicted in Figure 1, with a body frame  $\mathcal{B}$  attached to it. The  $z_b$  axis is the direction of actuation, in that the vehicle is propelled by a thrust vector directed opposite to  $z_b$ . This thrust vector has constant direction in the body frame, but its magnitude  $u_1$  can be freely controlled. It is assumed that the vehicle incorporates some mechanism that can induce torques  $u_2, u_3, u_4$  about the three body axes, as shown in the figure. The control inputs of our abstracted model are  $u_1, u_2, u_3, u_4$ . The actual physical inputs (e.g., rotor speeds) will depend on the vehicle design and the mechanism used to induce torques. In this paper, we consider the rotation matrix parameterization for attitude. We define the following states:

- $x \in \mathbb{R}^3$ : vehicle position expressed in frame  $\mathcal{I}$ .
- $v \in \mathbb{R}^3$ : vehicle linear velocity expressed in frame  $\mathcal{I}$ .
- $R \in \text{SO}(3)$ : vehicle attitude.
- $\Omega \in \mathbb{R}^3$ : vehicle angular velocity expressed in frame  $\mathcal{B}$ .

The state vectors is given by,

$$\chi = \text{col}(x, v, R, \Omega) \in \mathcal{X} := \mathbb{R}^3 \times \mathbb{R}^3 \times \text{SO}(3) \times \mathbb{R}^3.$$

The system configuration is specified by the pair  $(x, R)$  which can be identified with a homogeneous transformation matrix

$$H = \begin{bmatrix} R & x \\ 0 & 1 \end{bmatrix} \in \text{SE}(3),$$

and for this reason the configuration space of the vehicle is  $\text{SE}(3)$ . We now model the system dynamics. The model has

two components. A translational subsystem,

$$\begin{aligned} \dot{x} &= v \\ m\dot{v} &= mge_3 - u_1 Re_3 = mge_3 + T, \end{aligned} \quad (1)$$

and a rotational subsystem,

$$\begin{aligned} \dot{R} &= R\mathbf{S}(\Omega) \\ J\dot{\Omega} + \Omega \times J\Omega &= \begin{bmatrix} u_2 \\ u_3 \\ u_4 \end{bmatrix} = \tau. \end{aligned} \quad (2)$$

In the above,  $\tau := \text{col}(u_1, u_2, u_3)$  is the vector of external torques expressed in frame  $\mathcal{B}$  and  $J$  is the symmetric inertia matrix of the vehicle expressed in frame  $\mathcal{B}$ .

*Remark 2.1:* The model in (1)-(2) neglects disturbances and dissipative effects that are present in specific applications, and in this paper we will design feedbacks that ignore these effects. In specific vehicle applications, the experienced practitioner knows which effects can be ignored and which ones need modelling. In the latter case, the feedbacks we propose in this paper can be easily modified to compensate for those disturbances or dissipative effects whose model is partially known. As for effects whose model is not available or too complex to use, the practitioner has to rely on the intrinsic robustness of feedback.

A broad range of vehicles fit the class under consideration. These include space vehicles, unmanned aerial vehicles and automated underwater vehicles.

## III. STABILITY DEFINITIONS AND REDUCTION THEOREM

The solution of PCP will rely on some basic stability notions, presented next. Let  $\Sigma : \dot{\chi} = f(\chi)$  be a smooth dynamical system with state space a manifold  $\mathcal{X}$  endowed with a metric, and flow map  $\phi(t, \chi_0)$ . Let  $\Gamma \subset \mathcal{X}$  be a closed and positively invariant set for  $\Sigma$ .

*Definition 3.1:*  $\Gamma$  is *stable* for  $\Sigma$  if for any  $\epsilon > 0$  there exists a neighbourhood  $\mathcal{N}(\Gamma) \subset \mathcal{X}$  such that  $\phi(\mathbb{R}^+, \mathcal{N}(\Gamma)) \subset B_\epsilon(\Gamma)$ .  $\Gamma$  is *attractive* for  $\Sigma$  if there exists neighbourhood  $\mathcal{N}(\Gamma) \subset \mathcal{X}$  such that  $\lim_{t \rightarrow \infty} \|\phi(t, \chi_0)\|_\Gamma = 0$  for all  $\chi_0 \in \mathcal{N}(\Gamma)$ . The *domain of attraction of  $\Gamma$*  is the set  $\{\chi_0 \in \mathcal{X} : \lim_{t \rightarrow \infty} \|\phi(t, \chi_0)\|_\Gamma = 0\}$ .  $\Gamma$  is *asymptotically stable* if it is stable and attractive.

*Definition 3.2:* Let  $\Gamma_1 \subset \Gamma_2$  be two closed subsets of  $\mathcal{X}$  which are positively invariant for  $\Sigma$ . We say that  $\Gamma_1$  is *globally asymptotically stable relative to  $\Gamma_2$*  if it is asymptotically stable when initial conditions are restricted to lie in  $\Gamma_2$ , and its domain of attraction contains  $\Gamma_2$ .

*Definition 3.3:* The set  $\Gamma$  is *almost-globally asymptotically stable (AGAS)* for  $\Sigma$  if the set  $\Gamma$  is asymptotically stable for  $\Sigma$  with domain of attraction  $\mathcal{X} \setminus N$  where  $N \subset \mathcal{X}$  is a set of Lebesgue measure zero.

The following result is key to our development.

*Theorem 3.4 (Seibert-Florio [13]):* Let  $\Gamma_1$  and  $\Gamma_2$ ,  $\Gamma_1 \subset \Gamma_2 \subset \mathcal{X}$ , be two closed sets that are positively invariant for  $\Sigma$ , and suppose  $\Gamma_1$  is compact. Then,  $\Gamma_1$  is globally asymptotically stable if the following conditions hold:

- $\Gamma_1$  is globally asymptotically stable relative to  $\Gamma_2$ ,

- (ii)  $\Gamma_2$  is globally asymptotically stable,
- (iii) All trajectories of  $\Sigma$  are bounded.

The statement above is actually a corollary of a more general result by Seibert and Florio. See also [15]. We remark that the state space  $\mathcal{X}$  in Theorem 3.4 can be replaced by any positively invariant subset of  $\mathcal{X}$ .

#### IV. POSITION CONTROL PROBLEM

We now look to define formally the position control problem. Since the manifold  $\mathcal{X}$  is not contractible, we cannot globally asymptotically stabilize the point  $\bar{\chi} = (\bar{x}, 0, \bar{R}, 0)$  using a continuous feedback [12]. Therefore, we will look for an almost-global result. We are now ready to state the problem investigated in this paper.

**Position Control Problem (PCP):** Design smooth feedbacks  $u(\chi) = (u_1(\chi), \dots, u_4(\chi))$  for systems (1)-(2) that almost globally asymptotically stabilize a desired equilibrium  $\bar{\chi} = (\bar{x}, 0, \bar{R}, 0)$ .

We remark that in order for  $\bar{\chi} = (\bar{x}, 0, \bar{R}, 0)$  to be an equilibrium of the closed-loop system, the matrix  $\bar{R}$  must represent a rotation about the inertial axis  $z_i$ .

Our design is performed in two stages. A block diagram illustrating the approach is found in Figure 2. In the first stage, we design an outer loop controller for the translational subsystem assuming that the thrust vector is a control input. Then, in the second stage we design an inner loop attitude controller for the rotational subsystem which orients the thrust vector  $T$  of the vehicle to match the desired thrust designed in the first stage. Such an approach is not new in the literature. It is found prominently in the work of Tayebi and collaborators [3], [4], [5], [6] as well as in [7]. These papers, however, present specific position and attitude control designs, inextricably tied together through the technique of backstepping. The resulting controllers are complex, a feature that is typical of Lyapunov-based backstepping control. On the other hand, rather than relying on specific position control and attitude control designs, in this paper we show that *any* outer position control stage belonging to a suitable class can be combined with *any* inner attitude control stage in a suitable class in such a way that the resulting controller stabilizes the desired position almost-globally. Since we do not rely on Lyapunov methods, the combination of inner and outer controllers is transparent.

The technique presented in this paper has a number of useful features. First, by decoupling position control from attitude control, the complexity of the control design process is significantly reduced and the final control is intuitive and structured. Second, the proposed design is modular, in that one can replace either one of the control stages without having to redesign the remaining stage. As a result, one can leverage the rich literature on attitude control to systematically generate position controllers for thrust-propelled vehicles in SE(3). Finally, the modularity of our approach allows one to easily change the control specification for the outer control stage. For instance, one may swap the position controller with a path following controller for a point-mass system.

The results of this paper rely on the so-called reduction theorem for asymptotic stability of sets by P. Seibert and J.S. Florio in [13]. Some of the ideas presented here were explored in the context of co-axial helicopters in [14]. Note that the position control problem can be mapped to one of tracking if the vehicle is controlled to a series of way-points.

#### V. HIERARCHICAL SOLUTION OF PCP

As mentioned earlier, our control design relies on a two-stage approach, depicted in Figure 2 for the vehicle model. An outer loop position controller is designed for the translational subsystem (1) by viewing the thrust force  $T$  as a control input. The result is a feedback  $T_d(x, v)$  that globally asymptotically stabilizes the equilibrium  $(x, v) = (\bar{x}, 0)$  for (1). We then assign the thrust magnitude input by setting  $u_1 = \|T_d\|$ , and we compute the desired attitude  $R_d$  through a process called *attitude extraction* [3], [4], [6] which is standard in the literature. Specifically, we find a smooth function  $\mathbf{R} : (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^3 \rightarrow \text{SO}(3)$  such that

- (i)  $(\forall (T, x) \in (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^3) \|T\| \mathbf{R}(T, x) e_3 = -T$ ,
- (ii)  $\mathbf{R}(-mge_3, \bar{x}) = \bar{R}$ ,

and we let  $R_d = \mathbf{R}(T_d, x)$ . Identity (i) in (3) guarantees that when  $R = \mathbf{R}(T, x)$  and  $u_1 = \|T\|$ , the resulting thrust vector in (1) coincides with  $T$ . Identity (ii) in (3) guarantees that the attitude extraction function  $\mathbf{R}$  returns the desired equilibrium orientation  $\bar{R}$  when the vehicle hovers at the desired equilibrium position  $\bar{x}$ . There are infinitely many choices<sup>1</sup> of smooth functions  $\mathbf{R}$  satisfying (3). As a matter of fact, one can define  $\mathbf{R}(T, x)$  in such a way that the heading vector  $x_b$  is any arbitrary unit vector orthogonal to  $T$ .

The desired attitude  $R_d$  obtained at the first stage becomes the reference signal for the inner loop attitude controller at the second stage. The attitude controller assigns a body torque  $\tau$  making the point  $(R_d^{-1}R, \Omega - \Omega_d) = (I, 0)$  AGAS, for a suitable  $\Omega_d(t)$ . This control scheme is illustrated in Figure 2. We now present the main result of this paper.

*Theorem 5.1:* Consider smooth position and attitude controllers  $T_d(x, v) : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3 \setminus \{0\}$  and  $\tau_d(R, \Omega) : \text{SO}(3) \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$  satisfying the following properties:

- (i)  $\inf \|T_d(x, v)\| > 0$  and  $\sup \|T_d(x, v)\| < \infty$ .
- (ii) When  $T = T_d(x, v)$ , the equilibrium  $(x, v) = (\bar{x}, 0)$  is globally asymptotically stable for the translational subsystem (1).
- (iii) For any piecewise continuous function  $\rho : \mathbb{R} \rightarrow \mathbb{R}^3$  such that  $\rho(t) \rightarrow 0$ , letting  $T = T_d(x, v) + \rho(t)$  all solutions of the  $(x, v)$  subsystem (1) are bounded.
- (iv) When  $\tau = \tau_d(R, \Omega)$ , the point  $(R, \Omega) = (I, 0)$  is AGAS for the  $(R, \Omega)$  subsystem (2).

<sup>1</sup>This degree of freedom in the choice of  $\mathbf{R}$  is useful because it allows one to incorporate specifications on the heading vector  $x_b$ . For instance, one may want a camera on the vehicle to fixate on a point during motion, in which case  $x_b$  would depend on  $x$ , hence the dependence of  $\mathbf{R}$  on  $x$ .

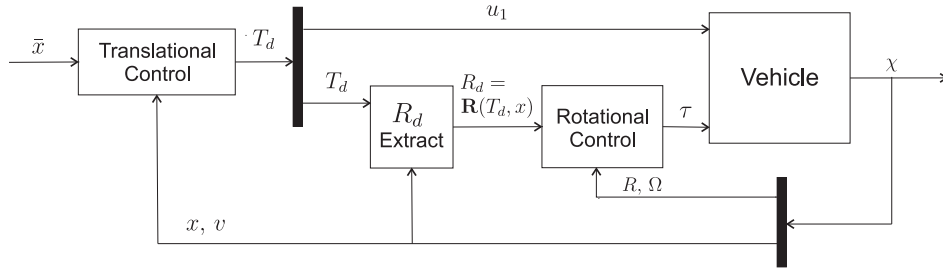


Fig. 2: Block diagram of position control system. The outer loop assigns a thrust vector reference  $T_d$ . The inner loop converts  $T_d$  into a reference attitude, which is then used by an attitude controller to assign the vehicle torques.

Then, letting

$$\begin{aligned}\tilde{R} &:= \mathbf{R}^{-1}(T_d(x, v), x)R \\ \tilde{\Omega} &:= \Omega - \tilde{R}^{-1}\mathbf{\Omega}(x, v, R) \\ \mathbf{\Omega}(x, v, R) &:= \mathbf{S}^{-1}\left(\mathbf{R}^{-1}(T_d(x, v), x)\dot{\mathbf{R}}(x, v, R)\right),\end{aligned}$$

the smooth feedback

$$\begin{aligned}u_1 &= \|T_d(x, v)\| \\ \tau &= \tau_d(\tilde{R}, \tilde{\Omega}) - \tilde{\Omega} \times J\tilde{\Omega} + \Omega \times J\Omega \\ &\quad - J\left(\mathbf{S}(\tilde{\Omega})\tilde{R}^{-1}\mathbf{\Omega}(x, v, R) - \tilde{R}^{-1}\dot{\mathbf{\Omega}}(x, v, R, \Omega)\right),\end{aligned}\quad (4)$$

solves PCP for system (1), (2).

*Remark 5.2:* The functions  $\dot{\mathbf{R}}(x, v, R)$  and  $\dot{\mathbf{\Omega}}(x, v, R, \Omega)$  in the feedback above are the time derivatives of  $\mathbf{R}(T_d(x, v), x)$  and  $\mathbf{\Omega}(x, v, R)$  along (1)-(2) with  $u_1 = \|T_d(x, v)\|$ .

*Remark 5.3:* As mentioned earlier, the proposed control structure has two nested loops, depicted in Figure 2. The outer loop is the position controller  $T_d(x, v)$  for the translational subsystem. The inner loop generates reference signals  $\mathbf{R}(T_d(x(t), v(t)), x(t))$  and  $\mathbf{\Omega}(x(t), v(t), R(t))$ , and produces a torque feedback  $\tau$  in (4) making  $R(t)$  and  $\Omega(t)$  track these references. The definition of  $\tau$  in (4) has an intuitive explanation. Taking the time derivatives of the error signals  $\tilde{R}$  and  $\tilde{\Omega}$ , it is readily seen that

$$\begin{aligned}\dot{\tilde{R}} &= \tilde{R}\mathbf{S}(\tilde{\Omega}) \\ J\dot{\tilde{\Omega}} + \tilde{\Omega} \times J\tilde{\Omega} &= \tau_d(\tilde{R}, \tilde{\Omega}).\end{aligned}\quad (5)$$

So we see that  $\tau$  has been defined in such a way that, in error coordinates  $(\tilde{R}, \tilde{\Omega})$ , the proposed feedback reduces to  $\tau_d$ , an attitude stabilizer that makes the equilibrium  $(\tilde{R}, \tilde{\Omega}) = (I, 0)$  AGAS. This property implies that  $R(t) \rightarrow \mathbf{R}(T_d(x(t), v(t)), x(t))$  and  $\Omega(t) \rightarrow \mathbf{\Omega}(x(t), v(t), R(t))$ .

*Proof:* [Proof of Theorem 5.1] Consider first system (1)-(2), and define sets

$$\begin{aligned}\Gamma_1 &= \{(x, v, R, \Omega) = (\bar{x}, 0, \mathbf{R}(T_d(\bar{x}, 0), \bar{x}), 0)\} \\ \Gamma_2 &= \{\chi : \mathbf{R}(T_d(x, v), x)^{-1}R = I, \Omega - \tilde{R}^{-1}\mathbf{\Omega}(x, v, R) = 0\}.\end{aligned}$$

Assume for a moment that the closed-loop system has no finite escape times. By assumption (ii) in the theorem,  $T_d(x, v)$  is an almost global stabilizer of  $(x, v) = (\bar{x}, 0)$  for subsystem (1). This implies that  $T_d(\bar{x}, 0) = -mge_3$ . By assumption (i),  $\inf \|T_d(x, v)\| > 0$ , so the attitude extraction

function in (3) is well-defined. By property (ii) in (3),  $\mathbf{R}(T_d(\bar{x}, 0), \bar{x}) = \bar{R}$ , so that  $\Gamma_1 = \{\bar{\chi}\}$ , the equilibrium we wish to stabilize.  $\Gamma_2$  is the set where  $(\tilde{R}, \tilde{\Omega}) = (I, 0)$ . The dynamics of the  $(\tilde{R}, \tilde{\Omega})$  subsystem are given in (5), and by assumption (iv) the equilibrium  $(\tilde{R}, \tilde{\Omega}) = (I, 0)$  is AGAS. If we let  $\mathcal{X}$  denote its domain of attraction in  $\chi$  coordinates, then  $\mathcal{X}$  is positively invariant for the closed-loop system, and  $\Gamma_2$  is globally asymptotically stable relative to  $\mathcal{X}$ . Note that  $\mathcal{X}$  is a set of full measure in  $\mathbb{R}^3 \times \mathbb{R}^3 \times \text{SO}(3) \times \mathbb{R}^3$ . On  $\Gamma_2$ , we have  $R = \mathbf{R}(T_d(\bar{x}, v), x)$ . By property (i) of the attitude extraction function in (3), we have

$$-u_1 R e_3 = -\|T_d(x, v)\|\mathbf{R}(T_d(\bar{x}, v), x)e_3 = T_d(x, v).$$

Therefore, the motion on  $\Gamma_2$  is governed by

$$\begin{aligned}\dot{x} &= v \\ \dot{v} &= mge_3 + T_d(x, v).\end{aligned}$$

By assumption (ii) in the theorem,  $\Gamma_1$  is globally asymptotically stable relative to  $\Gamma_2$ . We will now show that all solutions of the closed-loop system originating in  $\mathcal{X}$  have no finite escape times and they are bounded. The translational subsystem can be written as

$$\begin{aligned}\dot{x} &= v \\ \dot{v} &= mge_3 + T_d(x, v) + (-\|T_d\|Re_3 - T_d(x, v)).\end{aligned}$$

In the above,  $Re_3$  has unit norm, and by assumption (i),  $T_d(x, v)$  is bounded. Hence,  $\dot{v}$  is bounded, and the  $(x, v)$  subsystem has no finite escape times. This in turn implies that the smooth function  $\mathbf{\Omega}(x, v, R)$  has no finite escape times. Finally, since  $R$  lives in  $\text{SO}(3)$ , a compact set, and  $\Omega - \mathbf{\Omega}(x, v, R)$  is bounded, we have that  $(R, \Omega)$  has no finite escape times. Now consider assumption (iii), and for an arbitrary  $\chi(0) \in \mathcal{X}$  let  $\rho(t) = -\|T_d(x(t), v(t))\|R(t)e_3 - T_d(x(t), v(t))$ . By property (i) in (3), and by the global asymptotic stability of  $\Gamma_2$ ,  $\rho(t) \rightarrow 0$ . Therefore,  $(x(t), v(t))$  are bounded, implying that the signal  $\mathbf{\Omega}(x(t), v(t), R(t))$  is bounded as well. Finally, the boundedness of  $\tilde{\Omega}(t)$  and that of  $\mathbf{\Omega}(x(t), v(t), R(t))$  imply that  $\Omega$  is bounded. Having shown that all solutions of the closed-loop system originating in  $\mathcal{X}$  are bounded, by Theorem 3.4 we conclude that  $\Gamma_1$  is globally asymptotically stable relative to  $\mathcal{X}$  or, what is the same, the equilibrium  $\chi = \bar{\chi}$  is AGAS for the closed-loop system. ■

*Remark 5.4:* The globality of Theorem 5.1 is inherited from the globality of the attitude control stage. For example,

we obtain a local result for a feedback  $\tau_d(R, \Omega)$  that asymptotically stabilizes the point  $(R, \Omega) = (I, 0)$  such as those with attitude parameterized by Euler angles.

## VI. SAMPLE IMPLEMENTATION

In section V, we developed a general framework for the solution of PCP. In this section we present a sample implementation.

### A. Stage 1: Position control

For the point-mass system

$$\begin{aligned}\dot{x} &= v \\ m\dot{v} &= mge_3 + T,\end{aligned}$$

we need to design a feedback  $T_d(x, v)$  that globally asymptotically stabilizes the equilibrium  $(x, v) = (\bar{x}, 0)$  and is such that  $\inf \|T_d(x, v)\| > 0$ ,  $\sup \|T_d(x, v)\| < \infty$ , and the solutions when  $T = T_d(x, v) + \rho(t)$ , with  $\rho(t) \rightarrow 0$  are bounded. There are many ways to design a bounded feedback meeting these specifications. We will use a nested-saturation controller developed in [16] (also see [17]),

$$T_d(x, v) = -m \left( ge_3 + \sigma_2 \left( K_2 v + \sigma_1 \left( K_1 (x - \bar{x}) + \frac{K_1}{K_2} v \right) \right) \right) \quad (6)$$

where  $K_1, K_2 > 0$  and  $\sigma_1, \sigma_2$  are smooth saturation functions satisfying,

- i  $\sigma_i(s) = (\sigma_{i1}(s_1), \sigma_{i2}(s_2), \sigma_{i3}(s_3))$  for  $i = 1, 2$
- ii  $s\sigma_{ij}(s) > 0$  when  $s \neq 0$  for  $i = 1, 2$   $j = 1, 2, 3$
- iii  $\dot{\sigma}_{ij}(0) \neq 0$  for  $i = 1, 2$   $j = 1, 2, 3$
- iv  $|\sigma_{ij}(s)| \leq M_{ij} \forall s \in \mathbb{R}$  where  $M_{1j} < \frac{M_{2j}}{2}$  for  $i = 1, 2$   $j = 1, 2, 3$ .

In particular, we choose  $\sigma_{ij}(s) = M_{ij} \tanh\left(\frac{1}{M_{ij}}s\right)$ . We impose the condition that  $M_{23} < g$  so that  $\inf \|T_d\| > 0$  at any time. It is also obvious that  $\sup \|T_d\| < \infty$ . Thus,  $T_d(x, v)$  satisfies assumption (i) of Theorem 5.1. Moreover, in [16] it was shown that  $T_d(x, v)$  above globally asymptotically stabilizes the equilibrium  $(x, v) = (\bar{x}, 0)$  for (1), and thus condition (ii) of Theorem 5.1 is satisfied. Finally, it is readily seen that (6) makes the equilibrium  $(x, v) = (\bar{x}, 0)$  exponentially stable. Using a standard Lyapunov analysis with a quadratic Lyapunov function arising from the linearization of the closed-loop system, it is easy to show that solutions of the system with vanishing input perturbations are bounded, so that assumption (iii) of Theorem 5.1 is satisfied.

### B. Stage 2: Attitude extraction

We begin the attitude control design by defining the attitude extraction function  $\mathbf{R}(T, x)$  satisfying the identities in (3). Let  $b_{1d}(T, x)$  be any smooth function  $\mathbb{R}^3 \times \mathbb{R}^3 \rightarrow S^2$  such that for all  $(T, x)$ ,  $b_{1d}(T, x)$  is orthogonal to  $T$  and  $b_{1d}(mg, \bar{x}) = \bar{R}e_1$  (this is the desired heading at the hovering equilibrium). Define  $b_{3d}(T, x) = -T/\|T\|$ . Then, the function

$$\mathbf{R}(T, x) := [b_{1d}(T, x) \quad b_{3d}(T, x) \times b_{1d}(T, x) \quad b_{3d}(T, x)]$$

satisfies the two identities in (3).

### C. Stage 2: Attitude control

Now we need to define the attitude controller  $\tau_d$  that achieves almost global stabilization of  $(R, \Omega) = (I, 0)$ . There is a vast literature on the subject of attitude stabilization, and our modular design allows one to pick from a multitude of designs. We pick the controller presented in [18],

$$\tau_d(R, \Omega) = -K_R \left( \sum_{i=1}^3 a_i e_i \times R e_i \right) - K_\Omega \Omega \quad (7)$$

where  $K_R, K_\Omega > 0$ , and  $a_i$  are distinct positive constants. From the analysis in [18], the point  $(R, \Omega) = (I, 0)$  is AGAS for the closed-loop system. Therefore, condition (iv) of Theorem 5.1 is satisfied, and PCP is solved.

## VII. SIMULATION RESULTS

In this section, we will provide simulation results for the sample implementation. The vehicle will be specified to travel from an initial to a desired position in  $\mathbb{R}^3$ . We will look at two cases. In case 1, the vehicle is initially upright and the desired heading is different from the initial heading. In case 2, the vehicle is initially upside-down and the desired heading is the same as the initial heading.

The initial conditions are taken as,

- $x_0 = (1, 1, 1)m$
- $v_0 = (0, 0, 0)m/s$
- $R_0 = I$  (upright) or  $R_0 = \text{diag}(1, -1, -1)$  (upside-down)

and the desired position is chosen to be  $x_d = (0, 0, 0)m$ . The desired heading while hovering is  $b_{1d}(-mge_3, \bar{x}) = (0, 1, 0)$  in case 1 and  $b_{1d}(-mge_3, \bar{x}) = (1, 0, 0)$  in case 2. The parameters are chosen as  $m = 2$  Kg and  $I_x, I_y, I_z = 1.2416$  Kg.m<sup>2</sup> and the gains for the translational controller are chosen as  $M_{1j} = 2$ ,  $M_{2j} = 5$ ,  $n_{1j} = 1$ ,  $n_{2j} = 1$  for  $j = 1, 2, 3$ . The rotational control gains are chosen as  $K_R = 200$ ,  $K_\Omega = 8$ ,  $a_1 = 0.9$ ,  $a_2 = 1$ ,  $a_3 = 1.1$ .

Figure 3 shows simulation results for case 1 and Figure 4 shows simulation results for case 2. The translational plots show the vehicle trajectory projected onto the  $x_i - y_i$ ,  $y_i - z_i$  and  $x_i - z_i$  planes, and the linear velocity given by  $\sqrt{v_1^2 + v_2^2 + v_3^2}$ . The attitude plot shows the three body axes plotted on a unit sphere.

For all the results, the vehicle successfully converges to the desired equilibrium point. In case 2, we see that the vehicle has some drift away from the desired equilibrium while it flips to an upright orientation. In figure 4 this drift is around  $6m$  for the particular choices of  $K_R$  and  $K_\Omega$ . It has been observed through simulation that increasing  $K_R$  has the effect of reducing this drift. Also, an increase in  $K_\Omega$  has the effect of reducing oscillation of the vehicle. Overall, we have obtained satisfactory performance with appropriately chosen gains  $K_R$  and  $K_\Omega$ .

## VIII. CONCLUSIONS

This paper presented a hierarchical approach to position control design for a class of thrust-propelled vehicles on SE(3). The main result of the paper is a set of conditions

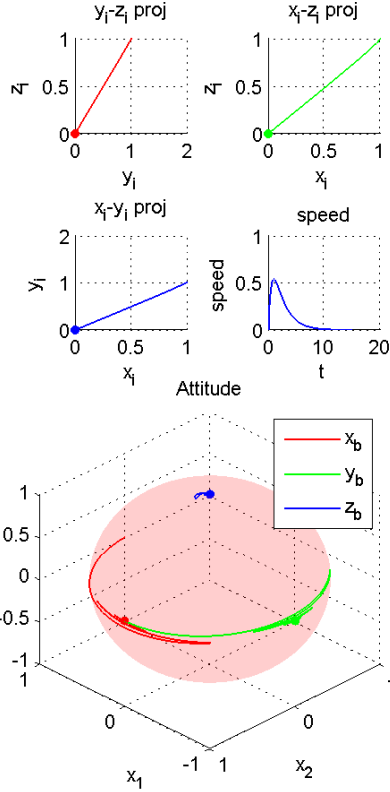


Fig. 3: Rotation matrix: simulation results for case 1.

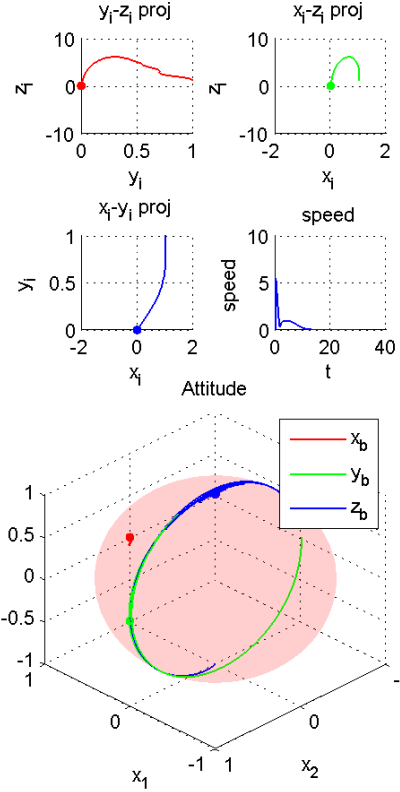


Fig. 4: Rotation matrix: simulation results for case 2.

under which a position controller designed for a point-mass system and an attitude controller can be combined to form a position controller for the vehicle. We tested our controller using a nested saturation feedback for the outer position control loop, and an attitude controller taken from [7]. Due to the modularity of our approach, it is possible to test a variety of available attitude control techniques.

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