A Smooth Distributed Feedback for Global Rendezvous of Unicycles

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***Abstract*— This paper presents a solution to the rendezvous control problem for a network of unicycles on the plane. A smooth, time-invariant control law is presented that drives the unicycles to a common position from arbitrary initial conditions. Each unicycle is equipped with an onboard camera and can measure its relative displacement to its neighbors in body frame. The feedback is a function only of these onboard measurements and no global positioning system is required, nor any information about the unicycles' orientations.**

I. Introduction

This paper presents a solution to the rendezvous control problem for a group of kinematic unicycles on the plane. The objective of the rendezvous control problem is to design the control inputs for each robot so as to drive the ensemble to a common position from arbitrary initial conditions. An important requirement is that the feedback be local and distributed. In other words, the feedback for each vehicle should depend only on its relative displacements to its neighbours measured in the vehicle's own body frame. In our formulation, the vehicles do not even have access to their relative orientation. The solution of the rendezvous problem proposed in this paper has the property of being local and distributed, continuously differentiable, and time-independent. For simplicity of exposition, the proposed solution relies on the assumption that the sensing graph of the unicycles is time-invariant, undirected, and connected.

The difficulty in solving the rendezvous control problem comes from the fact that the unicycles are nonholonomic, in that their velocity is restricted to be parallel to the vehicle's heading direction. To overcome this difficulty, the solution we present relies on a control structure made of two nested loops. An outer loop treats the vehicles as fully-actuated single integrators with a consensus controller as the velocity input. The desired velocity input computed by the outer loop becomes a reference signal for the inner loop, which assigns local and distributed feedbacks that solve the rendezvous control problem. This methodology is inspired by our previous work in [1], [2] for rendezvous of rigid bodies in three dimensions.

Several results exist that solve the rendezvous problem for a group of unicycle vehicles on the plane, however, they have drawbacks compared to our solution. In [3], the authors presented the first smooth, local and distributed solution to the rendezvous problem. However, the solution requires the use of time-varying feedbacks. In [4] the authors present a solution using a local and distributed, continuously differentiable, and time-independent feedback like us. However, this approach cannot be extended to rendezvous with directed graphs containing a reverse directed spanning tree. For example, in [5] it is shown that the feedback in [4] drives the unicycles to a circular formation when the sensing graph is a directed ring and therefore rendezvous is not achieved. On the other hand, in a submitted paper [6], we show that the feedback presented here does in fact solve the rendezvous problem for any directed sensing graph containing a reverse directed spanning tree. In [7] both positions and attitudes of the unicycles are synchronized using a time invariant distributed control. The graph is time-dependent and the authors assume an initially connected communication graph. The controller that is implemented, however, is discontinuous. In [8] a time-independent, local and distributed controller is presented. However, the authors make the assumption that whenever two vehicles get sufficiently close together they merge into a single vehicle, introducing a discontinuity in the control function.

This research was supported by the National Sciences and Engineering Research Council of Canada.

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II. Preliminaries and notation

We use interchangeably the notation $v = [v_1 \ \cdots \ v_n]^\top$ or $(v_1, \dots, v_n)$ for a column vector in $\mathbb{R}^n$. We denote by $\mathbf{1} \in \mathbb{R}^m$ the vector $(1, \dots, 1)$. If v, w are vectors in $\mathbb{R}^2$, we denote by $v \cdot w := v^\top w$ their Euclidean inner product, and by $\|v\| := (v \cdot v)^{1/2}$ the Euclidean norm of v. If $c \in \mathbb{R}$, we define

$$c^\times := \begin{bmatrix} 0 & -c \\ c & 0 \end{bmatrix}.$$

Let $\{e_1, e_2\}$ denote the natural basis of $\mathbb{R}^2$, $\mathsf{SO}(2) := \{M \in \mathbb{R}^{2\times 2} : M^{-1} = M^\top, \det(M) = 1\}$ and let S^1 denote the unit circle. If Γ is a closed subset of a Riemannian manifold $\mathcal{X}$, and $d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ is a distance metric on $\mathcal{X}$, we denote by $\|\chi\|_\Gamma := \inf_{\psi \in \Gamma} d(\chi, \psi)$ the point-to-set distance of $\chi \in \mathcal{X}$ to Γ. If $\varepsilon > 0$, we let $B_\varepsilon(\Gamma) := \{\chi \in \mathcal{X} : \|\chi\|_\Gamma < \varepsilon\}$ and by $\mathcal{N}(\Gamma)$ we denote a neighborhood of Γ in $\mathcal{X}$. If $A, B \subset \mathcal{X}$ are two sets, denote by $A \backslash B$ the set-theoretic difference of A and B. If $I = \{i_1, \dots, i_n\}$ is an index set, the ordered list of elements $(x_{i_1}, \dots, x_{i_n})$ is denoted by $(x_j)_{j \in I}$.

Let U, W be finite-dimensional vector spaces. A function $f : U \to W$ is *homogeneous of degree* r if, for all $\lambda > 0$ and for all $x \in V$, $f(\lambda x) = \lambda^r f(x)$. A function $f : U \times V \to W$, $f(x, y)$ is *homogeneous of degree* r *with respect to* x if for all $\lambda > 0$ and for all $(x, y) \in U \times V$, $f(\lambda x, y) = \lambda^r f(x, y)$.

The following stability definitions are taken from [9]. Let $\Sigma : \dot{\chi} = f(\chi)$ be a smooth dynamical system with state space a Riemannian manifold $\mathcal{X}$. Let $\phi(t, \chi_0)$ denote its local phase flow. Let $\Gamma \subset \mathcal{X}$ be a closed set that is positively invariant for Σ, i.e., for all $\chi_0 \in \Gamma$, $\phi(t, \chi_0) \in \Gamma$ for all $t > 0$ for which $\phi(t, \chi_0)$ is defined.

Definition 1: The set Γ is *stable* for Σ if for any $\varepsilon > 0$, there exists a neighborhood $\mathcal{N}(\Gamma) \subset \mathcal{X}$ such that, for all $\chi_0 \in \mathcal{N}(\Gamma)$, $\phi(t, \chi_0) \in B_\varepsilon(\Gamma)$, for all $t > 0$ for which $\phi(t, \chi_0)$ is defined. The set Γ is *attractive* for Σ if there exists neighborhood $\mathcal{N}(\Gamma) \subset \mathcal{X}$ such that for all $\chi_0 \in \mathcal{N}(\Gamma)$, $\lim_{t\to\infty} \|\phi(t, \chi_0)\|_\Gamma = 0$. The *domain of attraction of* Γ is the set $\{\chi_0 \in \mathcal{X} : \lim_{t\to\infty} \|\phi(t, \chi_0)\|_\Gamma = 0\}$. The set Γ is *globally attractive* for Σ if it is attractive with domain of attraction $\mathcal{X}$. The set Γ is *locally asymptotically stable (LAS)* for Σ if it is stable and attractive. The set Γ is *globally asymptotically stable* for Σ if it is stable and globally attractive. $\triangle$

III. Modeling and Rendezvous Control Problem

Let $\mathcal{I}$ be the common inertial frame for all robots. We denote the body frame for robot i by $\mathcal{B}_i = \{b_{ix}, b_{iy}\}$. The unicycle dynamics are given by,

$$\dot{x}_i = u_i R_i e_1 \tag{1}$$

$$\dot{R}_i = R_i(\omega_i)^\times, \quad i = 1, \ldots, n. \tag{2}$$

The position of the i-th robot is denoted by x_i. We define the relative position between robot i and j as $x_{ij} := x_j - x_i$. The attitude is represented by a rotation matrix R_i whose columns are the coordinate representations of b_{ix} and b_{iy} in frame $\mathcal{I}$, so that $R_i \in \mathsf{SO}(2)$. In this paper we adopt the convention that if $r \in \mathbb{R}^2$ is an inertial vector, the coordinate representation of r in frame $\mathcal{B}_i$ is denoted by r^i, that is, $r^i := R_i^{-1} r$. The quantity $u_i R_i e_1$ is the velocity of robot i with magnitude u_i and direction b_{ix}. The angular speed is denoted ω_i. The control inputs are the robot's speed input u_i and angular speed ω_i.

We define the *sensor graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a set of nodes labelled as $\{1, \ldots, n\}$, each representing a robot, $\mathcal{E}$ is the set of edges. An edge from node i to node j indicates that robot i can sense robot j and vice versa ($\mathcal{G}$ has no self-loops). A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is *connected* if for any two nodes $i, j \in \mathcal{V}$ there exists a path from i to j.

We denote by $\mathcal{N}_i \subset \mathcal{V}$ the set of neighbors of node i, i.e., the vehicles that robot i can sense. In this paper we assume that $\mathcal{N}_i$ is constant for each $i \in \{1, \ldots, n\}$ (and hence $\mathcal{G}$ is constant as well). If $j \in \mathcal{N}_i$, then we say that robot j is a *neighbour* of robot i. If this is the case, then robot i can sense the relative displacement of robot j in its own body frame, i.e., the quantity x_{ij}^i. Define the vector $y_i := (x_{ij})_{j \in \mathcal{N}_i}$. The relative displacements available to robot i are contained in the vector $y_i^i := (x_{ij}^i)_{j \in \mathcal{N}_i}$. A *local and distributed feedback* (u_i, ω_i) for robot i is a locally Lipschitz function of y_i^i.

We are now ready to define the Rendezvous Control Problem.

Rendezvous Control Problem: Consider system (1), (2) with undirected and connected sensor graph $\mathcal{G}$, and define the *rendezvous manifold*

$$\Gamma := \left\{(x_i, R_i)_{i \in \{1,\ldots,n\}} \in \mathbb{R}^{2n} \times \mathsf{SO}(2)^n : x_{ij} = 0, \ \forall i, j\right\}. \tag{3}$$

Find, if possible, local and distributed feedbacks $(u_i, \omega_i)_{i \in \{1,\ldots,n\}}$ that globally asymptotically stabilize Γ. $\triangle$

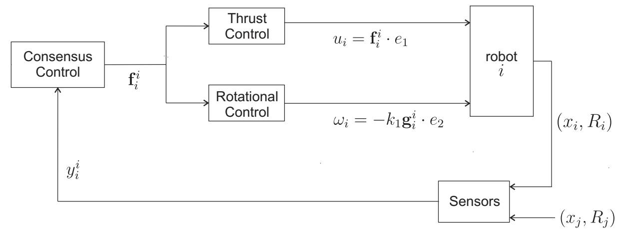

Fig. 1. Block diagram of the rendezvous control system for robot i.

IV. Solution of the Rendezvous Control Problem

In this section, we solve the rendezvous control problem for unicycles. Pick arbitrary real numbers $a_{ij} = a_{ji} > 0$, $i = 1, \ldots, n, j \in \mathcal{N}_i$, and define the function

$$\mathbf{f}_i(y_i) := \sum_{j \in \mathcal{N}_i} a_{ij} \|x_{ij}\| x_{ij}. \tag{4}$$

Let the unicycles' control inputs be defined as,

$$\begin{aligned} u_i &= \mathbf{f}_i(y_i^i) \cdot e_1, \\ \omega_i &= -k_1 \mathbf{g}_i(y_i^i) \cdot e_2, \ i = 1 \ldots n, \end{aligned} \tag{5}$$

where,

$$\mathbf{g}_i(y_i) := \frac{\sqrt{\|\mathbf{f}_i(y_i)\|}}{\|\mathbf{f}_i(y_i)\|} \mathbf{f}_i(y_i). \tag{6}$$

The feedback in (5) achieves rendezvous for the group of unicycles with an undirected sensing graph as presented in the next theorem.

Theorem 1: Consider system (1) and (2). Let u_i and ω_i be as in (5) with $\mathbf{f}_i(y_i)$ and $\mathbf{g}_i(y_i)$ as in (4) and (6), where $a_{ij} = a_{ji} > 0$, $i = 1, \ldots, n, j \in \mathcal{N}_i$. Assume that the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ representing the communication topology is undirected and connected. There exists $k_1^\star > 0$ such that for any $k_1 > k_1^\star$, feedback (5) solves the rendezvous control problem.

The proof of Theorem 1 is in Section V. The proposed control scheme is illustrated in the block diagram of Figure 1. There are two nested loops. The outer loop treats each robot as a single-integrator driven by the controller,

$$\dot{x}_i = \mathbf{f}_i(y_i), \ i = 1, \ldots, n. \tag{7}$$

By Theorem 3 in [10], the set $\left\{(x_i)_{i \in \{1,\ldots,n\}} \in \mathbb{R}^{2n} : x_{ij} = 0, \ i, j = 1, \ldots, n\right\}$ is globally asymptotically stable for (7). The consensus control $\mathbf{f}_i$ becomes a reference to the inner thrust and rotational controller blocks that assign the unicycle control inputs in (5). The intuition of these control inputs is

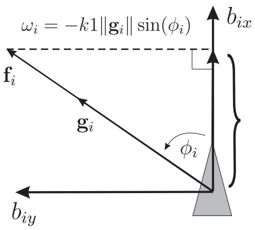

Fig. 2. Illustration of the control input u_i and reference angular speed ω_i in (5).

illustrated in Figure 2. The speed input u_i is the dot product $u_i = \mathbf{f}_i(y_i^i) \cdot e_1 = \mathbf{f}_i(y_i) \cdot b_{ix}$. That is, it is the projection of the reference $\mathbf{f}_i(y_i)$ onto the heading axis b_{ix} of robot i. The angular speed, on the other hand, is given by the dot product between the reference $\mathbf{g}_i(y_i)$ and the second body axis b_{iy}. These control inputs drive the robot velocity $u_i b_{ix}$ approximately to the reference $\mathbf{f}_i$. The convergence is approximate because the control inputs do not depend on the time derivative of $\mathbf{f}_i$. It is the difference in angle between $u_i b_{ix}$ and $\mathbf{f}_i$ as opposed to the difference in magnitude that is important for obtaining rendezvous. In Figure 2, one can see that $\omega_i = -k_1\|\mathbf{g}_i\|\sin(\phi_i)$ acts to reduce this angle with a rate proportional to the magnitude of $\mathbf{g}_i$. Since $\mathbf{g}_i(y_i)$ is homogeneous of degree one with respect to y_i, as the robots approach consensus, ω_i converges to zero slower than $\mathbf{f}_i(y_i)$ which is homogeneous of degree two. This allows ω_i to approximately close the gap between the vectors $u_i b_{ix}$ and $\mathbf{f}_i$ even as the robots converge to consensus (i.e., y_i approaches zero for all i).

V. Proof of Theorem 1

The feedback in (5) is local and distributed because it is a smooth function of y_i^i only. The proof relies on a coordinate transformation.

A. Coordinate Transformation

Define the average of the vehicle positions, $\beta = \sum_{j=1}^n x_j/n$, and the offset of robot i from β, $\delta_i = x_i - \beta$. Finally, define the relative offsets $\delta_{ij} := \delta_j - \delta_i$. One can consider $\delta_i,\ i = 1,\dots,n$ as new coordinates for the translational system. The control input u_i in new coordinates is given by $u_i = \sum_{j\in\mathcal{N}_i} a_{ij}\|\delta_{ij}\|\delta_{ij}\cdot e_1$ (analogous for ω_i). This yields the closed loop dynamics,

$$\dot{\delta}_i = u_i R_i e_1 - \frac{\sum_{j=1}^n u_j R_j e_1}{n} =: \hat{f}(\delta_i, R_i)_{i\in\{1,\dots,n\}}, \tag{8}$$

$$\dot{R}_i = R_i(\omega_i^i)^\times =: \hat{g}(\delta_i, R_i)_{i\in\{1,\dots,n\}} \tag{9}$$

for $i = 1,\dots,n$. The vehicles are at rendezvous if and only if $\delta_i = 0$ for all $i = 1,\dots,n$. Denote,

$$X := (\delta_i)_{i=1,\dots,n} \in \mathsf{X} := \mathbb{R}^{2n},$$
$$R := (R_i)_{i=1,\dots,n} \in \mathsf{R} := \mathsf{SO}(2)^n,$$

the new collective state is $(X, R) \in \mathsf{X}\times\mathsf{R}$. The rendezvous manifold in new coordinates is the set $\hat{\Gamma} := \{(X,R) \in \mathsf{X}\times\mathsf{R} : X = 0\}$. We may express the functions $\mathbf{f}_i(y_i)$, $\mathbf{g}_i(y_i)$ and $\mathbf{f}_i(y_i^i)$, $\mathbf{g}_i(y_i^i)$ in terms of (X,R). Accordingly, define $\hat{\mathbf{f}}_i : \mathsf{X}\to\mathbb{R}^2$, $\hat{\mathbf{g}}_i : \mathsf{X}\to\mathbb{R}^2$ and $\hat{\mathbf{f}}_i^i : \mathsf{X}\times\mathsf{SO}(2)\to\mathbb{R}^2$, $\hat{\mathbf{g}}_i^i : \mathsf{X}\times\mathsf{SO}(2)\to\mathbb{R}^2$ as follows:

$$\begin{aligned}
\hat{\mathbf{f}}_i(X) &:= \sum_{j\in\mathcal{N}_i} a_{ij}\|\delta_{ij}\|\delta_{ij} = \mathbf{f}_i(\delta_{ij})_{j\in\mathcal{N}_i},\\
\hat{\mathbf{g}}_i(X) &:= \frac{\sqrt{\|\hat{\mathbf{f}}_i(X)\|}}{\|\hat{\mathbf{f}}_i(X)\|}\hat{\mathbf{f}}_i(X) = \mathbf{g}_i(\delta_{ij})_{j\in\mathcal{N}_i},\\
\hat{\mathbf{f}}_i^i(X,R_i) &:= R_i^{-1}\left(\sum_{j\in\mathcal{N}_i} a_{ij}\|\delta_{ij}\|\delta_{ij}\right) = \mathbf{f}_i^i(\delta_{ij}^i)_{j\in\mathcal{N}_i},\\
\hat{\mathbf{g}}_i^i(X,R_i) &:= \frac{\sqrt{\|\hat{\mathbf{f}}_i^i(X,R_i)\|}}{\|\hat{\mathbf{f}}_i^i(X,R_i)\|}\hat{\mathbf{f}}_i^i(X,R_i) = \mathbf{g}_i^i(\delta_{ij}^i)_{j\in\mathcal{N}_i}.
\end{aligned} \tag{10}$$

We remark that $\hat{\mathbf{f}}_i$ and $\hat{\mathbf{f}}_i^i$ are homogeneous of degree two with respect to X, and $\hat{\mathbf{g}}_i$ and $\hat{\mathbf{g}}_i^i$ are homogeneous of degree one with respect to X.

In the new (X,R) coordinates, it needs to be shown that the set $\hat{\Gamma}$ is globally asymptotically stable.

B. Lyapunov function

Consider the function $W : \mathsf{X}\times\mathsf{R}\to\mathbb{R}$ defined as

$$W(X,R) = \alpha W_{\mathsf{tran}}(X) + W_{\mathsf{rot}}(X,R), \tag{11}$$

where $\alpha > 0$ is a design parameter and

$$\begin{aligned}
W_{\mathsf{tran}}(X) &= \sqrt{V(X)},\\
W_{\mathsf{rot}}(X,R) &= \sum_{i=1}^n \hat{\mathbf{g}}_i^i(X,R_i)\cdot e_1,
\end{aligned} \tag{12}$$

with $V(X) = \sum_{i=1}^n \delta_i^\top\delta_i$. We remark that $V(X)$ is the Lyapunov function employed in [10] for consensus of single integrators. $V(X)$ is positive definite in X coordinates. Define the function $\mu : \mathsf{X}\backslash 0 \to \mu(\mathsf{X}\backslash 0)$, $\mu(X) := X/\sqrt{V(X)}$. Since this function is homogeneous of degree zero with respect to X, the co-domain $\mu(\mathsf{X}\backslash 0)$ is bounded.

Lemma 1: Consider the continuous function $W(X,R)$ defined in (11). There exists $\alpha^\star > 0$ such that, for all $\alpha > 2\alpha^\star$, the following properties hold:

(i) $W \geq 0$ and $W^{-1}(0) = \{(X,R) : X = 0\}$.
(ii) For all $c > 0$, the sublevel set $W_c := \{(X,R) : W(X,R) \leq c\}$ is bounded.

The proof is in the appendix. From now on we assume $\alpha > 2\alpha^\star$.

C. Stability analysis

Next we compute the derivative of W. The next lemma will be useful to prove our main result.

Lemma 2: For system (8), (9) the time derivatives of $W_{\mathsf{tran}}(X)$ and $W_{\mathsf{rot}}(X,R)$ in (12) satisfy,

$$\dot{W}_{\mathsf{tran}} \leq V(X)\left[-M_2 + \sum_{i=1}^n M_1\left|\hat{\mathbf{f}}_i^i(\mu(X),R_i)\cdot e_2\right|\right]$$

$$\dot{W}_{\mathsf{rot}} \leq V(X)\left[-k_1\sum_{i=1}^n\left|\hat{\mathbf{g}}_i^i(\mu(X),R_i)\cdot e_2\right|^2 + M_3\right].$$

The proof of Lemma 2 is presented in the appendix. From Lemma 2, the derivative of W satisfies,

$$\begin{aligned}\dot{W} =&\alpha \dot{W}_{\mathsf{tran}} + \dot{W}_{\mathsf{rot}} \\ \leq& V(X)\Bigg[-\alpha M_2 + \alpha M_1 \sum_{i=1}^{n} \left|\hat{\mathbf{f}}_i^i(\mu(X), R_i)\cdot e_2\right| \\ &-k_1 \sum_{i=1}^{n} \left|\hat{\mathbf{g}}_i^i(\mu(X), R_i)\cdot e_2\right|^2 + M_3\Bigg].\end{aligned}$$

It will be shown next that there exists $k_1^\star$ such that choosing $k_1 > k_1^\star$ implies $\dot{W} \leq 0$ with equality if and only if $V(X) = 0$. Choosing $\alpha > 3M_3/M_2$, we obtain,

$$\begin{aligned}\dot{W} \leq& V(X)\Bigg[-2M_3 + \alpha M_1 \sum_{i=1}^{n} \left|\hat{\mathbf{f}}_i^i(\mu(X), R_i)\cdot e_2\right| \\ &-k_1 \sum_{i=1}^{n} \left|\hat{\mathbf{g}}_i^i(\mu(X), R_i)\cdot e_2\right|^2\Bigg].\end{aligned} \tag{13}$$

From (10), $\hat{\mathbf{g}}_i^i(\mu(X), R_i) = \frac{\sqrt{\|\hat{\mathbf{f}}_i^i(\mu(X),R_i)\|}}{\|\hat{\mathbf{f}}_i^i(\mu(X),R_i)\|}\hat{\mathbf{f}}_i^i(\mu(X), R_i)$. Plugging this into (13) leads to,

$$\begin{aligned}\dot{W} \leq& V(X)\Bigg[-2M_3 + \alpha M_1 \sum_{i=1}^{n} \left|\hat{\mathbf{f}}_i^i(\mu(X), R_i)\cdot e_2\right| \\ &-k_1 \sum_{i=1}^{n} \left(\frac{\sqrt{\|\hat{\mathbf{f}}_i^i(\mu(X), R_i)\|}}{\|\hat{\mathbf{f}}_i^i(\mu(X), R_i)\|}\hat{\mathbf{f}}_i^i(\mu(X), R_i)\cdot e_2\right)^2\Bigg] \\ \leq& V(X)\Bigg[-2M_3 + \alpha M_1 \sum_{i=1}^{n} \left|\hat{\mathbf{f}}_i^i(\mu(X), R_i)\cdot e_2\right| \\ &-k_1 \sum_{i=1}^{n} \frac{1}{\|\hat{\mathbf{f}}_i^i(\mu(X), R_i)\|}\left|\hat{\mathbf{f}}_i^i(\mu(X), R_i)\cdot e_2\right|^2\Bigg].\end{aligned}$$

Since $\hat{\mathbf{f}}_i^i(\mu(X), R_i)$ is a continuous function of its arguments and $\mu(X)$ lies on a bounded set, $\left|\hat{\mathbf{f}}_i^i(\mu(X), R_i)\right|$ obtains a finite supremum M_4. This implies that,

$$\begin{aligned}\dot{W} \leq& V(X)\Bigg[-2M_2 + \alpha M_1 \sum_{i=1}^{n} \left|\hat{\mathbf{f}}_i^i(\mu(X), R_i)\cdot e_2\right| \\ &-k_1 \sum_{i=1}^{n} \frac{1}{M_4}\left|\hat{\mathbf{f}}_i^i(\mu(X), R_i)\cdot e_2\right|^2\Bigg].\end{aligned}$$

Denote $\boldsymbol{\beta}_i(\mu(X), R_i) := \left|\hat{\mathbf{f}}_i^i(\mu(X), R_i)\cdot e_2\right|$, and $\boldsymbol{\beta} := (\boldsymbol{\beta}_i(\mu(X), R_i))_{i\in\{1,\ldots,n\}}$. Then,

$$\begin{aligned}\dot{W} &\leq V(X)\left[-2M_2 + \alpha M_1 \mathbf{1}^\top \boldsymbol{\beta} - \frac{k_1}{M_4}|\boldsymbol{\beta}|^2\right] \\ &\leq V(X)\begin{bmatrix}\mathbf{1}^\top & \boldsymbol{\beta}^\top\end{bmatrix}\begin{bmatrix}\frac{-2M_2}{n}I & \alpha\frac{M_1}{2}I \\ \alpha\frac{M_1}{2}I & \frac{-k_1}{M_4}I\end{bmatrix}\begin{bmatrix}\mathbf{1} \\ \boldsymbol{\beta}\end{bmatrix}.\end{aligned}$$

There exists $k_1^\star > 0$ such that choosing $k_1 > k_1^\star$, the matrix above is negative definite and therefore satisfies,

$$\dot{W} \leq -\sigma V(X),\ \sigma > 0,\ k_1 > k_1^\star, \tag{14}$$

and as such $\dot{W} \leq 0$, with equality if and only if $V(X) = 0$, or equivalently, $X = 0$. By Lemma 1, all level sets of $W(X, R)$ are compact and $W^{-1}(0) = \{(X, R) : X = 0\}$. This implies $\hat{\Gamma}$ is globally asymptotically stable. $\square$

VI. Conclusion

We have presented the first solution to the rendezvous problem for a group of unicycle vehicles on the plane using continuous, static feedbacks that are local and distributed. The solution assumes a constant, undirected communication topology and relies on a control structure made of two nested loops. It can be shown that the proposed feedback solves the rendezvous problem for the more general class of directed sensing graphs containing a spanning tree. The proof of this fact is more involved than the one presented in this paper.

Appendix

A. Proof of Lemma 1

Recall the definition of $W(X, R)$,

$$\begin{aligned}W &= \alpha\sqrt{V(X)} + \sum_{i=1}^{n} \hat{\mathbf{g}}_i^i(X, R_i)\cdot e_1 \\ &= \sqrt{V(X)}\left(\alpha + \frac{\sum_{i=1}^{n}\hat{\mathbf{g}}_i^i(X, R_i)\cdot e_1}{\sqrt{V(X)}}\right).\end{aligned}$$

Using the fact that $\hat{\mathbf{g}}_i^i(X, R_i)$ is homogeneous with respect to its first argument, we have

$$W = \sqrt{V(X)}\left(\alpha + \sum_{i=1}^{n} \hat{\mathbf{g}}_i^i\left(\mu(X), R_i\right)\cdot e_1\right).$$

Since $\hat{\mathbf{g}}_i^i$ is continuous, $\mu(X)$ is bounded, and $R \in \mathsf{R}$, a compact set, it follows that the function $\sum_{i=1}^{n}\left|\hat{\mathbf{g}}_i^i\left(\mu(X), R_i\right)\cdot e_3\right|$ has a bounded supremum. Accordingly, let

$$\alpha^\star = \sup_{(X,R)\in\mathsf{X}\times\mathsf{R}} \sum_{i=1}^{n}\left|\hat{\mathbf{g}}_i^i\left(\mu(X), R_i\right)\cdot e_1\right|.$$

For all $\alpha > 2\alpha^\star$, we have

$$W(X, R) \geq \underline{W}(X, R) := \alpha^\star\sqrt{V(X)} \geq 0.$$

The above inequality implies that $W \geq 0$ and $W^{-1}(0) \subset \underline{W}^{-1}(0)$. But $\underline{W} = 0$ if and only if $V(X) = 0$ (i.e., $X = 0$). Thus $W^{-1}(0) \subset \{(X, R) : X = 0\}$. Conversely, on the set $\{(X, R) : X = 0\}$ it holds that $X = 0$ and hence $W = 0$, and therefore $\{(X, R) : X = 0\} \subset W^{-1}(0)$. It follows that $W^{-1}(0) = \{(X, R) : X = 0\}$ proving part (i) of the lemma.

For part (ii), note that for all $c > 0$, $W_c \subset \{(X, R) : \underline{W}(X, R) \leq c\}$. Since the sublevel sets of $\underline{W}$ are compact in X coordinates and $R \in \mathsf{R}$, a compact set, the set W_c is bounded. Continuity of W implies that W_c is compact. $\square$

B. Proof of Lemma 2

We compute the inequalities $\dot{W}_{\mathsf{tran}}$ and $\dot{W}_{\mathsf{rot}}$ in Lemma 2 for system (8) and (9). Using the fact that $\sum_{j=1}^{n}\hat{\mathbf{f}}_j(X) = 0$

(since the graph is undirected and since, by design, $a_{ij} = a_{ji}$), the dynamics of δ_i in (8) are given by,

$$
\begin{aligned}
\dot{\delta}_i &= u_i R_i e_1 - \frac{\sum_{j=1}^n u_j R_j e_1}{n} \\
&= u_i R_i e_1 - \frac{\sum_{j=1}^n u_j R_j e_1}{n} + \frac{\sum_{j=1}^n \hat{\mathbf{f}}_j(X)}{n} \\
&= u_i R_i e_1 - \frac{\sum_{j=1}^n (u_j R_j e_1 - \hat{\mathbf{f}}_j(X))}{n}.
\end{aligned}
$$

For simplicity of notation, we drop the arguments of $\hat{\mathbf{f}}_i(X)$ and $\hat{\mathbf{f}}_i^i(X, R_i)$. Adding and subtracting $\hat{\mathbf{f}}_i$ to the previous expression yields,

$$
\begin{aligned}
\dot{\delta}_i &= \hat{\mathbf{f}}_i + (u_i R_i e_1 - \hat{\mathbf{f}}_i) - \frac{\sum_{j=1}^n (u_j R_j e_1 - \hat{\mathbf{f}}_j)}{n} \\
&= \hat{\mathbf{f}}_i + \frac{\sum_{j=1}^n (u_i R_i e_1 - \hat{\mathbf{f}}_i)}{n} - \frac{\sum_{j=1}^n (u_j R_j e_1 - \hat{\mathbf{f}}_j)}{n}.
\end{aligned}
$$

Replacing u_j and u_i by the assigned feedbacks in (5) and using the identity $R_i \hat{\mathbf{f}}_i^i = \hat{\mathbf{f}}_i$,

$$
\begin{aligned}
\dot{\delta}_i =& \hat{\mathbf{f}}_i + \frac{\sum_{j=1}^n R_i((\hat{\mathbf{f}}_i^i \cdot e_1) e_1 - \hat{\mathbf{f}}_i^i)}{n} \\
&- \frac{\sum_{j=1}^n R_j((\hat{\mathbf{f}}_j^j \cdot e_1) e_1 - \hat{\mathbf{f}}_j^j)}{n}.
\end{aligned}
$$

Here we denote,

$$
\begin{aligned}
a_i(X) :=& \hat{\mathbf{f}}_i \\
b_i(X, R) :=& \frac{\sum_{j=1}^n R_i((\hat{\mathbf{f}}_i^i \cdot e_1) e_1 - \hat{\mathbf{f}}_i^i)}{n} \\
&- \frac{\sum_{j=1}^n R_j((\hat{\mathbf{f}}_j^j \cdot e_1) e_1 - \hat{\mathbf{f}}_j^j)}{n}.
\end{aligned} \tag{15}
$$

Taking the time derivative of $W_{\mathsf{tran}} = \sqrt{V(X)}$ along the above vector field, we obtain,

$$
\begin{aligned}
&\dot{W}_{\mathsf{tran}} = \\
&\frac{1}{2\sqrt{V(X)}} \left[\sum_{i=1}^n \frac{\partial V(X)}{\partial \delta_i} (a_i(X) + b_i(X, R)) \right].
\end{aligned} \tag{16}
$$

The derivative of the first term, $\frac{\partial V(X)}{\partial \delta_i} a_i(X) = \frac{\partial V(X)}{\partial \delta_i} \hat{\mathbf{f}}_i$, is just the derivative of the W_{tran} along the nominal dynamics $\hat{\mathbf{f}}_i$. By the proof of Theorem 3 in [10], this is given by,

$$
\frac{\partial V(X)}{\partial \delta_i} \hat{\mathbf{f}}_i = - \sum_{(i,j) \in \mathcal{E}} a_{ij} \|\delta_{ij}\|^3 =: \mathbf{r}(X)
$$

which is less than or equal to zero, with equality if and only if $\delta_{ij} = 0$ for all $i, j \in \{1, \ldots, n\}$. Since $\sum_{i=1}^n \delta_i = 0$, this is equivalent to $\delta_i = 0$ for all $i \in \{1, \ldots, n\}$ and hence $X = 0$. Therefore the term $\mathbf{r}(X)$ is negative definite and homogeneous of degree three with respect to X. The derivative of the remaining term in the square brackets of (16) satisfies,

$$
\begin{aligned}
&\sum_{i=1}^n \frac{\partial V(X)}{\partial \delta_i} (b_i(X, R)) \\
&\quad \le \sum_{i=1}^n \frac{\partial V(X)}{\partial \delta_i} \left[\frac{\sum_{j=1}^n R_i((\hat{\mathbf{f}}_i^i \cdot e_1) e_1 - \hat{\mathbf{f}}_i^i)}{n} \right. \\
&\qquad \left. - \frac{\sum_{j=1}^n R_j((\hat{\mathbf{f}}_j^j \cdot e_1) e_1 - \hat{\mathbf{f}}_j^j)}{n} \right] \\
&\qquad \le \sum_{i=1}^n \frac{1}{n} \frac{\partial V(X)}{\partial \delta_i} \left[\sum_{j=1}^n \left\| (\hat{\mathbf{f}}_j^j \cdot e_1) e_1 - \hat{\mathbf{f}}_j^j \right\| \right. \\
&\qquad \left. + \sum_{j=1}^n \left\| (\hat{\mathbf{f}}_i^i \cdot e_1) e_1 - \hat{\mathbf{f}}_i^i \right\| \right].
\end{aligned}
$$

We claim that $\|(\hat{\mathbf{f}}_i^i(X, R_i) \cdot e_1) e_1 - \hat{\mathbf{f}}_i^i(X, R_i)\| = \left| \hat{\mathbf{f}}_i^i(X, R_i) \cdot e_2 \right|$. Indeed, writing $\hat{\mathbf{f}}_i^i = (\hat{\mathbf{f}}_i^i \cdot e_1) e_1 + \hat{\mathbf{f}}_i^i - (\hat{\mathbf{f}}_i^i \cdot e_1) e_1$, we have $\hat{\mathbf{f}}_i^i \cdot e_2 = (\hat{\mathbf{f}}_i^i - (\hat{\mathbf{f}}_i^i \cdot e_1) e_1) \cdot e_2$. Since the vector $\hat{\mathbf{f}}_i^i - (\hat{\mathbf{f}}_i^i \cdot e_1) e_1$ is parallel to e_2, $\left| (\hat{\mathbf{f}}_i^i - (\hat{\mathbf{f}}_i^i \cdot e_1) e_1) \cdot e_2 \right| = \|\hat{\mathbf{f}}_i^i - (\hat{\mathbf{f}}_i^i \cdot e_1) e_1\|$, so that $\left| \hat{\mathbf{f}}_i^i \cdot e_2 \right| = \|\hat{\mathbf{f}}_i^i - (\hat{\mathbf{f}}_i^i \cdot e_1) e_1\|$. This yields,

$$
\begin{aligned}
&\sum_{i=1}^n \frac{\partial V(X)}{\partial \delta_i} (b_i(X, R)) \\
&\quad \le \sum_{i=1}^n \frac{1}{n} \left\| \frac{\partial V(X)}{\partial \delta_i} \right\| \left(\sum_{j=1}^n \left| \hat{\mathbf{f}}_j^j \cdot e_2 \right| + \sum_{j=1}^n \left| \hat{\mathbf{f}}_i^i \cdot e_2 \right| \right) \\
&\quad \le \sum_{i=1}^n \frac{1}{n} \left\| \frac{\partial V(X)}{\partial \delta_i} \right\| \left(\sum_{j=1}^n \left| \hat{\mathbf{f}}_j^j \cdot e_2 \right| + n \left| \hat{\mathbf{f}}_i^i \cdot e_2 \right| \right)
\end{aligned}
$$

which is homogeneous of degree three with respect to X since $\frac{\partial V(X)}{\partial \delta_i}$ is homogeneous of degree one and $\hat{\mathbf{f}}_i^i$ is homogeneous of degree two with respect to X for all i. Putting everything together, $\dot{W}_{\mathsf{tran}}$ is bounded by,

$$
\begin{aligned}
\dot{W}_{\mathsf{tran}} \le& \frac{1}{2\sqrt{V(X)}} \left[\mathbf{r}(X) + \sum_{i=1}^n \frac{1}{n} \left\| \frac{\partial V(X)}{\partial \delta_i} \right\| \right. \\
&\left. \left(\sum_{j=1}^n \left| \hat{\mathbf{f}}_j^j \cdot e_2 \right| + n \left| \hat{\mathbf{f}}_i^i \cdot e_2 \right| \right) \right].
\end{aligned} \tag{17}
$$

Since $\mathbf{r}(X)$ is homogeneous of degree three, one can write,

$$
\begin{aligned}
\mathbf{r}(X) &= \frac{\sqrt{V(X)} V(X)}{\sqrt{V(X)} V(X)} \mathbf{r}(X) \\
&= \sqrt{V(X)} V(X) \mathbf{r} \left(\frac{X}{\sqrt{V(X)}} \right) \\
&= \sqrt{V(X)} V(X) \mathbf{r} \left(\mu(X) \right).
\end{aligned}
$$

Analogous operations can be performed with the remaining term in the square bracket of (17) yielding,

$$\dot{W}_{\mathsf{tran}} \leq \frac{V(X)}{2}\Big[\mathbf{r}(\mu(X)) + \sum_{i=1}^{n}\frac{1}{n}\left\|\frac{\partial V(\mu(X))}{\partial \delta_i}\right\|\left(\sum_{j=1}^{n}\left|\hat{\mathbf{f}}_j^j(\mu(X),R_j)\cdot e_2\right| + n\left|\hat{\mathbf{f}}_i^i(\mu(X),R_i)\cdot e_2\right|\right)\Big].$$

Since $\mathbf{r}$ is continuous and $\mu(X)$ lies on a bounded set S_1, it follows that $\mathbf{r}(\mu(X))/2$ is bounded with supremum $-M_2 < 0$. Similarly, the function $\left\|\frac{\partial V(\mu(X))}{\partial \delta_i}\right\|$ has a bounded supremum. Letting $M_1 := \sup_{\substack{\theta\in S_1\\ i\in\{1,\ldots,n\}}}\left\|\frac{\partial V(\theta)}{\partial \delta_i}\right\|$, we obtain,

$$\begin{aligned}\dot{W}_{\mathsf{tran}} \leq & V(X)\Big[-M_2 + \frac{M_1}{2n}\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}\left|\hat{\mathbf{f}}_j^j(\mu(X),R_j)\cdot e_2\right| + n\left|\hat{\mathbf{f}}_i^i(\mu(X),R_i)\cdot e_2\right|\Big)\Big]\\ \leq & V(X)\Big[-M_2 + \frac{M_1}{2n}\sum_{i=1}^{n}\Big(n\left|\hat{\mathbf{f}}_i^i(\mu(X),R_i)\cdot e_2\right| + n\left|\hat{\mathbf{f}}_i^i(\mu(X),R_i)\cdot e_2\right|\Big)\Big]\\ \leq & V(X)\Big[-M_2 + M_1\sum_{i=1}^{n}\left|\hat{\mathbf{f}}_i^i(\mu(X),R_i)\cdot e_2\right|\Big]\end{aligned}$$

This proves the first inequality in Lemma 2. We now turn to the second. Recall the definition of W_{rot},

$$W_{\mathsf{rot}}(X,R) = \sum_{i=1}^{n}\hat{\mathbf{g}}_i^i(X,R_i)\cdot e_1.$$

The time derivative of W_{rot} along the vector field in (8)-(9) is

$$\dot{W}_{\mathsf{rot}} = \sum_{i=1}^{n}\left(\frac{d}{dt}\hat{\mathbf{g}}_i^i\right)\cdot e_1.$$

To express $(d/dt)\hat{\mathbf{g}}_i^i$, recall that $\hat{\mathbf{g}}_i^i(X,R_i) = R_i^{-1}\hat{\mathbf{g}}_i(X)$. Then,

$$\frac{d}{dt}\hat{\mathbf{g}}_i^i = \left(\frac{d}{dt}R_i^{-1}\right)\hat{\mathbf{g}}_i + R_i^{-1}\frac{d\hat{\mathbf{g}}_i}{dt}.$$

We will denote the derivative of $\hat{\mathbf{g}}_i(X) = \frac{\sqrt{\|\hat{\mathbf{f}}_i(X)\|}}{\|\hat{\mathbf{f}}_i(X)\|}\hat{\mathbf{f}}_i(X)$ by,

$$\begin{aligned}\mathbf{h}_i(X,R) &:= (d/dt)\hat{\mathbf{g}}_i(X)\\ &= \sum_{j=1}^{n}\frac{\partial\hat{\mathbf{g}}_i(X)}{\partial\delta_j}\left[a_j(X) + b_j(X,R)\right]\end{aligned}$$

which is homogeneous of degree two with respect to X because $\frac{\partial\hat{\mathbf{g}}_i}{\partial\delta_j}$ is homogeneous of degree zero and both $a_j(X)$ and $b_j(X,R)$ are homogeneous of degree two with respect to X. Consistently with our notational convention, we will let $\mathbf{h}_i^i(X,R) := R_i^{-1}\mathbf{h}_i(X,R)$. Returning to the derivative of $\hat{\mathbf{g}}_i^i$, we have

$$\begin{aligned}\frac{d}{dt}\hat{\mathbf{g}}_i^i &= -(\omega_i^i)^{\times}R_i^{-1}\hat{\mathbf{g}}_i(X) + R_i^{-1}\mathbf{h}_i(X,R)\\ &= -\begin{bmatrix}0 & -\omega_i^i\\ \omega_i^i & 0\end{bmatrix}\hat{\mathbf{g}}_i^i(X,R_i) + \mathbf{h}_i^i(X,R).\end{aligned}$$

Substituting the above identity in the expression for $\dot{W}_{\mathsf{rot}}$, we get

$$\begin{aligned}\dot{W}_{\mathsf{rot}} &= \sum_{i=1}^{n}\left(-e_1^\top\begin{bmatrix}0 & -\omega_i^i\\ \omega_i^i & 0\end{bmatrix}\hat{\mathbf{g}}_i^i(X,R_i) + \mathbf{h}_i^i(X,R)\cdot e_1\right)\\ &= \sum_{i=1}^{n}\left((\hat{\mathbf{g}}_i^i(X,R_i)\cdot e_2)\omega_i^i + \mathbf{h}_i^i(X,R)\cdot e_1\right).\end{aligned}$$

Substituting the controller $\omega_i^i = -k_1(\hat{\mathbf{g}}_i^i\cdot e_2)$ and taking norms, we arrive at the inequality

$$\dot{W}_{\mathsf{rot}} \leq \sum_{i=1}^{n}\left[-k_1\left|\hat{\mathbf{g}}_i^i(X,R_i)\cdot e_2\right|^2 + \mathbf{h}_i^i(X,R)\cdot e_1\right].$$

Note that $\sum_{i=1}^{n}\left|\hat{\mathbf{g}}_i^i(X,R_i)\cdot e_2\right|^2$ and $\mathbf{h}_i^i(X,R)$ are homogeneous of degree two with respect to X. This yields,

$$\begin{aligned}\dot{W}_{\mathsf{rot}} \leq & V(X)\Big[-k_1\sum_{i=1}^{n}\left|\hat{\mathbf{g}}_i^i(X/\sqrt{V(X)},R_i)\cdot e_2\right|^2 + \mathbf{h}_i^i(X/\sqrt{V(X)},R)\cdot e_2\Big]\\ \leq & V(X)\Big[-k_1\sum_{i=1}^{n}\left|\hat{\mathbf{g}}_i^i(\mu(X),R_i)\cdot e_2\right|^2 + \mathbf{h}_i^i(\mu(X),R)\cdot e_2\Big].\end{aligned}$$

The function $\left|\mathbf{h}_i^i(\mu(X),R)\cdot e_2\right|$ has a bounded supremum. Letting $M_3 = \sup_{(\theta,R)\in S_1\times \mathsf{R}}\left(\left|\mathbf{h}_i^i(\mu(X),R)\cdot e_2\right|\right)$, we conclude that

$$\dot{W}_{\mathsf{rot}} \leq V(X)\left[-k_1\sum_{i=1}^{n}\left|\hat{\mathbf{g}}_i^i(\mu(X),R_i)\cdot e_2\right|^2 + M_3\right]$$

as required. This concludes the proof of Lemma 2. □

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