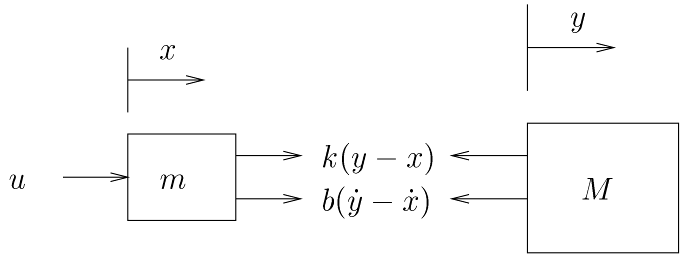
ECE311 Control & Systems

Problem Set No. 1

Problem 2.7

The forces acting on the masses are shown in the following figure:



Using Newton's law, the dynamics of the system can be described by

$$\begin{aligned} m\ddot{x} &= u + k(y-x) + b(\dot{y}-\dot{x}) \\ M\ddot{y} &= -k(y-x) - b(\dot{y}-\dot{x}), \end{aligned}$$

or

$$\ddot{x} = \frac{k}{m}(y-x) + \frac{b}{m}(\dot{y}-\dot{x}) + \frac{1}{m}u, \tag{1}$$

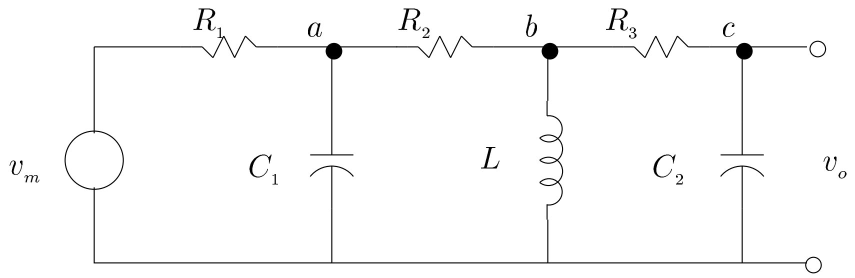
$$\ddot{y} = -\frac{k}{M}(y-x) - \frac{b}{M}(\dot{y}-\dot{x}). \tag{2}$$

Now, choose the state vector $\boldsymbol{x} = [\dot{x}\ x\ \dot{y}\ y]^T$ and write equations (1) and (2) in the standard form

$$\begin{aligned} \dot{\boldsymbol{x}} &= \begin{bmatrix} -\frac{b}{m} & -\frac{k}{m} & \frac{b}{m} & \frac{k}{m} \\ 1 & 0 & 0 & 0 \\ \frac{b}{M} & \frac{k}{M} & -\frac{b}{M} & -\frac{k}{M} \\ 0 & 0 & 1 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} \frac{1}{m} \\ 0 \\ 0 \\ 0 \end{bmatrix} u, \\ y &= [0\ 0\ 0\ 1]\boldsymbol{x}. \end{aligned}$$

$\square$

Problem 2.14



Using nodal-analysis, one can write KCL equations for nodes a, b, and c with the corresponding voltages v_1, v_2, and v_3:

$$\begin{aligned}
\frac{v_1 - v_m}{R_1} + C_1\dot{v}_1 + \frac{v_1 - v_2}{R_2} &= 0 \quad \text{for node } a \\
\frac{v_2 - v_1}{R_2} + \frac{v_2 - v_3}{R_3} + \frac{1}{L}\int v_2 dt &= 0 \quad \text{for node } b \\
\frac{v_3 - v_2}{R_3} + C_2\dot{v}_3 &= 0, \quad \text{for node } c
\end{aligned}$$

with the output $v_o = v_3$.
Now, rearrange the equations and differentiate the equation for node b to have

$$\begin{aligned}
\dot{v}_1 &= \frac{1}{C_1}[-(\frac{1}{R_1} + \frac{1}{R_2})v_1 + \frac{1}{R_2}v_2 + \frac{1}{R_1}v_m] && (3) \\
(\frac{1}{R_2} + \frac{1}{R_3})\dot{v}_2 - \frac{1}{R_2}\dot{v}_1 - \frac{1}{R_3}\dot{v}_3 &= -\frac{1}{L}v_2 && (4) \\
\dot{v}_3 &= \frac{1}{C_2}[\frac{1}{R_3}v_2 - \frac{1}{R_3}v_3]. && (5)
\end{aligned}$$

One can substitute for $\dot{v}_1$ and $\dot{v}_3$ from equations (3), (5) into (4), to obtain the differential equation for $\dot{v}_2$. Or, one may write equations (3)-(5) in the form of

$$\boldsymbol{E}\dot{\boldsymbol{x}} = \boldsymbol{F}'\boldsymbol{x} + \boldsymbol{G}'u,$$

where

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{R_2} & \frac{1}{R_2} + \frac{1}{R_3} & -\frac{1}{R_3} \\ 0 & 0 & 1 \end{bmatrix}}_{\triangleq \boldsymbol{E}} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{1}{C_1}(\frac{1}{R_1} + \frac{1}{R_2}) & \frac{1}{C_1 R_2} & 0 \\ 0 & -\frac{1}{L} & 0 \\ 0 & \frac{1}{C_2 R_3} & -\frac{1}{C_2 R_3} \end{bmatrix}}_{\triangleq \boldsymbol{F}'} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{R_1} \\ 0 \\ 0 \end{bmatrix}}_{\triangleq \boldsymbol{G}'} v_m$$

with

$$v_o = [0\ 0\ 1]\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

The standard state-space equations in the form

$$\dot{\boldsymbol{x}} = \boldsymbol{F}\boldsymbol{x} + \boldsymbol{G}u,$$

can be obtained if one defines

$$\begin{aligned} \boldsymbol{F} &= \boldsymbol{E}^{-1}\boldsymbol{F}' \\ \boldsymbol{G} &= \boldsymbol{E}^{-1}\boldsymbol{G}'. \end{aligned}$$

(Note that $\boldsymbol{E}$ is nonsingular and $\boldsymbol{E}^{-1}$ exists). □

Remark: **Typically, the inductor current would be used as a state variable and not v_2. Since v_2 is a linear combination of the inductor current and other states, it is also a valid selection as a state variable. In fact,**

$$i_L = \frac{v_1 - v_2}{R_2} - \frac{v_2 - v_3}{R_3}.$$

Problem 2.24

a) Assume the tank area and the fluid density are s and ρ, respectively. The differential equations describing the water-tank system are (see Section 2.5.2)

$$\begin{aligned} \dot{h}_1 &= \frac{\omega_{in} - \omega_A}{\rho s} & \dot{h}_2 &= \frac{\omega_A - \omega_C}{\rho s} \\ \omega_A &= \frac{1}{R}(p_1 - p_a)^{1/2} & p_1 &= \rho g(h_1 - h_3) + p_a \\ \omega_C &= \frac{1}{R}(p_2 - p_a)^{1/2} & p_2 &= \rho g h_2 + p_a, \end{aligned}$$

where p_a is the atmospheric pressure (note $\alpha = 1/2$). Combining above equations yields

$$\begin{aligned} \dot{h}_1 &= \frac{\omega_{in} - \omega_A}{\rho s} = \frac{\omega_{in} - \frac{1}{R}\sqrt{\rho g(h_1 - h_3)}}{\rho s}, \\ \dot{h}_2 &= \frac{\omega_A - \omega_C}{\rho s} = \frac{\sqrt{\rho g}}{\rho R s}(\sqrt{h_1 - h_3} - \sqrt{h_2}). \end{aligned}$$

For $h_2 = 10$cm, $\omega_C = 200$ gr/min=10/3 gr/sec, $\rho_{water} = 1$gr/cm^3, $g \simeq 10^3$cm/sec^2, one can compute R as

$$R = \frac{\sqrt{\rho g h_2}}{\omega_C} = 30 \frac{1}{\sqrt{\text{gr} \times \text{cm}}}$$

Therefore,

$$\begin{aligned} \dot{h}_1 &= \frac{1}{s}(\omega_{in} - 1.05\sqrt{h_1 - h_3}), &(6) \\ \dot{h}_2 &= \frac{1.05}{s}(\sqrt{h_1 - h_3} - \sqrt{h_2}). &(7) \end{aligned}$$

b) Assuming $\boldsymbol{h} = [h_1\ h_2]^T$, equations (6) and (7) can be written as

$$\dot{\boldsymbol{h}} = \underbrace{\begin{bmatrix} -\frac{1.05}{s}\sqrt{h_1 - h_3} \\ \\ \frac{1.05}{s}\sqrt{h_1 - h_3} - \frac{1.05}{s}\sqrt{h_2} \end{bmatrix}}_{\triangleq \boldsymbol{F}(\boldsymbol{h})} + \begin{bmatrix} \frac{1}{s} \\ \\ 0 \end{bmatrix} \omega_{in} \tag{8}$$

First determine the equilibrium point(s) of (8). Assume

$$\begin{aligned} \omega_{in} &= \bar{\omega}_{in} & &\text{(constant) corresponds to an equilibrium point} \\ h_1 &= \bar{h}_1 & &\text{(constant)} \\ h_2 &= \bar{h}_2 & &\text{(constant)} \\ h_3 &= \bar{h}_3 & &\text{(constant)} \end{aligned}$$

From (8), we have

$$\begin{aligned} 0 &= \bar{\omega}_{in} - 1.05\sqrt{\bar{h}_1 - \bar{h}_3} &(9) \\ 0 &= 1.05(\sqrt{\bar{h}_1 - \bar{h}_3} - \sqrt{\bar{h}_2}) &(10) \end{aligned}$$

Since $\bar{h}_1 = 30$cm and $\bar{h}_2 = 10$cm, in solving for $\bar{h}_3$ and $\bar{\omega}_{in}$ from (9)-(10) we obtain the following equilibrium point

$$\boxed{\bar{h}_1 = 30\text{cm}, \quad \bar{h}_2 = 10\text{cm}, \quad \bar{h}_3 = 20\text{cm}, \text{ and } \bar{\omega}_{in} = 1.05\sqrt{10} = 3.32\text{cm}}$$

Now linearize (8) about the equilibrium point to obtain

$$
\begin{aligned}
\delta\dot{h}_1 &= -\frac{1.05}{2s}\frac{1}{\sqrt{\bar{h}_1-\bar{h}_3}}\,\delta h_1 + 0\ \delta h_2 + \frac{1}{s}\ \delta\omega_{in}\\
\delta\dot{h}_2 &= \frac{1.05}{2s}\frac{1}{\sqrt{\bar{h}_1-\bar{h}_3}}\,\delta h_1 - \frac{1.05}{2s}\frac{1}{\sqrt{\bar{h}_2}}\,\delta h_2
\end{aligned}
$$

or

$$
\delta\dot{\boldsymbol{h}} = \frac{1.05}{2\sqrt{10}s}\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}\delta\boldsymbol{h} + \begin{bmatrix} \frac{1}{s} \\ 0 \end{bmatrix}\ \delta\omega_{in}.
$$

c) When hole A is closed and holes B and C are open, we get

$$
\omega_B = \frac{1}{R}(p_1 - p_2)^{1/2} = \frac{1}{R}\sqrt{\rho g(h_1 - h_2)}
$$

while ω_C is the same as part **(a)**. In this case, the system equations are

$$
\begin{aligned}
\dot{h}_1 &= \frac{\omega_{in}-\omega_B}{\rho s} = \frac{\omega_{in}-\frac{1}{R}\sqrt{\rho g(h_1-h_2)}}{\rho s}, & (11)\\
\dot{h}_2 &= \frac{\omega_B-\omega_C}{\rho s} = \frac{\sqrt{\rho g}}{\rho R s}(\sqrt{h_1-h_2}-\sqrt{h_2}). & (12)
\end{aligned}
$$

Determine the equilibrium point of (11)-(12). As in part (**a**), $R = 30\frac{1}{\sqrt{\text{gr} \times \text{cm}}}$ and $\frac{\sqrt{\rho g}}{R} = 1.05$. Therefore, from (11)-(12)

$$
\begin{aligned}
0 &= \bar{\omega}_{in} - 1.05\sqrt{\bar{h}_1-\bar{h}_2}\\
0 &= \frac{\sqrt{\rho g}}{\rho R s}(\sqrt{\bar{h}_1-\bar{h}_2}-\sqrt{\bar{h}_2}),
\end{aligned}
$$

we get

$$
\begin{aligned}
\bar{\omega}_{in} &= 1.05\sqrt{\bar{h}_1-\bar{h}_2}\\
\bar{h}_1 &= 2\bar{h}_2.
\end{aligned}
$$

With $\bar{h}_1 = 30$cm, we have $\bar{h}_2 = 15$cm and $\bar{\omega}_{in} = 1.05\sqrt{15} = 4.06$.

$$
\boxed{\bar{h}_1 = 30\text{cm},\ \ \bar{h}_2 = 15\text{cm},\ \text{and}\ \ \bar{\omega}_{in} = 4.06\text{cm}}
$$

On linearization of (11)-(12), we obtain

$$
\begin{aligned}
\delta\dot{h}_{1} &= -\frac{\sqrt{\rho g}}{2R\rho s}\frac{1}{\sqrt{\bar{h}_{1}-\bar{h}_{2}}}\left(\delta h_{1}-\delta h_{2}\right)+\frac{1}{\rho s}\ \delta\omega_{in} \\
\delta\dot{h}_{2} &= \frac{\sqrt{\rho g}}{2\rho R s}\left\{\frac{1}{\sqrt{\bar{h}_{1}-\bar{h}_{2}}}\left(\delta h_{1}-\delta h_{2}\right)-\frac{\delta h_{2}}{\sqrt{\bar{h}_{2}}}\right\}
\end{aligned}
$$

or

$$
\delta\dot{\boldsymbol{h}} = \frac{1.05}{2\sqrt{15}s}\begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}\delta\boldsymbol{h} + \begin{bmatrix} \frac{1}{s} \\ 0 \end{bmatrix}\ \delta\omega_{in}.
$$

□