

Given the system $y(s) = \frac{1}{s^3} u(s)$, find a controller which solves the RSP for constant disturbances / tracking signals.

(a) State Feedback.

In general

For checking stabilizability and detectability: for $\forall \lambda \in \text{sp}(A)$.

$\text{rank} \begin{pmatrix} A - \lambda I & B \\ C & 0 \end{pmatrix} = \text{Full rank.}$

↑
detectability

stabilizability.

$\text{rank} \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = \text{rank} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} = 4$

⇒ stabilizable & detectable

⇒ sol'n exist for RSP.

For a special case of constant signal

servo compensator has the form

According to internal model theorem

$\dot{\eta} = 0 \eta + e$
||
 (y_{ref} - y)

For state-feedback, we will assume $e = x$.

⇒ $\dot{\eta} = x$.

∴ We can put the compensator and system together

$$\begin{pmatrix} \dot{x} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ \eta \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} u$$

$$\eta = \underbrace{\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}}_{\hat{C}} \begin{pmatrix} x \\ \eta \end{pmatrix}$$

Now our objective is to find a controller.

$u = (R_0 \ R_1) \begin{pmatrix} x \\ \eta \end{pmatrix} = R_0 x + R_1 \eta$ s.t.

Minimize the cost $J = \int_0^{\infty} (\eta' \eta + \epsilon u' u) dt$.

$$\Rightarrow Q = \hat{C}^T \hat{C} \quad R = E = I \quad \text{if we let } E=1$$

Now we need to solve the Riccati Eqⁿ.

$$\hat{A}'P + P\hat{A} + \underbrace{\hat{C}^T \hat{C}}_Q - P \underbrace{\hat{B} R^{-1} \hat{B}'}_I P = 0$$

Solve for P

$$\Rightarrow U = -R^{-1} \hat{B}' P(x)$$

$$= (-2.6131 \quad -3.4142 \quad -2.6131 \quad -1) \begin{pmatrix} x \\ \eta \end{pmatrix}$$

\Rightarrow Our State Feedback Controller:

$$U = (-2.6131 \quad -3.4142 \quad -2.6131) x - \eta$$

$$\underline{NB} \quad \dot{\eta} = (y - y_{ref}) \Rightarrow \eta = \int_0^t (y - y_{ref}) d\tau$$

$$\therefore U = (-2.6131 \quad -3.4142 \quad -2.6131) x - \int_0^t (y - y_{ref}) d\tau$$

② If we don't observe internal state, then we need to construct an observer, for system

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u$$

$$y = (1 \ 0 \ 0) x$$

Suppose observer pole @ $(-50, -50, -50)$.

\Rightarrow pole-placement techniques.

then find Θ

$$\Theta = \mathcal{O}(A', C') = (C', A'C', A'^2 C') = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\det(\lambda I - A) = \det \begin{pmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & 0 & \lambda \end{pmatrix} = \lambda^3 \Rightarrow \begin{matrix} a_1 = 0 \\ a_2 = 0 \\ a_3 = 0 \end{matrix}$$

Desired: $(\lambda + 50)^3 = \lambda^3 + 3\lambda^2 \times 50 + 3\lambda \times 50^2 + 50^3$

$$\begin{matrix} a_1^* = -50^3 \\ a_2^* = -3 \times 50^2 \\ a_3^* = -3 \times 50 \end{matrix}$$

$$\hat{\Theta} = \begin{pmatrix} -a_2 & -a_3 & 1 \\ -a_3 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\Theta \hat{\Theta} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = (\hat{\Theta} \Theta)^{-1}$$

$$[a_1^* - a_1 \quad a_2^* - a_2 \quad a_3^* - a_3] (\hat{\Theta} \Theta)^{-1}$$

$$- \Lambda' = \begin{bmatrix} -50^3 & -3 \times 50^2 & -3 \times 50 \end{bmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$- \Lambda' = \begin{bmatrix} -3 \times 50 & -3 \times 50^2 & -50^3 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 3 \times 50 \\ 3 \times 50^2 \\ 50^3 \end{bmatrix}$$

Controller:

$$\begin{aligned} \dot{u} &= (-2.6131 \quad -3.4142 \quad -2.6131) \hat{x} - u \\ \dot{u} &= (y - y_{ref}) \\ \dot{\hat{x}} &= (A - \Lambda C) \hat{x} + \Lambda y + B u \end{aligned}$$