FLOW FUNCTIONS, CONTROL FLOW FUNCTIONS, AND THE REACH CONTROL PROBLEM

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Abstract. The paper studies the reach control problem (RCP) to make trajectories of an affine system defined on a polytopic state space reach and exit a prescribed facet of the polytope in finite time without first leaving the polytope. We introduce the notion of a flow function, which provides the analog of a Lyapunov function for the equilibrium stability problem. A flow function comprises a functional that decreases along closed-loop trajectories, and it provides a necessary and sufficient condition for closed-loop trajectories to exit the polytope. In turn this provides to the designer an analysis tool for determining if a specific instance of RCP is solved, without resorting to exhaustive simulation of the closed-loop system. Results include a variant of the LaSalle Principle tailored to RCP. An open problem is to identify suitable classes of flow functions. We explore functions of the form \( V(x) = \max \{ V_i(x) \} \), and we give evidence that these functions arise naturally when RCP is solved using continuous piecewise affine feedbacks. Next we introduce the notion of a control flow function. It is shown that the Artstein-Sontag theorem of control Lyapunov functions has direct analogies to RCP via control flow functions. These results convert an analysis tool based on flow functions into a synthesis tool based on control flow functions. Finally, examples illustrate the new ideas of the paper.

1. Introduction

We study the reach control problem (RCP) for affine systems on polytopes. The problem is to find a feedback control to make the closed-loop trajectories of an affine system defined on a polytopic state space reach and exit a prespecified facet of the polytope in finite time. Unlike [23, 24], we do not require the state trajectories to leave the polytope at the first time they reach the exit facet. The problem sits within a family of reachability problems for hybrid systems. A hybrid system is a dynamical system that combines both discrete event and continuous time behavior [9, 19, 34]. Our interest lies in a subclass of hybrid systems, piecewise affine hybrid systems. A piecewise affine hybrid system is a discrete automaton for which each discrete mode is equipped with its own continuous-time affine dynamics defined on a polytope. When the continuous state crosses a facet of a polytope, the system is transferred to a new discrete mode. The reachability analysis for piecewise affine hybrid systems at the continuous level reduces to studying RCP for an affine system on a polytope [25]. Interesting applications of RCP can include motion of robots in complex environments [6], aircraft and underwater vehicles [7], anesthesia [21], genetic networks [8], smart buildings, process control [26], among others [19].

The preponderance of literature on RCP regards simplices because their remarkable structure allows to focus on the essence of the reachability problem [24, 25, 38, 12, 13, 14, 15, 3, 4, 39]. Moreover, the search for feedback classes to solve RCP on simplices is narrowed due to their natural fit with affine feedback [24]. First results on simplices focused on necessary

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and sufficient conditions for a given affine feedback to solve RCP [25, 38]. Attention then turned to feedback synthesis using geometric methods [12, 13, 14, 15, 3]. In these papers a preferred triangulation of the polytopic state space was adopted to better expose the problem's geometric structure. In [12] it was shown that RCP is solvable by affine feedback if and only if it is solvable by continuous state feedback. In [15] it was shown that (a slightly stronger version of) RCP is solvable by discontinuous PWA feedback if and only if it is solvable by open-loop controls. Finally, in [3], time-varying affine feedback was shown to provide an alternative to discontinuous PWA feedback when the problem is solvable by open-loop controls. These results effectively closed the search for feedback classes to solve RCP under the preferred triangulation [4]. When the preferred triangulation is not adopted, then [39] initiated an exploration of other system structure that makes RCP solvable on a simplex by affine feedback. The results remain focused on simplices.

In contrast with simplices, the status for polytopes is more fragmentary. In [25] a method we call the simplex-based method was proposed. It involves first triangulating the polytope, then solving an RCP on each simplex, and then use dynamic programming on a graph to route trajectories through simplices to reach the exit facet of the polytope. In [33] various triangulation methods were explored but only for affine hypersurface systems (systems with $n - 1$ inputs, where $n$ is the system dimension). In [27], the geometric tools of [12] were extended from simplices to polytopes and a variant of RCP called the monotonic reach control problem (MRCP) was formulated. An in depth comparison of MRCP and simplex-based methods is found in [27]; a summary is presented in Section 6.

The simplex-based method and MRCP are the only known synthesis methods for solving RCP on polytopes. Both use PWA feedback. In our opinion, it is unlikely that the geometric tools in [39, 13, 15] can be extended to polytopes due to the inherent combinatorial complexity of polytopes [22]. One then turns to numerical approaches for synthesizing PWA feedbacks [28]. Unfortunately, we encounter examples not solvable by either the simplex-based method or MRCP, yet a PWA feedback is numerically obtained and simulations show it solves RCP. This observation sets the stage for this paper.

We require an analysis tool that allows to diagnose rigorously if a candidate PWA (or continuous state) feedback solves RCP, without resorting to exhaustive simulation. One immediately recognizes an analogy with Lyapunov analysis for the equilibrium stability problem [31]. But does RCP have an inherent notion of a function that acts like a Lyapunov function? Indeed it does. It was camouflaged as a flow condition in [38]. The flow condition is reinterpreted in this paper as a linear functional $V$ called a flow function that strictly decreases along closed-loop trajectories in the polytope $P$.

The contributions of the paper are as follows. First, we introduce the notion of a flow function, which is a functional bounded from below on $P$ and strictly decreasing along closed-loop trajectories in $P$. Flow functions provide a necessary and sufficient condition that all trajectories initiated in $P$ leave it in finite time. The fact that trajectories only leave $P$ through the exit facet is verified independently via so-called invariance conditions [24, 38]. Second, we focus our attention on locally Lipschitz flow functions, and this allows us to obtain a set of results that can be used to verify leaving $P$ in finite time, without resorting to exhaustive simulation of the closed-loop system. This includes a variant of the LaSalle Principle for RCP. Then we focus on PWA feedback, which is widely used to solve RCP on polytopes [25, 27, 28]. We present several results which play the role of converse Lyapunov theorems. The aim is to identify a class of flow functions that naturally emerges when solving
RCP by PWA feedback. We provide an LP-based numerical procedure for computing flow functions within the proposed class. Finally, the analogy with Lyapunov theory is deepened as we explore the Artstein-Sontag theorem for control Lyapunov functions within the context of RCP. Control Lyapunov functions continue to be intensively studied due to important emerging applications in hybrid systems and robotics [20, 1]. We are lead to the notion of control flow functions, and we propose a “universal formula” for RCP. These results extend what is a verification tool based on flow functions to a synthesis tool based on control flow functions.

The paper is organized as follows. Section 2 provides preliminaries on nonsmooth analysis. Section 3 reviews the standard problem statement and definitions for RCP [25, 38, 12]. Flow functions are introduced in Section 4, and the theme is continued in Section 5 with a variant of the LaSalle Principle for RCP. Section 6 identifies a suitable class of flow functions when using PWA feedback. In Section 7 we introduce the notion of control flow functions. Section 8 provides illustrative examples. Finally, Section 9 concludes the paper. A preliminary version of this paper appeared in [29]. Here we include all proofs, we generalize our results on PWA feedback to chains of simplices (Section 6), we introduce the notion of control flow functions (Section 7), and we provide novel illustrative examples (Section 8).

**Notation.** Let $\mathcal{K} \subset \mathbb{R}^n$ be a set. The closure is $\overline{\mathcal{K}}$, the interior is $\mathcal{K}^0$, and the boundary is $\partial \mathcal{K} := \overline{\mathcal{K}} \setminus \mathcal{K}^2$, where the notation $\mathcal{K}_1 \setminus \mathcal{K}_2$ denotes elements of the set $\mathcal{K}_1$ not contained in the set $\mathcal{K}_2$. The notation $T_{\mathcal{K}}(x)$ denotes the Bouligand tangent cone to the set $\mathcal{K}$ at point $x$. For $x \in \mathbb{R}^n$, $B_{\delta}(x)$ denotes the open ball in $\mathbb{R}^n$ centered at $x$ with radius $\delta$. The notation $0$ denotes the subset of $\mathbb{R}^n$ containing only the zero vector. The notation $\text{co}\{v_1, v_2, \ldots\}$ denotes the convex hull of a set of points $v_i \in \mathbb{R}^n$. The notation $\mathbb{R}_+$ denotes the set of non-negative real numbers. A function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be of class $\mathcal{C}^k$ if all its partial derivatives up to order $k$ exist and are continuous. The notation $L_f V(x) = \frac{\partial f}{\partial x} f(x)$ denotes the Lie derivative of $\mathcal{C}^1$ function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ with respect to function $f : \mathbb{R}^n \rightarrow \mathbb{R}$.

## 2. Background

We require some notions from nonsmooth analysis [17, 37]. Let $\mathcal{P} \subset \mathbb{R}^n$ and let $h : \mathcal{P} \rightarrow (-\infty, +\infty]$ be an extended real-valued function. We say that $h$ is upper semi-continuous at $x \in \mathcal{P}$ if for each $\epsilon > 0$ there exists $\delta > 0$ such that if $y \in \mathcal{P}$ and $\|y - x\| < \delta$, then $h(y) < h(x) + \epsilon$. If $h(x) = -\infty$, then $h(x)$ is upper semi-continuous at $x$ if for all $c < 0$, there exists $\delta > 0$ such that if $y \in \mathcal{P}$ and $\|y - x\| < \delta$, then $h(y) < c$. Now let $F : \mathcal{P} \rightarrow 2^{\mathbb{R}^n}$ be a set-valued map. We say $F$ is lower semi-continuous at $x \in \mathcal{P}$ if for any open subset $\mathcal{W} \subset \mathbb{R}^n$ such that $\mathcal{W} \cap F(x) \neq \emptyset$, there exists $\delta > 0$ such that for all $x' \in B_{\delta}(x) \cap \mathcal{P}$, $\mathcal{W} \cap F(x') \neq \emptyset$.

Next consider

$$\dot{x} = f(x) \quad (1)$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a locally Lipschitz function. Let $\phi(t, x_0)$ denote the unique solution of (1) starting at $x_0$. Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be a locally Lipschitz function. The upper right Dini derivative of $V(\phi(t, x_0))$ with respect to $t$ is

$$D^+ V(\phi(t, x_0)) := \limsup_{\tau \to 0^+} \frac{V(\phi(t + \tau, x_0)) - V(\phi(t, x_0))}{\tau}.$$
We can also define the upper Dini derivative of $V$ with respect to $f$ given by

$$D^+_f V(x) := \limsup_{\tau \to 0^+} \frac{V(x + \tau f(x)) - V(x)}{\tau}. \quad (2)$$

It was shown by Yoshizawa [44] that for $V$ locally Lipschitz

$$D^+V(\phi(t,x_0)) = D^+_f V(x), \quad (3)$$

where $x = \phi(t,x_0)$.

Finally, we use some notions from algebraic topology [35]. An $n$-dimensional simplex $S := \text{co} \{v_0, \ldots, v_n\}$ is the convex hull of $(n + 1)$ affinely independent points $\{v_0, \ldots, v_n\}$ in $\mathbb{R}^n$. A face of $S$ is any simplex spanned by a subset of $\{v_0, \ldots, v_n\}$. A proper face of $S$ is any face of $S$ different from $S$ itself. A facet of $S$ is an $(n - 1)$-dimensional face. The union of the proper faces of $S$ is called the boundary of $S$, denoted $\text{bd}(S)$. The interior of $S$ is $\text{int}(S) = S \setminus \text{bd}(S)$. An $n$-dimensional polytope $P := \text{co} \{v_1, \ldots, v_p\}$ is the convex hull of $p$ points $\{v_1, \ldots, v_p\}$ in $\mathbb{R}^n$ whose affine hull has dimension $n$. A triangulation $T$ of an $n$-dimensional polytope $P$ is a finite collection of $n$-dimensional simplices $S_1, \ldots, S_L$ such that (i) $P = \bigcup_{i=1}^L S_i$, (ii) For all $i, j \in \{1, \ldots, L\}$ with $i \neq j$, the intersection $S_i \cap S_j$ is either empty or a common face of $S_i$ and $S_j$. Let $T$ be a triangulation of $P$. A point $x \in P$ lies in the interior of precisely one simplex $S_x$ in $T$ whose vertices are, say, $v_1, \ldots, v_k$ (note that $S_x$ is not necessarily an $n$-dimensional simplex). Then $x = \sum_{i=1}^k \beta_i v_i$, where $\beta_i > 0$ and $\sum_i \beta_i = 1$. Coefficients $\beta_1, \ldots, \beta_k$ are called the barycentric coordinates of $x$. If $w$ is a vertex of $T$, the star of $w$ in $T$, denoted by $\text{st}(w)$, is the union of the interiors of those simplices in $T$ that have $w$ as a vertex. It is an open set in $\mathbb{R}^n$. The closure of $\text{st}(w)$, denoted $\overline{\text{st}}(w)$, is called the closed star of $w$ in $T$.

3. Reach Control Problem

Consider an $n$-dimensional polytope in $\mathbb{R}^n$

$$P := \text{co} \{v_1, \ldots, v_p\}$$

with vertex set $V := \{v_1, \ldots, v_p \mid v_i \in \mathbb{R}^n\}$ and $(n - 1)$-dimensional facets $F_0, F_1, \ldots, F_r$. The exit facet is designated to be the facet $F_0$ of $P$. Let $h_i$ be the unit normal to each facet $F_i$ pointing outside the polytope. Define the index sets $I := \{1, \ldots, p\}$, $J := \{1, \ldots, r\}$, and $J(x) := \{j \in J \mid x \in F_j\}$. For each $x \in P$, define the closed, convex cone

$$C(x) := \{ y \in \mathbb{R}^n \mid h_j \cdot y \leq 0, j \in J(x) \}.$$ 

Example 3.1. Consider the 2-dimensional polytope $P = \text{co} \{v_1, \ldots, v_6\}$ shown in Figure 1. The vertex set of $P$ is $V = \{v_1, \ldots, v_6\}$ and the facets of $P$ are the 1-dimensional faces $F_0, \ldots, F_5$. The outward unit normal vectors are labeled as $h_0, \ldots, h_5$. The exit facet is $F_0$. We have $J(v_6) = \{5\}$ whereas $J(v_5) = \{4, 5\}$. The cones $C(x)$ for the vertices $v_i$ are depicted as shaded cones attached at each $v_i$. Of course these cones have their apex at 0, but they are depicted as attached at the base point $v_i$ since they will be used to characterize tangent vectors. Notice that for any $x \in P \setminus F_0$, $C(x)$ is exactly the Bouligand tangent cone to $P$ at $x$, $\mathcal{T}_P(x)$ [17]. Instead, at points $x \in F_0$, $C(x)$ includes directions pointing out of $P$. Indeed the definition of $C(x)$ does not involve $h_0$ because of $F_0$’s role as the exit facet. \(<
We consider the affine control system defined on $\mathcal{P}$:
\[
\dot{x} = Ax + Ba + a, \quad x \in \mathcal{P},
\]
where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $A \in \mathbb{R}^{n \times n}$, $a \in \mathbb{R}^n$, $B \in \mathbb{R}^{n \times m}$, and $\text{rank}(B) = m$. Let $B = \text{Im} B$, the image of $B$. Also, let $\phi_u(t, x_0)$ be the trajectory of (4) under a control law $u$ starting from $x_0 \in \mathcal{P}$. We are interested in studying reachability of the exit facet $F_0$ from $\mathcal{P}$ by feedback control.

**Problem 3.1 (Reach Control Problem (RCP)).** Consider system (4) defined on $\mathcal{P}$. Find a state feedback $u(x)$ such that:

(i) for each $x_0 \in \mathcal{P}$ there exist $T \geq 0$ and $\gamma > 0$ such that $\phi_u(t, x_0) \in \mathcal{P}$ for all $t \in [0, T]$, $\phi_u(T, x_0) \in F_0$, and $\phi_u(t, x_0) \notin \mathcal{P}$ for all $t \in (T, T + \gamma)$.

RCP says that trajectories of (4) starting from initial conditions in $\mathcal{P}$ reach and exit the facet $F_0$ in finite time, while not first leaving $\mathcal{P}$. Figure 1 illustrates the idea of RCP for a sample trajectory starting from an initial condition $x_0 \in \mathcal{P}$. Notice that the formulation of RCP does not impose that trajectories exit $\mathcal{P}$ at the first time they reach $F_0$, but does impose that each trajectory eventually exits $\mathcal{P}$ through $F_0$. Also, notice that condition (i) assumes that the dynamics (4) can be extended to a neighborhood of $\mathcal{P}$. A useful shorthand notation is to write $\mathcal{P} \xrightarrow{u} F_0$ by control $u(x)$ if RCP is solved using $u(x)$.

The class of continuous piecewise affine feedbacks is widely used to solve RCP on polytopes [24, 25, 27, 28]. Let $\mathcal{T}$ be a triangulation of $\mathcal{P}$. Given a state feedback $u(x)$ on $\mathcal{P}$, we say $u$ is a piecewise affine (PWA) feedback associated with $\mathcal{T}$ if for any $x \in \mathcal{P}$, $x = \sum \lambda_i v_i$ implies $u(x) = \sum \lambda_i u(v_i)$, where $\{v_i\}$ are the vertices of $S_x$ and the $\lambda_i$’s are the corresponding barycentric coordinates of $x$. One can show $u(x)$ is a continuous state feedback on $\mathcal{P}$ [35]. If $u(x)$ is a PWA feedback associated with $\mathcal{T}$, then for each $n$-dimensional simplex $S^k \in \mathcal{T}$, there exist $K_k \in \mathbb{R}^{m \times n}$ and $g_k \in \mathbb{R}^m$ such that $u$ takes the form $u(x) = K_k x + g_k$, $x \in S^k$. 

![Figure 1. Notation for the reach control problem.](image-url)
This paper will focus on continuous feedbacks and especially continuous PWA feedbacks that solve RCP. In the literature necessary conditions for a PWA feedback to solve RCP have been identified; they guarantee that closed-loop trajectories only exit \( \mathcal{P} \) through \( \mathcal{F}_0 \) [24].

**Definition 3.1.** We say the invariance conditions are solvable if for each \( v \in V \) there exists \( u \in \mathbb{R}^m \) such that

\[
Av + Bu + a \in \mathcal{C}(v).
\]

(5)

The conditions are stated only at the vertices of the polytope because the closed-loop vector field is convex using PWA feedback. Therefore the conditions hold at all points on the boundary of \( \mathcal{P} \). Given a general continuous state feedback \( u(x) \), convexity of the closed-loop vector field is lost, so the following stronger invariance conditions are required:

\[
Ax + Bu(x) + a \in \mathcal{C}(x), \quad x \in \mathcal{P}.
\]

(6)

**Example 3.2.** Returning to Figure 1 we illustrate the meaning of condition (5). Define the velocity vectors \( y_i := Av_i + Bu_i + a, i = 1, \ldots, 6 \). The control inputs \( u_i \in \mathbb{R}, i = 1, \ldots, 6 \) are selected so that \( y_i \in \mathcal{C}(v_i) \). At points \( x \) in the interior of \( \mathcal{P} \), the condition is vacuous because \( \mathcal{C}(x) = \mathbb{R}^n \). If a PWA feedback associated with a triangulation \( \mathcal{T} \) (the triangulation is constructed using only the vertices of \( \mathcal{P} \)), then conditions (5) imply (6). For this reason, conditions at boundary points of \( \mathcal{P} \) which are not vertices of \( \mathcal{P} \) are not explicitly stated. Instead, if only continuous state feedback is used, invariance conditions must be stated explicitly for each point in \( \mathcal{P} \) as in (6). This is illustrated in Figure 1 for a point in \( \mathcal{F}_4 \).

### 4. Flow Functions

Suppose we are presented with an instance of RCP on a polytope and we have in hand a continuous feedback \( u(x) \) as a candidate feedback solution such that the closed-loop system has unique solutions (for information on how to construct feedbacks on polytopes, see [24, 27, 28]). Since the invariance conditions (6) are necessary for solvability of RCP by continuous feedback, we assume that \( u(x) \) already achieves (6). We conclude that trajectories can only exit \( \mathcal{P} \) through \( \mathcal{F}_0 \). Then to verify if \( u(x) \) actually solves RCP on \( \mathcal{P} \), we only have to verify whether all trajectories initiated in \( \mathcal{P} \) leave it in finite time. Like Lyapunov theory, we hope to avoid a verification by exhaustive simulation.

In the literature on RCP for simplices and affine feedbacks this verification is performed using a flow condition comprising a linear functional that strictly decreases along closed-loop trajectories. Since the simplex is compact and the functional is continuous, the strictly decreasing condition means closed-loop trajectories must exit. The underpinning of the present work is the fact that such a linear functional always exists if RCP is solved on an \( n \)-dimensional simplex by a given affine feedback.

**Theorem 4.1.** [25, 38] Let \( \mathcal{S} = \text{co} \{v_0, \ldots, v_n\} \) be an \( n \)-dimensional simplex and consider the affine system (4) defined on \( \mathcal{S} \). Let \( u(x) = Kx + g \) be an affine feedback where \( K \in \mathbb{R}^{m \times n}, g \in \mathbb{R}^m \), and \( u_0 = u(v_0), \ldots, u_n = u(v_n) \). Then \( \mathcal{S} \xrightarrow{\mathcal{S}} \mathcal{F}_0 \) using \( u(x) \) if and only if

(a) The invariance conditions (5) hold.

(b) There exists \( \xi \in \mathbb{R}^n \) such that \( \xi \cdot (Av_i + Bu_i + a) < 0, i \in \{0, \ldots, n\} \).
The inequality in (b) is called a flow condition [38]. Observe that if we define the linear functional \( V(x) := \xi \cdot x \), then the flow condition, involving the Lie derivative of \( V \) along solutions, has the interpretation to require that the function \( V \) decreases along solutions of the closed loop system \( \dot{x} = (A + BK)x + Bg + a \) so long as they lie in \( S \). The meaning of conditions (a) and (b) is now very clear. Condition (a) says trajectories only exit \( S \) via \( F_0 \). Condition (b) says that trajectories do exit.

While the results for simplices are transparent, unfortunately, linear functionals are too restrictive as a class when verifying feedback solutions on polytopes [27]. Indeed, we have many examples where a continuous feedback \( u(x) \) is verified to solve RCP via exhaustive simulation, but no linear functional exists. These examples highlight the need for a more general tool to verify that trajectories leave a polytope in finite time. Turning to the literature, there are well-known results providing general tests for trajectories to leave compact sets.

**Proposition 4.2 ([37], Ch. 7).** Let \( P \) be a compact set in \( \mathbb{R}^n \). Let (1) be a dynamical system defined on \( P \) where \( f : \mathbb{R}^n \to \mathbb{R}^n \) is continuous, and let \( V : \mathbb{R}^n \to \mathbb{R} \) be a \( C^1 \) function defined on a neighborhood of \( P \). If \( L_f V(x) \neq 0 \), \( x \in P \), then all trajectories starting in \( P \) leave it in finite time.

To summarize, on the one hand, we have specific forms of the flow conditions (linear flow conditions) matching specific forms of the system and feedback (affine systems and feedbacks), in the same way that quadratic Lyapunov functions fit with linear systems and feedbacks. On the other hand, we have general forms of the flow condition requiring only certain differentiability assumptions. Combining these two situations into one framework, we are lead to the following definition.

**Definition 4.1.** Let \( P \) be an \( n \)-dimensional polytope and (1) a dynamical system defined on \( P \). A flow function \( V : \mathbb{R}^n \to \mathbb{R} \) is a functional bounded from below on \( P \) and strictly decreasing in \( P \) along trajectories of (1).

An open problem is to identify the most useful classes of flow functions for RCP. We begin with the most general context. Suppose we have a feedback \( u(x) \) such that the closed-loop vector field \( f(x) := Ax + Bu(x) + a \) is locally Lipschitz on a neighborhood of \( P \). Suppose we have a functional \( V(x) \) bounded from below on \( P \) and satisfying

\[
V(\phi_u(t,x_0)) \leq V(x_0) - t \tag{7}
\]

for all \( x_0 \in P \) and \( t \geq 0 \) such that \( \phi_u(\tau,x_0) \in P, \tau \in [0,t] \). It is obvious from (7) that trajectories must exit \( P \) in finite time. Conversely, suppose that using \( u(x) \), all trajectories initiated in \( P \) leave it in finite time. Then for each \( x_0 \in P \), there exist \( T_{x_0} \geq 0 \) and \( \gamma_{x_0} > 0 \) such that \( \phi_u(t,x_0) \in P \) for all \( t \in [0,T_{x_0}] \), and \( \phi_u(t,x_0) \notin P \) for all \( t \in (T_{x_0}, T_{x_0} + \gamma_{x_0}) \). Define the map \( T : P \mapsto \mathbb{R}_+ \) by \( T(x) := T_x, x \in P \). By uniqueness of solutions, \( T \) is a well-defined (single-valued) function. Also \( T(x) \geq 0 \) on \( P \). By the semi-group property,

\[
T(\phi_u(t,x_0)) = T(x_0) - t, \quad t \in [0,T(x_0)].
\]

Thus, we have proved the following straightforward but fundamental result showing that existence of a function bounded from below on \( P \) and satisfying (7) is a necessary and sufficient condition for leaving \( P \) in finite time.

**Theorem 4.3.** Consider the system (4) defined on a polytope \( P \). Let \( u(x) \) be a continuous state feedback such that the closed-loop vector field \( f(x) \) is locally Lipschitz on a neighborhood
of $\mathcal{P}$. All closed-loop trajectories starting in $\mathcal{P}$ leave it in finite time if and only if there exists $V : \mathcal{P} \rightarrow \mathbb{R}$ such that $V(x)$ is bounded from below on $\mathcal{P}$ and (7) holds.

So far we have not placed any smoothness requirements on $V$. Now we consider the case when $V$ is locally Lipschitz; here only sufficient conditions can be obtained.

**Theorem 4.4.** Consider the system (4) defined on a polytope $\mathcal{P}$. Let $u(x)$ be a continuous state feedback such that the closed-loop vector field $f(x)$ is locally Lipschitz on a neighborhood of $\mathcal{P}$. All closed-loop trajectories starting in $\mathcal{P}$ leave it in finite time if there exists a function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ that is locally Lipschitz on a neighborhood of $\mathcal{P}$ and satisfies

$$D^+_f V(x) \leq -1, \quad x \in \mathcal{P}. \quad (8)$$

**Proof.** Since $V(x)$ is continuous and $\mathcal{P}$ is compact, $V(x)$ is bounded from below on $\mathcal{P}$. Since $D^+_f V(\phi_u(t, x_0)) = D^+_f V(x)$ with $x = \phi_u(t, x_0)$ as per (3), we can apply the Comparison Lemma [31] to deduce that (8) gives

$$V(\phi_u(t, x_0)) \leq V(x_0) - t \quad (9)$$

for all $x_0 \in \mathcal{P}$ and $t \geq 0$ such that $\phi_u(t, x_0) \in \mathcal{P}$, $\tau \in [0, t]$. Then we can apply Theorem 4.3. \hfill \Box

Now we focus on a particular form of $V$ that appears to have special relevance to RCP. Let $I_0 = \{1, \ldots, L\}$, and suppose for each $i \in I_0$, $V_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is a $\mathcal{C}^1$ function. Define

$$V(x) := \max_{i \in I_0} \{V_i(x)\}. \quad (10)$$

Also, for each $x \in \mathbb{R}^n$ define the index set

$$I_0(x) := \{i \in I_0 \mid V_i(x) = V(x)\}.$$

**Lemma 4.5** ([18]). Consider the system (1) and let $V(x)$ be as in (10). Then $V(x)$ is locally Lipschitz and

$$D^+_f V(x) = \max_{i \in I_0(x)} L_f V_i(x).$$

**Theorem 4.6.** Consider the system (4) defined on a polytope $\mathcal{P}$. Let $u(x)$ be a continuous state feedback such that the closed-loop vector field $f(x)$ is locally Lipschitz on a neighborhood of $\mathcal{P}$. Let $V$ be as in (10). All closed-loop trajectories starting in $\mathcal{P}$ leave it in finite time if

$$D^+_f V(x) < 0, \quad x \in \mathcal{P}. \quad (11)$$

**Proof.** Consider the sets $\Gamma_i := \{x \in \mathcal{P} \mid i \in I_0(x)\}$, $i \in I_0$. Since each $V_i$ is continuous and $\Gamma_i = \bigcap_{j \neq i} \{x \in \mathcal{P} \mid V_i(x) - V_j(x) \geq 0\}$, each $\Gamma_i$ is compact. By assumption and Lemma 4.5, $L_f V_i(x) < 0$ for $x \in \Gamma_i$, $i \in I_0$. Since $L_f V_i(x)$ is continuous and $\Gamma_i$ is compact, there exists $\epsilon_i > 0$ such that $L_f V_i(x) < -\epsilon_i$ for $x \in \Gamma_i$, $i \in I_0$. Let $\epsilon := \min_{i \in I_0} \epsilon_i$. Define $\widehat{V}(x) = \frac{V(x)}{\epsilon}$. Then for any $x \in \mathcal{P}$

$$D^+_f \widehat{V}(x) = \frac{1}{\epsilon} \max_{i \in I_0(x)} L_f V_i(x) < \frac{1}{\epsilon} \max_{i \in I_0(x)} \{-\epsilon_i\} \leq -1.$$

By Lemma 4.5, $\widehat{V}(x)$ is locally Lipschitz. The result follows from Theorem 4.4. \hfill \Box

We remark that the three previous theorems also hold for compact non-convex sets. This fact is useful in solving examples (see for instance Example 8.1). In Section 6 we explore in greater depth the nature of flow functions that arise out of solving RCP by PWA feedbacks.
5. Lasalle Principle for RCP

In this section we study the case when a flow function has not been found, but we have identified a locally Lipschitz function $V$ satisfying $D^+_fV(x) \leq 0$ for all $x \in \mathcal{P}$. The question is whether this information is enough to deduce that closed-loop trajectories exit $\mathcal{P}$. For this we use an argument similar to LaSalle’s Principle, but we use it in the opposite way to how LaSalle’s Principle is normally applied. The LaSalle Principle is used in Lyapunov theory in the case when a positive definite Lyapunov function is not available, but some function that is non-increasing along solutions is available [31]. It allows to show that trajectories tend to an invariant set. Instead, we use the LaSalle Principle in the case when a flow function is not available, but some function that is non-increasing along solutions is available. We use this information to show that trajectories exit from $\mathcal{P}$ if there is no invariant set in a particular subset of $\mathcal{P}$. Thus, the novelty and the contribution are in showing that a LaSalle Principle is meaningful in the context of RCP, despite RCP imposing a completely different requirement from equilibrium stability. As such, the proof method is almost identical to the standard LaSalle Principle, so it is omitted.

**Theorem 5.1 (LaSalle).** Consider the system (4) defined on a polytope $\mathcal{P}$. Let $u(x)$ be a continuous state feedback such that the closed-loop vector field $f(x)$ is locally Lipschitz on a neighborhood of $\mathcal{P}$. Suppose there exists $V : \mathbb{R}^n \to \mathbb{R}$ that is locally Lipschitz on a neighborhood of $\mathcal{P}$ and satisfies $D^+_fV(x) \leq 0$, $x \in \mathcal{P}$.

Let

$$\mathcal{M} := \{x \in \mathcal{P} \mid D^+_fV(x) = 0\}.$$ 

If $\mathcal{M}$ does not contain an invariant set, then all trajectories starting in $\mathcal{P}$ leave it in finite time.

**Remark 5.1.** A more direct statement of our version of the LaSalle Principle could be as follows: if $\mathcal{P}$ contains no invariant set of the closed loop system, then all closed-loop trajectories exit $\mathcal{P}$. The proof would be a straightforward application of Birkhoff’s theorem [31]. This simpler statement is far less useful than Theorem 5.1. This is because searching for invariant sets is difficult, and Theorem 5.1 restricts that search to $\mathcal{M}$. We will see in Example 8.2 this can significantly simplify the analysis problem.

6. PWA feedback

In this section we focus on (continuous) PWA feedback, a widely studied feedback class to solve RCP on polytopes [24, 25, 27, 28], not to mention other control problems [10, 16]. In the literature there are currently two techniques to solve RCP on polytopes by PWA feedback: the monotonic reach control problem (MRCP) [27, 28] and so-called simplex-based methods [25]. A detailed comparison of the two methods was given in [27]. Here we highlight the main findings.

MRCP contains the same problem statement as RCP but it additionally imposes that a linear flow function strictly decreases along the closed-loop trajectories. Comparing with condition (b) of Thereom 4.1, MRCP subsumes in its definition what is a necessary condition for solvability of RCP by affine feedback on simplices. However, because the existence of a linear flow function is no longer necessary for solvability of RCP on polytopes [27], MRCP is a more restrictive problem. On the other hand, to solve MRCP by PWA feedback it is
Figure 2. A triangulation of $\mathcal{P}$ into two simplices

necessary that the invariance conditions of $\mathcal{P}$ hold, but it is not necessary that invariance conditions of individual simplices of a triangulation on which a PWA feedback is defined also hold. For example, consider Figure 2 which depicts a polytope $\mathcal{P}$ consisting of two simplices $\mathcal{S}^1$ and $\mathcal{S}^2$. The control objective is to drive closed-loop trajectories out of $\mathcal{P}$ through $\mathcal{F}_0$. We observe that the cone $C^1(v_3)$ characterizing the invariance conditions at $v_3$ for $\mathcal{S}^1$ is smaller than the cone $C(v_1)$ characterizing the invariance conditions at $v_1$ for $\mathcal{P}$. In this sense, RCP on $\mathcal{S}^1$ is a more restrictive problem (and may not be solvable) than RCP on $\mathcal{P}$.

The second method to solve RCP on polytopes is called here simplex-based methods or simply simplex methods and was introduced in [25]. The first step of the method is to propitiously triangulate the polytope (using any a priori knowledge about the system dynamics). The second step is to solve RCP for each simplex $\mathcal{S}^i$ of the triangulation using affine feedback $u^i(x) = K^i x + g^i$. A dynamic programming algorithm on a graph that represents simplices and their neighbors is employed to decide what sequence of RCP’s must be solved. This method is appealing because it builds up hierarchically from available tools for synthesizing feedbacks on simplices [24, 25, 38]. However, as already discussed, it is necessary that invariance conditions for individual simplices are satisfied, and these conditions may be more restrictive than those for the overall polytope. On the other hand, the simplex-based method relaxes the requirement for the existence of a linear flow function, which we already know is too restrictive.

It is clear from the foregoing discussion that the two available methods are complementary. Indeed, we have found through examples that one technique may work when the other fails, and vice versa. Also, we have found examples in which both techniques fail; nevertheless via exhaustive simulation we verify that a PWA feedback solves RCP on a polytope. Evidently existing techniques are not general enough to explain why a given continuous PWA feedback solves RCP.
We return to our immediate inquiry to classify flow functions for verifying if a candidate PWA feedback solves RCP on a polytope. In the context of MRCP, there is nothing to be done - the flow function has been ordained to be linear. More interesting is the question of the relationship between flow functions and the simplex-based method. That is, what class of flow functions emerges when RCP is solved by the simplex-based method? The answer may give clues about the preferred classes of flow functions for PWA feedback.

6.1. Two Simplices. In this section we study the question of what class of flow functions naturally arises when solving RCP via PWA feedback. Because this problem has not been formulated before, we study it in the simplest possible context. We consider the case when the polytope \( \mathcal{P} \) consists of two simplices \( S^1 \) and \( S^2 \). Let \( \mathcal{T} = \{S^1, S^2\} \) denote the triangulation of \( \mathcal{P} \). See Figure 2. Saying that RCP is solved by the simplex-based method using a PWA feedback \( u(x) \) on \( \mathcal{T} \) implies

\[
\begin{align*}
&\bullet S^1 \rightarrow S_0 \text{ using } u^1(x) = K^1 x + g^1, \\
&\bullet S^2 \rightarrow S_1 \text{ using } u^2(x) = K^2 x + g^2, \text{ where } \mathcal{F} = S^1 \cap S^2.
\end{align*}
\]

Moreover, the controller
\[
u(x) = \begin{cases} u_1(x), & x \in S^1 \\ u_2(x), & x \in S^2 \setminus S^1 \end{cases}
\]
is continuous. What form does the flow function take in this case?

**Theorem 6.1.** Consider the system (4) defined on an \( n \)-dimensional polytope \( \mathcal{P} \), a triangulation \( \mathcal{T} = \{S^1, S^2\} \) of \( \mathcal{P} \), and a continuous PWA feedback \( u(x) \) defined on \( \mathcal{T} \). Let \( f(x) := Ax + Bu(x) + a \). If \( S^1 \rightarrow \mathcal{F}_0 \) and \( S^2 \rightarrow \mathcal{F} \) using \( u(x) \), then there exist affine functions \( V_1: \mathbb{R}^n \rightarrow \mathbb{R} \) and \( V_2: \mathbb{R}^n \rightarrow \mathbb{R} \) such that
\[
V(x) = \max\{V_1(x), V_2(x)\} \tag{11}
\]
satisfies \( D_x^+ V(x) < 0 \) for all \( x \in \mathcal{P} \).

**Proof.** Let \( S^1 = \text{co} \{v_1, \ldots, v_{n+1}\} \) and \( S^2 = \text{co} \{v_2, \ldots, v_{n+2}\} \). The exit facet of \( S^1 \) is \( \mathcal{F}_0 = \text{co} \{v_1, \ldots, v_n\} \) and the exit facet of \( S^2 \) is \( \mathcal{F} = S^1 \cap S^2 = \text{co} \{v_2, \ldots, v_{n+1}\} \). See Figure 2. Let \( h \) be the unit normal vector to \( \mathcal{F} \) pointing out of \( S^2 \), and define \( \alpha := h \cdot x \), where \( x \) is any point in \( \mathcal{F} \). By Theorem 4.1, there exists \( \xi_1 \in \mathbb{R}^n \) such that
\[
\xi_1 \cdot (Ax + Bu(x) + a) < 0, \quad x \in S^1. \tag{12}
\]
We choose
\[
V_1(x) := \xi_1 \cdot x.
\]
By the geometry of the simplex (see Lemma 2.1(2) of [24]), there exist \( \lambda_i > 0 \) such that
\[
h = -\lambda_2 h_2 - \cdots - \lambda_{n+1} h_{n+1}, \tag{13}
\]
where \( h_i \) is the outward unit normal vector of the facet of \( S^2 \) not containing vertex \( v_i \) for \( i = 2, \ldots, n+1 \). Then because \( S^2 \rightarrow \mathcal{F} \), the invariance conditions (5) are satisfied at \( v_{n+2} \) using \( u(x) \); that is
\[
h_j \cdot (Av_{n+2} + Bu(v_{n+2}) + a) \leq 0, \quad j = 2, \ldots, n + 1. \tag{14}
\]
Now by Lemma 2.1(1) of [24], \( \{h_2, \ldots, h_{n+1}\} \) span \( \mathbb{R}^n \). Hence, there must be some inequality among those in (14) which holds strictly (for if not, \( Av_{n+2} + Bu(v_{n+2}) + a = 0 \), which contradicts that \( S^2 \xrightarrow{S^2 \to F} \)). Suppose w.l.o.g. that

\[
h_2 \cdot (Av_{n+2} + Bu(v_{n+2}) + a) < 0. \tag{15}\]

Combining (13), (14), and (15), we obtain

\[
(-h) \cdot (Av_{n+2} + Bu(v_{n+2}) + a) < 0. \tag{16}\]

Because \( u(x) \) is continuous and \( S^1 \xrightarrow{S^1 \to F_0} \), the invariance conditions (5) for \( S^1 \) hold at vertices \( v_i \in F \); in particular,

\[
(-h) \cdot (Av_i + Bu(v_i) + a) \leq 0, \quad i = 2, \ldots, n+1. \tag{17}\]

Now define

\[
\xi_2 := \xi_1 - ch \tag{18}\]

where, using (16), \( c > 0 \) is selected sufficiently large so that

\[
\xi_2 \cdot (Av_{n+2} + Bu(v_{n+2}) + a) < 0. \tag{19}\]

Using (12), (17), and (18), we get

\[
\xi_2 \cdot (Av_i + Bu(v_i) + a) < 0, \quad i = 2, \ldots, n+1. \tag{20}\]

Since \( u(x) \) is affine on \( S^2 \), (19) and (20) together imply

\[
\xi_2 \cdot (Ax + Bu(x) + a) < 0, \quad x \in S^2. \tag{21}\]

We choose

\[
V_2(x) := \xi_2 \cdot x + ca,
\]

and we let \( V(x) \) be as in (11).

It remains to show \( D^+_f V(x) < 0 \) for \( x \in P \). Our results above give

\[
\begin{align*}
L_f V_1(x) &= \xi_1 \cdot (Ax + Bu(x) + a) < 0, \quad x \in S^1 \tag{22a} \\
L_f V_2(x) &= \xi_2 \cdot (Ax + Bu(x) + a) < 0, \quad x \in S^2. \tag{22b}
\end{align*}
\]

Recall that \( ch \cdot x = ca \) for \( x \in S^1 \cap S^2 \) and that \( h \) points outside of \( S^2 \). Thus

\[
\begin{align*}
ch \cdot x &= (\xi_1 - \xi_2) \cdot x \leq ca, \quad x \in S^2 \tag{23a} \\
ch \cdot x &= (\xi_1 - \xi_2) \cdot x \geq ca, \quad x \in S^1. \tag{23b}
\end{align*}
\]

Define the sets

\[
\begin{align*}
\Gamma^1 &= \{ x \in P \mid \xi_1 \cdot x \geq \xi_2 \cdot x + ca \} = \{ x \in P \mid V_1(x) \geq V_2(x) \} \tag{24a} \\
\Gamma^2 &= \{ x \in P \mid \xi_1 \cdot x \leq \xi_2 \cdot x + ca \} = \{ x \in P \mid V_1(x) \leq V_2(x) \}. \tag{24b}
\end{align*}
\]

By (23), \( \Gamma^i = S^i, \ i = 1, 2 \). Then according to Lemma 4.5, \( D^+_f V(x) = \max_{i \in I_0(x)} L_f V_i(x) \).

Therefore, by (22), we conclude \( D^+_f V(x) < 0 \) for all \( x \in P \), as desired. \( \square \)
6.2. **Numerical Procedure.** The goal of the previous section was to discover a form of the flow function that naturally arises from solving RCP via simplex-based methods. The result plays the same role as a converse Lyapunov theorem. As with certain converse Lyapunov theorems such as existence of quadratic Lyapunov functions for stable linear systems, Theorem 6.1 may appear to be primarily of theoretical interest. For if we know that \( S^1 \xrightarrow{\xi} F_0 \) and \( S^2 \xrightarrow{\xi} F \), then we know that RCP is solved. However, the result is of practical interest when simplex methods fail, yet a flow function of the form (11) may still be relevant. A typical scenario is when simplex methods fail because the invariance conditions of \( S^1 \) are not solvable at some vertices on \( F \). For instance, in Figure 2 the invariance conditions of \( \mathcal{P} \) are solvable at \( v_3 \). However, for any \( u_3 \) we select, \( A v_3 + B u_3 + a \) points outside \( S^1 \), so \( \xrightarrow{\xi} F_0 \) always fails for any PWA feedback \( u(x) \) on \( T \). Despite this failure, the overall problem to exit the polytope may still be solved by the same \( u(x) \), and by verifying the existence of a flow function of the form (11). In light of these remarks, there is practical interest to have a numerical procedure to construct a flow function of the form (11), even if some invariance conditions for \( S^1 \) are not achievable.

**Corollary 6.2.** Consider the system (4) defined on a polytope \( \mathcal{P} \), and a triangulation \( T = \{ S^1, S^2 \} \) of \( \mathcal{P} \), where \( S^1 = \text{co} \{ v_1, \ldots, v_{n+1} \} \) and \( S^2 = \text{co} \{ v_2, \ldots, v_{n+2} \} \). Let \( u(x) \) be a continuous PWA feedback on \( T \) and let \( f(x) := Ax + Bu(x) + a \). Suppose that \( u(x) \) satisfies the invariance conditions (5) of \( \mathcal{P} \), and for some \( 2 \leq k \leq n + 2 \) it satisfies (5) of \( S^1 \) at vertices \( v_2, \ldots, v_{k-1} \). Suppose the following linear programming (LP) problem is solvable

\[
\begin{bmatrix}
f(v_1)^T & 0 \\
\vdots & \vdots \\
f(v_{n+1})^T & 0 \\
f(v_k)^T & -h \cdot f(v_k) \\
\vdots & \vdots \\
f(v_{n+2})^T & -h \cdot f(v_{n+2}) \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
\xi_1 \\
\xi_2 \\
\vdots \\
\xi_k
\end{bmatrix} < 0.\tag{25}
\]

Then there exist affine functions \( V_1 : \mathbb{R}^n \rightarrow \mathbb{R} \), \( V_2 : \mathbb{R}^n \rightarrow \mathbb{R} \), and a function \( V \) of the form (11) such that \( D^T f V(x) < 0 \) for all \( x \in \mathcal{P} \).

**Proof.** Let \( \xi_1, c \) be a solution of (25). Since \( u(x) \) is affine on \( S^1 \), the first \( n+1 \) inequalities in (25) imply

\[
\xi_1 \cdot (Ax + Bu(x) + a) < 0, \quad x \in S^1.\tag{26}
\]

Choose \( V_1(x) = \xi_1 \cdot x \). Then we define

\[
\xi_2 := \xi_1 - ch.\tag{27}
\]

The second to last inequality in (25) implies

\[
\xi_2 \cdot (Av_{n+2} + Bu(v_{n+2}) + a) < 0.\tag{28}
\]

By assumption, \( u(x) \) satisfies the invariance conditions of \( S^1 \) at the vertices \( v_2, \ldots, v_{k-1} \). This implies

\[
(-h) \cdot (Av_i + Bu(v_i) + a) \leq 0, \quad i = 2, \ldots, k - 1.\tag{29}
\]

By (25), \( c > 0 \). Then by (26), (27), and (29)

\[
\xi_2 \cdot (Av_i + Bu(v_i) + a) < 0, \quad i = 2, \ldots, k - 1.\tag{30}
\]
Also, from (27) and (25), we get
\[ \xi_2 \cdot (Av_i + Bu(v_i) + a) < 0, \quad i = k, \ldots, n + 1. \quad (31) \]
Since \( u(x) \) is affine on \( S^2 \), (28), (30), and (31) imply
\[ \xi_2 \cdot (Ax + Bu(x) + a) < 0, \quad x \in S^2. \quad (32) \]
We choose \( V_2(x) = \xi_2 \cdot x + c_\alpha \), and we let \( V(x) \) be as in (11). Finally, by the same argument used in the last part of the proof of Theorem 6.1, we get \( D^+_x V(x) < 0, x \in \mathcal{P} \), as desired. \( \square \)

Corollary 6.2 provides a simple tool for verifying that all closed-loop trajectories initiated in \( \mathcal{P} \) leave it in finite time for the case where existing techniques fail.

6.3. Chain of Simplices. In this section we explore topologies when the previous result for two simplices can be extended to multiple simplices. In particular we study chains of simplices that form possibly non-convex polyhedra.

**Definition 6.1.** Let \( I_0 := \{1, \ldots, L\} \). Let \( \{\mathcal{S}^k \mid k \in I_0\} \) be a collection of \( n \)-dimensional simplices. Define \( \mathcal{P} := S^1 \cup \cdots \cup S^L \). We say that \( \mathcal{P} \) is a chain if the following hold:

(i) If \( \mathcal{S}^k \cap \mathcal{S}^j \neq \emptyset \) for \( k, j \in I_0 \), then \( \mathcal{S}^k \cap \mathcal{S}^j \) is a face of \( \mathcal{S}^k \) and of \( \mathcal{S}^j \).

(ii) For each \( k \in I_0 \), denote the exit facet of \( \mathcal{S}^k \) by \( \mathcal{F}_0^k \). Then for \( k = 2, \ldots, L, \mathcal{F}_0^k := \mathcal{S}^k \cap \mathcal{S}^{k-1} \).

(iii) The exit facet of \( \mathcal{P} \) is \( \mathcal{F}_0^1 \).

We denote by \( h^k_0 \) the unit normal vector of \( \mathcal{F}_0^k \) pointing out of \( \mathcal{S}^k \). Also let \( \alpha_k \in \mathbb{R} \) be such that \( h^k_0 \cdot x = \alpha_k \) for all \( x \in \mathcal{F}_0^k \). Figures 3 and 4 illustrate the notion of a chain. It is noted that such topologies are of practical interest because of their applications in reach-avoid control problems [25, 36] such as motion control of robots in complex environments [6] and anaesthesia [21]. Now we place an additional restriction on the types of chains to be studied.

**Assumption 6.1.** Let \( \mathcal{P} := S^1 \cup \cdots \cup S^L \) be a chain. We assume that
\[
\begin{align*}
\{x \in \mathcal{P} \mid h^k_0 \cdot x \leq \alpha_k, h^{k+1}_0 \cdot x \geq \alpha_{k+1}\} &= \mathcal{S}^k, \quad k = 1, \ldots, L - 1, \\
\{x \in \mathcal{P} \mid h^L_0 \cdot x \leq \alpha_L\} &= \mathcal{S}^L.
\end{align*}
\]

**Example 6.1.** Assumption 6.1 says that only one simplex \( \mathcal{S}^k \) can lie in the intersection of the two half-spaces \( \{x \in \mathcal{P} \mid h^k_0 \cdot x \leq \alpha_k\} \) and \( \{x \in \mathcal{P} \mid h^{k+1}_0 \cdot x \geq \alpha_{k+1}\} \). Figure 3 shows examples in which Assumption 6.1 is satisfied. For instance, in Figure 3(a), it can be seen that \( \{x \in \mathcal{P} \mid h^{k+1}_0 \cdot x \geq \alpha_{k+1}\} \cap \{x \in \mathcal{P} \mid h^k_0 \cdot x \leq \alpha_k\} = \mathcal{S}^k \) for \( k = 1, \ldots, 4 \) and \( \{x \in \mathcal{P} \mid h^5_0 \cdot x \leq \alpha_5\} = \mathcal{S}^5 \). On the other hand, Figure 4 shows an example in which Assumption 6.1 is not satisfied because \( \{x \in \mathcal{P} \mid h^8_0 \cdot x \leq \alpha_8\} \neq \mathcal{S}^8 \). Intuitively, Assumption 6.1 requires that the given chain does not make a circulation in the state space. Such a circulation may be required by the control specification. Notice in this case the chain can be divided into two or more chains which do satisfy Assumption 6.1. Hence, Assumption 6.1 is not a significant restriction. \( \diamond \)

**Theorem 6.3.** Consider the system (4) defined on a chain \( \mathcal{P} := S^1 \cup \cdots \cup S^L \), and suppose that Assumption 6.1 holds. Let \( u(x) \) be a continuous PWA feedback which is affine on each \( S^i, i \in I_0 \), and let \( f(x) := Ax + Bu(x) + a \). If \( S^i \xrightarrow{\mathcal{S}^i} \mathcal{F}_0^i \) for \( i \in I_0 \) using \( u(x) \), then there exist affine functions \( V_i : \mathbb{R}^n \to \mathbb{R}, i \in I_0 \) such that
\[
V(x) = \max_{i \in I_0} \{V_i(x)\} \quad (33)
\]
satisfies $D_f^+ V(x) < 0$ for all $x \in \mathcal{P}$.

Proof. Because $S^1 \rightarrow \mathcal{F}_0^1$, by Theorem 4.1 there exists $\xi_1 \in \mathbb{R}^n$ such that

$$\xi_1 \cdot (A x + B u(x) + a) < 0, \quad x \in S^1.$$ \hfill (34)

We choose

$$V_1(x) := \xi_1 \cdot x.$$ 

Consider $S^k$ for $k \in \{2, \ldots, L\}$ and let $v^k_0$ be the vertex of $S^k$ not in $\mathcal{F}_0^k$. See Figure 5. By the same argument as in the proof of Theorem 6.1, we obtain

$$(-h^k_0) \cdot (A v^k_0 + B u(v^k_0) + a) < 0, \quad k \in \{2, \ldots, L\}.$$ \hfill (35)
Recall that $h^0 = \xi - c_0h^0_0$. Let

$$\xi_k := \xi_{k-1} - c_k h^0_k, \quad k \in \{2, \ldots, L\}. \quad (36)$$

Using (35), we choose $c_k > 0$ sufficiently large such that

$$\xi_k \cdot (Av^k_0 + Bu(v^k_0) + a) < 0, \quad k \in \{2, \ldots, L\}. \quad (37)$$

Now we show that for each $k \in I_0$ and for all $x \in S^k$, $\xi_k \cdot (Ax + Bu(x) + a) < 0$. We argue by induction. For the base step, we have (34). Next, suppose that

$$\xi_k \cdot (Ax + Bu(x) + a) < 0, \quad x \in S^k. \quad (38)$$

We must show $\xi_{k+1} \cdot (Ax + Bu(x) + a) < 0$ for all $x \in S^{k+1}$. Referring to Figure 5, because $F^k_{0+1}$ is a facet of $S^k$ which is not its exit facet, $u(x)$ is continuous, and $S^k \to F^k_0$, the invariance conditions (5) for $S^k$ hold at vertices $v_i \in F^k_{0+1}$. That is,

$$(-h^{k+1}_0 \cdot (Av_i + Bu(v_i) + a) \leq 0, \quad v_i \in F^k_{0+1}. \quad (39)$$

Using (36), (38), and (39), we get

$$\xi_{k+1} \cdot (Av_i + Bu(v_i) + a) < 0, \quad v_i \in F^k_{0+1}. \quad (40)$$

Since $u(x)$ is affine on $S^{k+1}$ and $S^{k+1} = \{v^k_{0+1}, v_i \mid v_i \in F^k_{0+1}\}$, (37) and (40) together imply $\xi_{k+1} \cdot (Ax + Bu(x) + a) < 0$ for all $x \in S^{k+1}$, as desired. Now we choose

$$V_k(x) := \xi_k \cdot x + \sum_{j=1}^{k-1} c_{j+1} \alpha_{j+1}, \quad k \in \{2, \ldots, L\} \quad (41)$$

and we let $V(x)$ be as in (33).

It remains to show $D^2_f V(x) < 0$ for all $x \in \mathcal{P}$. Our results above give

$$L_f V_k(x) = \xi_k \cdot (Ax + Bu(x) + a) < 0, \quad x \in S^k, \quad k \in I_0. \quad (42)$$

Recall that $h^0_k \cdot x = \alpha_k$ for $x \in F^k_0$ and that $h^0_k$ points outside of $S^k$. Also, by definition $V_k(x) - V_{k+1}(x) = (\xi_k - \xi_{k+1}) \cdot x - c_{k+1} \alpha_{k+1}$. Therefore for any $x \in \mathcal{P}$,

$$V_k(x) \geq V_{k+1}(x) \iff h^{k+1}_0 \cdot x \geq \alpha_{k+1}, \quad k \in \{1, \ldots, L - 1\} \quad (43a)$$

$$V_k(x) \geq V_{k-1}(x) \iff h^k_0 \cdot x \leq \alpha_k, \quad k \in \{2, \ldots, L\}. \quad (43b)$$
Define the sets
\[ \Gamma^k = \{ x \in \mathcal{P} \mid V_k(x) \geq V_j(x), \ j \in I_0 \} , \ \ k \in I_0. \] (44)

If we compare (43) with Assumption 6.1, we find that \( \Gamma^k \subseteq S^k \), \( k \in I_0 \). Consider any \( x \in \mathcal{P} \) and suppose \( k \in I_0(x) \). By the definition of \( I_0(x) \), \( V_k(x) \geq V_j(x) \), \( j \in I_0 \), which implies \( x \in \Gamma^k \subseteq S^k \). It follows from (42) that for all \( x \in \mathcal{P} \), if \( k \in I_0(x) \), then \( L_f V_k(x) < 0 \). According to Lemma 4.5, \( D^+_f V(x) = \max_{i \in I_0(x)} L_f V_i(x) \). We conclude \( D^+_f V(x) < 0 \) for all \( x \in \mathcal{P} \), as desired. \( \square \)

Analogous to the case of two simplices and Corollary 6.2, it is possible to verify existence of a flow function of the form (33) by solving an LP problem in the decision variables \( \xi_1, \xi_2, \ldots, \xi_L \). An illustrative example is given in Section 8.

### 7. Control Flow Functions

We have emphasized an analogy between Lyapunov functions for the equilibrium stability problem and flow functions for the reach control problem. The analogy will be deepened in this section, where we examine the Artstein-Sontag theorem based on control Lyapunov functions and we reinterpret it in the context of RCP. We introduce the notion of a control flow function. Like control Lyapunov functions, control flow functions convert a tool for analysis, flow functions, into a tool for synthesis. We begin with a non-constructive result on synthesis of PWA feedback following [2]. We then turn to constructive methods - the inspiration is the universal formulas of [30, 43, 42].

Let \( V : \mathbb{R}^n \rightarrow \mathbb{R} \) be a \( C^1 \) function. We say that \( V \) is a control flow function if for each \( x \in \mathcal{P} \) there exists \( u \in \mathbb{R}^m \) such that \( Ax + Bu + a \in \mathcal{C}(x) \) and
\[ \frac{\partial V}{\partial x}(Ax + Bu + a) < 0. \] (45)

Suppose it is known that a control flow function exists for the system (4) on \( \mathcal{P} \). We are interested in the question of whether this implies that RCP is solvable on \( \mathcal{P} \). Second, if it is solvable, is it possible to construct a feedback law? To that end, for each \( x \in \mathcal{P} \) define
\[ \mathcal{U}(x) := \{ u \in \mathbb{R}^m \mid Ax + Bu + a \in \mathcal{C}(x) \} \]
\[ \mathcal{U}^o(x) := \{ u \in \mathbb{R}^m \mid Ax + Bu + a \in \mathcal{C}^o(x) \} \]
\[ \mathcal{U}^{\text{flow}}(x) := \{ u \in \mathcal{U}(x) \mid \frac{\partial V}{\partial x}(Ax + Bu + a) < 0 \} \]
\[ (\mathcal{U}^{\text{flow}})^o(x) := \{ u \in \mathcal{U}^o(x) \mid \frac{\partial V}{\partial x}(Ax + Bu + a) < 0 \}. \]

Assuming \( \mathcal{U}(x) \neq \emptyset \) for all \( x \in \mathcal{P} \), then \( \mathcal{U} : \mathcal{P} \rightarrow 2^{\mathbb{R}^m} \) is a set-valued map with closed, convex values. If \( (\mathcal{U}^{\text{flow}})^o(x) \) is non-empty, then it is a convex set consisting of all control inputs satisfying both (45) and a strict form of the invariance conditions. In particular, velocity vectors must lie in the interior of the \( n \)-dimensional cone \( \mathcal{C}(x) \). In the sequel we assume that for each \( x \in \mathcal{P} \), \( \mathcal{U}^{\text{flow}}(x) \neq \emptyset \). Moreover in Theorems 7.2 and 7.5 we assume \( (\mathcal{U}^{\text{flow}})^o(x) \neq \emptyset, x \in \mathcal{P} \). The additional requirement to satisfy strict invariance conditions is needed when the flow function is not sufficiently smooth; whether it can be removed is an open problem.

We begin with a fact about \( \mathcal{C}(x) \). We have already discussed that \( \mathcal{C}(x) = T_f(x) \), assuming \( x \) is not in \( \mathcal{F}_0 \); they differ at points in \( \mathcal{F}_0 \). On compact, convex sets \( \mathcal{X}, x \mapsto T_x(x) \) is a lower semi-continuous set-valued map [17]. Not surprisingly, this also holds for \( x \mapsto \mathcal{C}(x) \).
Lemma 7.1. The map \( x \mapsto \mathcal{C}(x) \) is lower semi-continuous on \( \mathcal{P} \). Moreover, for each \( x \in \mathcal{P} \), there exists \( \delta > 0 \) such that for all \( x' \in \mathcal{B}_\delta(x) \cap \mathcal{P}, \mathcal{C}(x) \subset \mathcal{C}(x') \).

Proof. The second statement implies the first one, so we only prove the second statement. Let \( x \in \mathcal{P} \). If \( x \in \mathcal{P}^0 \) then there exists \( \delta > 0 \) such that for all \( x' \in \mathcal{B}_\delta(x) \cap \mathcal{P}, \mathcal{C}(x) = \mathcal{C}(x') = \mathbb{R}^n \). If \( x \in \partial \mathcal{P} \), suppose w.l.o.g. \( x \in \cap_{i=1}^k \mathcal{F}_i \) for some \( 1 \leq k \leq r \). Then \( h_j \cdot x < \alpha_j \) for \( j = k + 1, \ldots, r \), where \( \{ z \mid h_j \cdot z = \alpha_j \} \) is the hyperplane containing \( \mathcal{F}_j \), and \( \mathcal{C}(x) = \{ y \in \mathbb{R}^n \mid h_j \cdot y \leq 0, j = 1, \ldots, k \} \). Therefore, there exists \( \delta > 0 \) such that for all \( x' \in \mathcal{B}_\delta(x) \cap \mathcal{P}, h_j \cdot x' < \alpha_j \), for \( j = k + 1, \ldots, r \). That is, w.l.o.g. \( x' \in \cap_{i=1}^k \mathcal{F}_i \) for some \( 1 \leq k' \leq k \). Hence, \( \mathcal{C}(x) \subset \mathcal{C}(x') \).

Theorem 7.2. Consider the system (4) defined on a polytope \( \mathcal{P} \). Suppose there exists a \( \mathcal{C}^1 \) function \( V : \mathbb{R}^n \to \mathbb{R} \) such that for each \( x \in \mathcal{P} \), \( (U_{\text{flow}})^\circ(x) \neq \emptyset \). Then \( \mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0 \) by continuous PWA feedback.

Proof. The proof follows the construction in [2]. Let \( u_0(x) \) be any selection from the multi-valued map \( (U_{\text{flow}})^\circ(x), x \in \mathcal{P} \). Because \( V \) is \( \mathcal{C}^1 \), \( Ax + Bu + a \) is smooth in \( x \), and by Lemma 7.1, it follows that for each \( x \in \mathcal{P} \) and \( y \in \mathcal{P} \) sufficiently close to \( x \), \( \frac{\partial V}{\partial x}(y)(Ay + Bu_0(x) + a) < 0 \) and \( Ay + Bu_0(x) + a \in \mathcal{C}(x) \subset \mathcal{C}(y) \). Consequently, for each \( x \in \mathcal{P} \) and \( y \in \mathcal{P} \) sufficiently close to \( x \), \( u_0(x) \in (U_{\text{flow}})^\circ(y) \). In particular, for each \( x \in \mathcal{P} \), we can find a ball \( \mathcal{B}_0(x) \cap \mathcal{P} \) centered at \( x \) and open in \( \mathcal{P} \), such that for all \( y \in \mathcal{B}_0(x) \cap \mathcal{P} \), \( u_0(x) \in (U_{\text{flow}})^\circ(y) \). Then \( \{ \mathcal{B}_0(x) \} \) is an open cover of \( \mathcal{P} \) and since \( \mathcal{P} \) is compact, there exists a finite subcover \( \{ \mathcal{B}_0^n \mid i \in I_0 \} \), where \( I_0 = \{1, \ldots, L \} \) and \( \mathcal{B}_0 = \mathcal{B}_0(x') \) for some \( x' \in \mathcal{P} \). By Theorem 16.4 of [35] there exists a triangulation \( \mathcal{T} \) of \( \mathcal{P} \) with vertex set \( \mathbb{T}_0 \) such that \( \mathcal{T} \) refines the open cover \( \{ \mathcal{B}_0^n \mid i \in I_0 \} \). That is, for each \( w \in \mathbb{T}_0 \), \( \mathcal{S}(w) \subset \mathcal{B}_0^n \) for some \( j \in I_0 \).

Next consider any \( x \in \mathcal{T}_0 \). Then \( x = \sum_{i=1}^l \beta_i^x w_{j_i} \) where \( (\beta_1^x, \ldots, \beta_l^x) \) are the barycentric coordinates of \( x \) and \( \{ w_{j_i} \} \) are the vertices of \( \mathcal{S}_x \) as reviewed in Section 2. Define the control \( u(x):= \sum_{i=1}^l \beta_i^x u(w_{j_i}) \). Clearly, \( u : \mathcal{P} \to \mathbb{R}^m \) is a continuous PWA feedback on \( \mathcal{P} \). Moreover, because \( x \in \mathcal{S}(w_{j_i}) \subset \mathcal{B}_0^{\kappa(w_{j_i})} \), we have \( u(w_{j_i}) = u_0(x^{\kappa(w_{j_i})}) \in (U_{\text{flow}})^\circ(x), i = 1, \ldots, l \). By convexity of \( (U_{\text{flow}})^\circ(x) \) we conclude that \( u(x) \in (U_{\text{flow}})^\circ(x) \).

In sum, we have shown there exists a continuous PWA feedback \( u(x) \) on \( \mathcal{P} \) such that for all \( x \in \mathcal{P} \),

\[
Ax + Bu(x) + a \in \mathcal{C}(x),
\]

\[
\frac{\partial V}{\partial x}(x)(Ax + Bu(x) + a) < 0.
\]

The PWA feedback \( u(x) \) can be affinely extended to a neighborhood of \( \mathcal{P} \) such that it is locally Lipschitz on this neighborhood [11]. Therefore the closed-loop vector field \( Ax + Bu(x) + a \) is locally Lipschitz on a neighborhood of \( \mathcal{P} \). By Theorem 4.1 of [24], closed-loop trajectories cannot exit \( \mathcal{P} \) from non-exit facets. By Proposition 4.2, closed-loop trajectories must exit \( \mathcal{P} \) in finite time. We conclude \( \mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0 \) using \( u(x) \). □
Remark 7.1. Although the additional requirement of achieving the strict invariance conditions was imposed in Theorem 7.2 for technical reasons, this requirement is also important in practice. In particular, it was shown in [40] that achieving the strict invariance conditions is necessary for robustness of the RCP solution against small perturbations in system parameters.

The next problem we investigate is whether a control flow function can be used to explicitly construct the feedback solving RCP. We seek a “universal formula” associated with RCP. Because this problem has not been posed before, the main result (Theorem 7.4) is stated only for single-input systems. Already the formulas in [30, 43, 42] provide feedbacks that ensure that a function \( V \) is strictly decreasing along closed-loop trajectories. We are not able to directly adopt those formulas because in RCP we have the added requirement that the invariance conditions must hold for the proposed feedback. The latter involves a more careful analysis of the range of control values permissible for each \( x \in \mathcal{P} \). To that end, we begin with the following technical result.

Lemma 7.3. Consider the system (4) defined on a polytope \( \mathcal{P} \). Suppose there exists a \( \mathcal{C}^1 \) function \( V : \mathbb{R}^n \to \mathbb{R} \) such that for each \( x \in \mathcal{P} \), \( \mathcal{U}^{flow}(x) \neq \emptyset \). Then there exists \( \alpha > 0 \) such that for each \( x \in \mathcal{P} \), there exists \( u \in \mathbb{R}^m \) satisfying

\[
\frac{\partial V}{\partial x}(x)(Ax + Bu + a) \leq -\alpha \quad (46a)
\]

\[
Ax + Bu + a \in \mathcal{C}(x). \quad (46b)
\]

Proof. Define the extended real-valued function \( \chi : \mathcal{P} \to [-\infty, +\infty] \) by

\[
\chi(x) := \inf_{w \in \mathcal{U}^{flow}(x)} \frac{\partial V}{\partial x}(x)(Ax + Bu + a), \quad x \in \mathcal{P}.
\]

Since \( \mathcal{U}^{flow}(x) \neq \emptyset \) for all \( x \in \mathcal{P} \), we know that \( \chi(x) \in [-\infty, 0) \), \( x \in \mathcal{P} \). We claim that \( \chi(x) \) is an upper semi-continuous function on \( \mathcal{P} \).

First consider \( x_0 \in \mathcal{P} \) such that \( -\infty < \chi(x_0) < 0 \). We must show that for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that if \( x \in \mathcal{B}_\delta(x_0) \cap \mathcal{P} \), then \( \chi(x) < \chi(x_0) + \epsilon \). It can be shown that there exists \( u_0 \in \mathcal{U}(x_0) \) such that \( \frac{\partial V}{\partial x}(x_0)(Ax_0 + Bu_0 + a) = \chi(x_0) \); we omit the argument. By assumption there exist \( u_i \in \mathbb{R}^m \), \( i = 1, \ldots, p \), an assignment of control inputs at the vertices of \( \mathcal{P} \) satisfying (5). (Note that if \( x_0 = v_i \), a vertex of \( \mathcal{P} \), then assign \( u_i := u_0 \).)

If \( x_0 \) is not a vertex of \( \mathcal{P} \), consider the point set \( \{x_0, v_1, \ldots, v_p\} \). If \( x_0 \) is a vertex of \( \mathcal{P} \), then consider the point set \( \{v_1, \ldots, v_p\} \). We can construct a triangulation \( \mathcal{T} \) of \( \mathcal{P} \) based on the selected point set which satisfies that for each simplex \( \mathcal{S}_j \in \mathcal{T} \), if \( x_0 \in \mathcal{S}_j \), then \( x_0 \) is a vertex of \( \mathcal{S}_j \) [32]. Based on the control inputs \( \{u_0, u_1, \ldots, u_p\} \) (or \( \{u_1, \ldots, u_p\} \) if \( x_0 \) is a vertex of \( \mathcal{P} \), we form on each \( n \)-dimensional simplex in the triangulation the unique affine feedback [24]. This yields a continuous PWA feedback on \( \mathcal{P} \); denote it \( u'(x) \). By a standard convexity argument, \( u'(x) \) satisfies the invariance conditions (6) of \( \mathcal{P} \). Also, \( u'(x_0) = u_0 \), so \( \frac{\partial V}{\partial x}(x_0)(Ax_0 + Bu'(x_0) + a) = \chi(x_0) < 0 \). Because \( V(x) \) is \( \mathcal{C}^1 \) and \( u' \) is continuous on \( \mathcal{P} \), \( \frac{\partial V}{\partial x}(x)(Ax + Bu'(x) + a) \) is continuous on \( \mathcal{P} \). Hence, for each \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for all \( x \in \mathcal{B}_\delta(x_0) \cap \mathcal{P} \), \( \frac{\partial V}{\partial x}(x)(Ax + Bu'(x) + a) < \chi(x_0) + \epsilon \). Since \( u'(x) \) satisfies (6), \( u'(x) \in \mathcal{U}(x) \). Then by definition of \( \chi(x) \),

\[
\chi(x) \leq \frac{\partial V}{\partial x}(x)(Ax + Bu'(x) + a) < \chi(x_0) + \epsilon, \quad x \in \mathcal{B}_\delta(x_0) \cap \mathcal{P}.
\]
Second, consider \( x_0 \in \mathcal{P} \) such that \( \chi(x_0) = -\infty \). We must show that for each \( c < 0 \) there exists \( \delta > 0 \) such that for all \( x \in \mathcal{B}_\delta(x_0) \cap \mathcal{P}, \chi(x) < c \). Because \( \chi(x_0) = -\infty \), there exists a sequence \( \{u^k\} \) such that \( Ax_0 + Bu^k + a \neq 0 \) and

\[
\frac{\partial V}{\partial x}(x_0)(Ax_0 + Bu^k + a) \longrightarrow -\infty \tag{47a}
\]

Define

\[
b^k := \frac{Ax_0 + Bu^k + a}{\|Ax_0 + Bu^k + a\|} \in \mathcal{C}(x_0).
\]

Since \( \{b^k\} \) is a bounded sequence, it has a convergent subsequence \( \{b^{k_j}\} \) such that \( b^{k_j} \rightarrow \overline{b} \). Since \( \mathcal{C}(x_0) \) is closed, \( \overline{b} \in \mathcal{C}(x_0) \). However, (47a) implies \( \|Ax_0 + Bu^{k_j} + a\| \rightarrow \infty \) so \( \frac{Ax_0 + a}{\|Ax_0 + Bu^{k_j} + a\|} \rightarrow 0 \). Then \( \overline{b} = \lim_{j \to \infty} \frac{Bu^{k_j}}{\|Ax_0 + Bu^{k_j} + a\|} \in \mathcal{B} \cap \mathcal{C}(x_0) \). Also by (47a), \( \frac{\partial V}{\partial x}(x_0)\overline{b} < 0 \). Fix \( c < 0 \). By Lemma 7.1, there exists \( \delta_1 > 0 \) such that for all \( x \in \mathcal{B}_{\delta_1}(x_0) \cap \mathcal{P}, \mathcal{C}(x_0) \subset \mathcal{C}(x) \). Hence \( \overline{b} \in \mathcal{B} \cap \mathcal{C}(x) \). Also, there exists \( \delta_2 > 0 \) such that for all \( x \in \mathcal{B}_{\delta_2}(x_0) \cap \mathcal{P}, \frac{\partial V}{\partial x}(x)\overline{b} < 0 \). Let \( \delta := \min\{\delta_1, \delta_2\} \). For \( x \in \mathcal{B}_{\delta}(x_0) \cap \mathcal{P} \), let \( u' \) be such that \( Ax + Bu' + a \in \mathcal{C}(x) \). Then define \( u = u' + c\overline{b}, \) where \( c' > 0 \) and \( \overline{b} = B\overline{a} \). Then for all \( c' > 0 \)

\[
Ax + Bu + a = (Ax + Bu' + a) + c'\overline{b} \in \mathcal{C}(x)
\]

and for \( c' > 0 \) sufficiently large

\[
\frac{\partial V}{\partial x}(x)(Ax + Bu + a) = \frac{\partial V}{\partial x}(x)(Ax + Bu' + a) + c' \left( \frac{\partial V}{\partial x}(x)\overline{b} \right) < c.
\]

We conclude for each \( c < 0 \) there exists \( \delta > 0 \) such that for all \( x \in \mathcal{B}_{\delta}(x_0) \cap \mathcal{P}, \chi(x) < c, \) as desired.

We have shown \( \chi(x) \) is upper semi-continuous on \( \mathcal{P} \). Since \( \mathcal{P} \) is compact, by Theorem 2.1 of [41], \( \chi(x) \) attains a maximum on \( \mathcal{P} \). That is, there exists \( \alpha > 0 \) such that \( \chi(x) \leq -\alpha \) for all \( x \in \mathcal{P} \). \( \square \)

The next result provides a universal formula for RCP for the case of single-input systems.

**Theorem 7.4.** Consider the system (4) defined on a polytope \( \mathcal{P} \). Suppose that \( m = 1 \). Also, suppose there exists a \( \mathcal{C}^2 \) function \( V : \mathbb{R}^n \to \mathbb{R} \) such that for each \( x \in \mathcal{P}, U_{\text{flow}}(x) \neq \emptyset \). Then \( \mathcal{P} \overset{\mathcal{P}}{\longrightarrow} \mathcal{F}_0 \) by continuous state feedback.

**Proof.** By Lemma 7.3, there exists \( \alpha > 0 \) such that for each \( x \in \mathcal{P} \) there exists \( u \in \mathbb{R} \) such that (46) hold. Hence, there exist inputs \( u_i \in \mathcal{U}(v_i), i = 1, \ldots, p \). Triangulate \( \mathcal{P} \) into simplices using only vertices of \( \mathcal{P} \), and form on each simplex the unique affine feedback based on the control values \( u_i, i = 1, \ldots, p \). This yields a continuous PWA feedback \( u^{\text{inv}}(x) \) such that, by convexity, \( Ax + Bu^{\text{inv}}(x) + a \in \mathcal{C}(x) \) for all \( x \in \mathcal{P} \). Apply the feedback transformation \( u = u^{\text{inv}}(x) + w \) to (4) to obtain the new system

\[
\dot{x} = Ax + Bu^{\text{inv}}(x) + Bw + a.
\]

We claim there exists \( \alpha > 0 \) such that for each \( x \in \mathcal{P} \), there exists \( w \in \mathbb{R} \) such that

\[
\frac{\partial V}{\partial x}(x)(Ax + Bu^{\text{inv}}(x) + Bw + a) \leq -\alpha \tag{49a}
\]

\[
Ax + Bu^{\text{inv}}(x) + Bw + a \in \mathcal{C}(x).
\]

(49b)
Proof of claim: Fix $x \in \mathcal{P}$ and let $u \in \mathbb{R}$ satisfy (46). Define $w := u - u^{\text{inv}}(x)$. Then (49) immediately follow.

Let $f(x) := Ax + Bu^{\text{inv}}(x) + a$ and $g(x) := B$. Define the feedback $^1$

$$u^{\text{flow}}(x) = \begin{cases} 0, & L_f V(x) \leq -\frac{\alpha}{2}, \\ -(L_f V(x) + \frac{\alpha}{2}), & L_f V(x) > -\frac{\alpha}{2}. \end{cases}$$

Observe that by (49a), if $L_g V(x) = 0$, then $L_f V(x) \leq -\alpha$. That is,

$$L_f V(x) > -\alpha \implies L_g V(x) \neq 0.$$  

Using this fact, it can be shown that $u^{\text{flow}}(x)$ is continuous and locally Lipschitz. Define the feedback

$$u(x) = u^{\text{inv}}(x) + u^{\text{flow}}(x).$$

We show that for each $x \in \mathcal{P}$,

$$\frac{\partial V}{\partial x}(Ax + Bu(x) + a) \leq -\frac{\alpha}{2} \quad \text{(52a)}$$

$$Ax + Bu(x) + a \in \mathcal{C}(x).$$  

(52b)

There are two cases:

(i) $L_f V(x) \leq -\frac{\alpha}{2}$. In this case $u(x) = u^{\text{inv}}(x)$. Then $\frac{\partial V}{\partial x}(Ax + Bu(x) + a) = L_f V(x) \leq -\frac{\alpha}{2}$. Also, by construction of $u^{\text{inv}}(x)$, $Ax + Bu(x) + a \in \mathcal{C}(x)$.

(ii) $L_f V(x) > -\frac{\alpha}{2}$. In this case $u(x) = u^{\text{inv}}(x) - \frac{(L_f V(x) + \frac{\alpha}{2})}{L_g V(x)}$. By direct substitution,

$$\frac{\partial V}{\partial x}(Ax + Bu(x) + a) = -\frac{\alpha}{2}.$$  

Next, there exists $w \in \mathbb{R}$ satisfying (49). If $L_g V(x) > 0$, then

$$w \leq \frac{-(L_f V(x) + \alpha)}{L_g V(x)} < \frac{-(L_f V(x) + \frac{\alpha}{2})}{L_g V(x)} = u^{\text{flow}}(x) < 0.$$  

Otherwise if $L_g V(x) < 0$, then

$$0 < u^{\text{flow}}(x) = \frac{-(L_f V(x) + \frac{\alpha}{2})}{L_g V(x)} < \frac{-(L_f V(x) + \alpha)}{L_g V(x)} \leq w.$$  

Consequently, in either case there exists $\lambda \in (0, 1)$ such that $u^{\text{flow}}(x) = \lambda w$, so

$$u(x) = (1 - \lambda)u^{\text{inv}}(x) + \lambda(u^{\text{inv}}(x) + w).$$

By the construction of $u^{\text{inv}}(x)$, (49b), and convexity of $\mathcal{C}(x)$, $Ax + Bu(x) + a \in \mathcal{C}(x)$.

By (52a), the continuous function $V(x)$ is strictly decreasing along the closed-loop trajectories of (4) in compact $\mathcal{P}$. Hence, all closed-loop trajectories starting in $\mathcal{P}$ leave it in finite time. Because $u(x)$ is locally Lipschitz on a neighborhood of $\mathcal{P}$, so is the closed-loop vector field. Then by Theorem 4.1 of [24], trajectories that leave $\mathcal{P}$ do so only through $\mathcal{F}_0$. We conclude $\mathcal{P} \xrightarrow{\mathcal{F}} \mathcal{F}_0$ by the continuous state feedback $u = u^{\text{inv}} + u^{\text{flow}}$. \qed

The feedback $u = u^{\text{inv}} + u^{\text{flow}}$ consists of a continuous PWA feedback $u^{\text{inv}}$ and a locally Lipschitz feedback $u^{\text{flow}}$ so the closed-loop vector field $Ax + Bu(x) + a$ is locally Lipschitz. This fact relies on $V$ being a $\mathcal{G}^2$ function. When only a $\mathcal{G}^1$ control flow function is available, then $u^{\text{flow}}$ and the closed-loop vector field are continuous. In this case solutions exist for each initial condition in $\mathcal{P}$ [31], but uniqueness of solutions is not guaranteed. Results such

\footnote{Here we adopt the same notation used in the universal formulas in [43, 42].}
as Proposition 4.2 do not require uniqueness of solutions. On the other hand, Theorem 4.1 of [24], which tells us that closed-loop trajectories cannot exit \( \mathcal{P} \) from non-exit facets when \( Ax + Bu(x) + a \in \mathcal{C}(x), \ x \in \mathcal{P} \), cannot be used because it requires that the closed-loop vector field be locally Lipschitz on a neighborhood of \( \mathcal{P} \). In order to overcome this obstacle, we introduce in the next result the stronger requirement, already used in Theorem 7.2, that \( Ax + Bu(x) + a \in \mathcal{C}^0(x), \ x \in \mathcal{P} \). The requirement can be relaxed if it is known that \( u^{\text{flow}} \) is locally Lipschitz.

**Theorem 7.5.** Consider the system (4) defined on a polytope \( \mathcal{P} \). Suppose that \( m = 1 \). Also, suppose there exists a \( \mathcal{C}^1 \) function \( V : \mathbb{R}^n \to \mathbb{R} \) such that for each \( x \in \mathcal{P} \), \( (u^{\text{flow}})^c(x) \neq \emptyset \). Then \( \mathcal{P} \xrightarrow{\mathcal{P}} \mathcal{F}_0 \) by continuous state feedback.

**Proof.** The proof closely follows that of Theorem 7.4, so we only highlight the differences. By Lemma 7.3, there exists \( \alpha' > 0 \) such that for each \( x \in \mathcal{P} \), there exists \( u' \in \mathbb{R} \) such that

\[
\frac{\partial V}{\partial x}(x)(Ax + Bu' + a) \leq -\alpha' \\
Ax + Bu' + a \in \mathcal{C}(x) .
\]

Also by assumption for each \( x \in \mathcal{P} \), \( \mathcal{U}^c(x) \neq \emptyset \). In particular, choose \( u''_i \in \mathcal{U}^c(v_i), \ i \in I \), and based on these selected control inputs construct a continuous PWA feedback \( u''(x) \) on \( \mathcal{P} \) such that, by convexity, \( u''(x) \in \mathcal{U}^c(x), \ x \in \mathcal{P} \) [24]. Since \( \frac{\partial V}{\partial x}(x)Bu''(x) \) is continuous on \( \mathcal{P} \) and \( \mathcal{P} \) is compact, there exists \( M > 0 \) such that \( \frac{\partial V}{\partial x}(x)Bu''(x) \leq M, \ x \in \mathcal{P} \). Let \( \epsilon := \frac{\alpha'}{2M} \) and define \( u := u' + \epsilon u''(x) \). Then \( Ax + Bu + a \in \mathcal{C}^0(x) \) and

\[
\frac{\partial V}{\partial x}(x)(Ax + Bu + a) = \frac{\partial V}{\partial x}(x)(Ax + Bu' + a) + \epsilon \frac{\partial V}{\partial x}(x)Bu''(x) \leq -\alpha' + \epsilon M = -\frac{\alpha'}{2} .
\]

Let \( \alpha := \frac{\alpha'}{2} \). We conclude there exists \( \alpha > 0 \) such that for each \( x \in \mathcal{P} \) there exists \( u \in \mathbb{R} \) such that

\[
\frac{\partial V}{\partial x}(x)(Ax + Bu + a) \leq -\alpha \quad (53a) \\
Ax + Bu + a \in \mathcal{C}^0(x) . \quad (53b)
\]

Now the proof proceeds exactly as in the proof of Theorem 7.4 to obtain a feedback \( u = u^{\text{inv}} + u^{\text{flow}} \); the only difference is that \( u^{\text{inv}} \) is constructed to satisfy \( Ax + Bu^{\text{inv}}(x) + a \in \mathcal{C}^0(x), \ x \in \mathcal{P} \). Hence, here \( u \) satisfies \( Ax + Bu(x) + a \in \mathcal{C}^0(x), \ x \in \mathcal{P} \). We verify that all closed-loop trajectories exit \( \mathcal{P} \) by applying Proposition 4.2.

Finally, we must show trajectories exit from \( \mathcal{F}_0 \). Let \( \mathcal{F}_j = \{ x \in \mathbb{R}^n \mid h_j \cdot x = \alpha_j \} \cap \mathcal{P} \) for \( j \in J \). Let \( \phi_u(t, x_0) \) be any solution of \( \dot{x} = Ax + Bu(x) + a \) starting at \( x_0 \in \mathcal{P} \). We claim that for each \( j \in J \), if \( x_0 \in \mathcal{F}_j \), then there exists \( T_j > 0 \) such that for all \( t \in (0, T_j] \), \( \phi_u(t, x_0) \in \{ x \in \mathbb{R}^n \mid h_j \cdot x < \alpha_j \} \) (where it is assumed that \( Ax + Bu(x) + a \) is continuously extended to a neighborhood of \( \mathcal{P} \)). From this it follows that for each \( x_0 \in \mathcal{P} \), there exists \( T_0 > 0 \) such that for all \( t \in (0, T_0] \), \( \phi_u(t, x_0) \in \{ x \in \mathbb{R}^n \mid h_j \cdot x < \alpha_j, j \in J \} \). Therefore, if \( \phi_u(t, x_0) \) exits \( \mathcal{P} \), it must do so through \( \mathcal{F}_0 \).

**Proof of claim:** Fix \( j \in J \) and let \( x_0 \in \mathcal{F}_j \). Suppose by way of contradiction that for all \( T_j > 0 \), there exists \( t_j \in (0, T_j] \) such that \( \phi_u(t_j, x_0) \in \mathcal{H}_j := \{ x \in \mathbb{R}^n \mid h_j \cdot x \geq \alpha_j \} \). By Proposition 3.4.1 of [5], this implies \( Ax_0 + Bu(x_0) + a \in T_{\mathcal{H}_j}(x_0) \), where \( T_{\mathcal{H}_j}(x_0) \) is the
Remark 7.2. The proof of Theorem 7.4 introduces an interesting concept in feedback design for RCP. It proposes that the design may be carried out in two steps: first construct a continuous feedback $u$ in $[30, 43, 42]$ provided for RCP in the proof of Theorem 7.4 and the universal formulas for stabilization [38, 25, 27, 28] where the two requirements are treated together. Future work will explore the ramifications of this approach.

Remark 7.3. One can observe the analogy between the explicit continuous feedback $u(x)$ provided for RCP in the proof of Theorem 7.4 and the universal formulas for stabilization in [30, 43, 42]. However, while the universal formulas in [30, 43, 42] ensure that a $C^1$ function $V$ is strictly decreasing along closed-loop trajectories, they do not necessarily satisfy the invariance conditions of $P$, and so they do not solve RCP. Instead, $u(x)$ used in the proof of Theorem 7.4 was selected more carefully to simultaneously satisfy the invariance conditions and the requirement that a $C^1$ function $V$ is strictly decreasing along closed-loop trajectories. The cost we pay for the added requirement of achieving the invariance conditions is that unlike [42, 30], our explicit feedback $u(x)$ is not necessarily real analytic.

8. Examples

Example 8.1. In this example we show how to use our results in Section 6 to check if a given continuous PWA feedback solves RCP on a connected chain of simplices. Consider the system

\[
\dot{x} = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} -6 \\ 2 \end{bmatrix}
\]  

(54)

defined on the chain of simplices shown in Figure 6.

The vertices are $v_1 = (6, 6)$, $v_2 = (4, 6)$, $v_3 = (6, 4)$, $v_4 = (3, 4)$, $v_5 = (2, 5)$, $v_6 = (2, 3)$, $v_7 = (1, 4)$, $v_8 = (-1, 3)$, $v_9 = (1, 1)$, and $v_{10} = (0, 0)$. Let $P = S^1 \cup \cdots \cup S^8$, and the exit facet $F^1_0 = \text{co} \{v_1, v_2\}$. It is required to solve $P \xrightarrow{P} F^1_0$ by continuous PWA feedback.

We check solvability of the problem by simplex-based methods [25], [6]. In particular, it is required to achieve $S^i \xrightarrow{S^i} F^1_0$ by affine feedback, $i \in \{1, \cdots, 8\}$. We examine invariance conditions of the simplex $S^8$ at the vertex $v_8$. We have $f(v_8) = Av_8 + Bu_8 + a = (-8 + u_8, -2)$. The normal vector to the facet $F_7 = \text{co} \{v_6, v_8\}$ in $S^8$ is $h_7 = (0, -1)$. The invariance conditions of $S^i$ at $v_8$ yield $h_7 \cdot f(v_8) \leq 0$. Equivalently, $2 \leq 0$, which is impossible. Thus, $S^8 \xrightarrow{S^8} F^6_0$ is not solvable by affine feedback, and RCP is not solvable by simplex-based methods.

Despite this failure, we show that a continuous PWA feedback on the same triangulation solves RCP on $P$ by verifying the existence of a flow function on $P$. Suppose that the following control assignment is selected: $u(v_1) = -6$, $u(v_2) = 8$, $u(v_4) = -6$, $u(v_4) = 1$, $u(v_5) = 12$, $u(v_6) = 2.5$, $u(v_7) = 4.25$, $u(v_8) = 18$, $u(v_9) = 5$, and $u(v_{10}) = 5.5$. Then, the unique affine feedback is constructed on each simplex [24]. Now it is required to check if this continuous PWA feedback $u(x)$ solves RCP on $P$. Let $f(x) := Ax + Bu(x) + a$. 


Then \( f(v_1) = (0, 2) \), \( f(v_2) = (10, 0) \), \( f(v_3) = (0, 4) \), \( f(v_4) = (1, 1) \), \( f(v_5) = (10, -1) \),

\( f(v_6) = (0.5, 1) \), \( f(v_7) = (0.25, -1) \), \( f(v_8) = (10, -2) \), \( f(v_9) = (1, 2) \), and \( f(v_{10}) = (-0.5, 2) \).

It can be verified that the invariance conditions are satisfied, as shown by the velocity vectors \( f(v_i) \) shown in blue in Figure 6. Hence, all trajectories that leave \( \mathcal{P} \) do so via \( F_0^1 \). It remains to check if all closed-loop trajectories initiated in \( \mathcal{P} \) leave it in finite time.

First, we check if \( u(x) \) constructs a linear flow function on \( \mathcal{P} \). It can be easily verified that \( \frac{2}{3} f(v_7) + \frac{1}{3} f(v_{10}) = 0 \). Therefore, there does not exist \( \xi \in \mathbb{R}^n \) such that \( \xi \cdot f(x) < 0 \), \( x \in \mathcal{P} \).

Then we check the existence of a flow function \( V(x) \) in the form (33). It can be verified that \( h_0^3 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \), \( \alpha_2 = \frac{10}{\sqrt{2}} \), \( h_0^3 = (\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}) \), \( \alpha_3 = \frac{2}{\sqrt{5}} \), \( h_0^4 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \), \( \alpha_4 = \frac{7}{\sqrt{2}} \), \( h_0^5 = (1, 0) \), \( \alpha_5 = 2 \), \( h_0^6 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \), \( \alpha_6 = \frac{5}{\sqrt{2}} \), \( h_0^7 = (0, 1) \), \( \alpha_7 = 3 \), \( h_0^8 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \), and \( \alpha_8 = \sqrt{2} \). We solve the following LP to verify the existence of a flow function in the form (33).

Find \( \xi_1, c_2, \cdots, c_8 \) such that

\[
\xi_1 \cdot f(v_i) < 0, \ v_i \in S^1
\]

\[
(\xi_1 - \sum_{j=1}^{i-1} c_{j+1} h_0^{j+1}) \cdot f(v_{n+i}) < 0, \ i = 2, \cdots, 8
\]

\[
(\xi_1 - \sum_{j=1}^{5} c_{j+1} h_0^{j+1}) \cdot f(v_7) < 0
\]

\[
(\xi_1 - \sum_{j=1}^{6} c_{j+1} h_0^{j+1}) \cdot f(v_8) < 0
\]
The inequalities (55b) ensure that $c_i > 0$, $i = 2, \cdots, 8$, are selected sufficiently large as in the proof of Theorem 6.3. The inequalities (55c) and (55d) are added because $f(v_7)$ and $f(v_8)$ violate the invariance conditions of the simplices $S^5$ and $S^6$ respectively. The LP is solvable, and a solution is $\xi_1 = (-7.1782, -9.4655)$, $c_2 = 3.2771$, $c_3 = 24.9878$, $c_4 = 2.6109$, $c_5 = 5.2658$, $c_6 = 1.6826$, $c_7 = 79.2398$, and $c_8 = 73.3324$. Then by (36), we find $\xi_2 = (-9.4955, -11.7828)$, $\xi_3 = (-31.8452, -0.6079)$, $\xi_4 = (-33.6914, -2.4541)$, $\xi_5 = (-38.9572, -2.4541)$, $\xi_6 = (-40.1469, -3.6439)$, $\xi_7 = (-40.1469, -82.8837)$, and $\xi_8 = (-92.008, -134.7375)$. We define

$$V(x) = \max_{i \in \{1, \cdots, 8\}} \xi_i \cdot x + \sum_{j=1}^{i-1} c_{j+1} \alpha_{j+1}.$$  

For any $x \in \Gamma^1$, we have by definition $V_1(x) \geq V_2(x)$, which can be shown to be equivalent to $h_0^2 \cdot x \geq \alpha_2$. Hence, $x \in \{ x \in P \mid h_0^2 \cdot x \geq \alpha_2 \} = S^1$. Therefore, $\Gamma^1 \subseteq S^1$. Also, it can be verified that $L_f V_1(v_i) = \xi_1 \cdot f(v_i) < 0$, $i = 1, \cdots, 3$. Since $f(x)$ is affine on $S^1$, we have $L_f V_1(x) = \xi_1 \cdot f(x) < 0$, $x \in S^1$. In the same way, it can be verified that for $i \in \{2, \cdots, 8\}$, we have $\Gamma^i \subseteq S^i$ and $L_f V_i(x) < 0$, $x \in S^i$. This implies from Lemma 4.5 that $D_f^+ V(x) < 0$, $x \in P$. Then by Theorem 4.6 (which also applies to nonconvex sets), all closed-loop trajectories exit $P$ in finite time. Since the invariance conditions hold, they do so only through $F_0$. We conclude that $u(x)$ solves RCP on $P$.

Example 8.2. In this example we show how to use our proposed results in this paper to analyze reachability problems of multi-affine systems on rectangles, which receive special interest because of their applications in the control of multi-affine hybrid systems [7], [8]. Consider the multi-affine vector field

$$\dot{x} = f(x) = (-x_1 - x_1 x_2 + 1, -3 x_1 - 3 x_2 + 2 x_1 x_2 + 2)$$  

(56)

defined on a rectangle $P$. The rectangle is shown in Figure 7. The vertices of $P$ are $v_1 = (0, 0), v_2 = (1, 0), v_3 = (1, 1)$, and $v_4 = (0, 1)$. The exit facet is $F_0 = \text{co} \{v_1, v_2\}$. It is required to check if $P \xrightarrow{F_0} F_0$.

By direct computation, we get $f(v_1) = (1, 2)$, $f(v_2) = (0, -1)$, $f(v_3) = (-1, -2)$, and $f(v_4) = (1, -1)$. It can be verified that $f(v_i)$, $i = 1, \cdots, 4$, satisfy the invariance conditions (5). By Corollary 1 of [8], this implies that invariance conditions (6) are achieved. Therefore, trajectories that leave $P$ do so only via $F_0$.

Now it remains to check if all trajectories initiated in $P$ leave it in finite time. First, in [8] the sufficient condition $h_0 \cdot f(x) > 0$, $x \in P$, was used to check leaving $P$ in finite time. However, in this example it can be easily verified that $h_0 \cdot f(v_1) < 0$.

Secondly, we check if $f(x)$ admits a linear flow function on $P$. It can be verified that $0.5 f(v_1) + 0.5 f(v_3) = 0$, or $0 \in \text{co} \{ f(v_1), \cdots, f(v_4) \}$. This implies there does not exist $\xi \in \mathbb{R}^2$ such that $\xi \cdot f(x) < 0$, $x \in P$.

Now we show using LaSalle Principle for RCP that all trajectories initiated in $P$ leave it in finite time. Let $V(x) = \xi \cdot x$, where $\xi = (-1, 0.5)$. It can be verified that $\xi \cdot f(v_i) \leq 0$, $v_i \in P$. The equality holds only at $v_1$ and $v_3$. By the convexity of the multi-affine vector field on $P$ (Proposition 2 of [8]), we get $L_f V(x) = \xi \cdot f(x) \leq 0$, $x \in P$. 

\[ -c_i < 0, \ i = 2, \cdots, 8. \]  

(55e)
Then, we identify the set $\mathcal{M}$. The vector field $f(x)$ can be expressed as [8]:

$$f(x) = \sum_{i=1}^{4} \alpha_x^i f(v_i),$$

where $\alpha_x^1 = (1-x_1)(1-x_2)$, $\alpha_x^2 = x_1(1-x_2)$, $\alpha_x^3 = x_1x_2$, and $\alpha_x^4 = (1-x_1)x_2$. In particular, we have

$$\xi \cdot f(x) = \sum_{i=1}^{4} \alpha_x^i \xi \cdot f(v_i). \quad (57)$$

Recall that $\xi \cdot f(v_i) = 0$, $i = 1, 3$, and $\xi \cdot f(v_i) < 0$, $i = 2, 4$. Since for any $x \in \mathcal{P}$ $\alpha_x^i \geq 0$, $i = 1, \cdots, 4$, it follows from (57) that $\xi \cdot f(x) = 0$ is achieved only at $x \in \mathcal{P}$ satisfying $\alpha_x^2 = 0$ and $\alpha_x^4 = 0$ simultaneously. But, this happens only at $v_1$ and $v_3$. We conclude $\mathcal{M} = \{v_1\} \cup \{v_3\}$. Since $f(v_1) \neq 0$ and $f(v_3) \neq 0$, $\mathcal{M}$ does not contain an invariant set.

By Theorem 5.1, all trajectories initiated in $\mathcal{P}$ leave it in finite time. Since invariance conditions of $\mathcal{P}$ are achieved, we have $\mathcal{P} \xrightarrow{F} \mathcal{F}_0$.

It should be noted that in this example another method can be used to show that all trajectories initiated in $\mathcal{P}$ leave it in finite time; the method is to prove that $\mathcal{P}$ does not contain equilibrium points or closed orbits. However, our proposed method is simpler, and can be applied to examples of higher dimension ($n > 2$).
Example 8.3. In this example we use our results on control flow functions to find a continuous feedback solving RCP. Consider the system

\[
\dot{x} = \begin{bmatrix} 2.5 & 2.5 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix},
\]

defined on a polytope \( \mathcal{P} \). The polytope is shown in Figure 8. The vertices of \( \mathcal{P} \) are: \( v_1 = (0,0) \), \( v_2 = (1,0) \), \( v_3 = (1,1) \), and \( v_4 = (0,1) \). Let \( \mathcal{F}_0 := \text{co} \{ v_1, v_2 \} \), \( \mathcal{F}_1 := \text{co} \{ v_1, v_4 \} \), \( \mathcal{F}_2 := \text{co} \{ v_3, v_4 \} \), and \( \mathcal{F}_3 := \text{co} \{ v_2, v_3 \} \). First, we identify a control flow function for the system. Let \( \bar{f}(x) := Ax + a \) and \( g(x) := B \). Consider the \( \mathcal{C}^\infty \) function \( V(x) = -x_1 + 0.1x_2 \). It is easy to verify that \( L_f V(x) = -2.4x_1 - 2.4x_2 + 0.1 \) and \( L_g V(x) = -0.9 \). Then it can be verified that \( u = \frac{1}{0.9} \in \mathcal{U}^{\text{flow}}(x) := \{ u \in \mathbb{R} \mid Ax + Bu + a \in \mathcal{C}(x), \frac{\partial V}{\partial x}(Ax + Bu + a) < 0 \} \) for all \( x \in \mathcal{P} \setminus ( \mathcal{F}_2 \cup \mathcal{F}_3 ) \), \( u = -1 - x_1 - x_2 \in \mathcal{U}^{\text{flow}}(x) \) for all \( x \in \mathcal{F}_2 \setminus \mathcal{F}_3 \), and \( u = -2.5x_1 - 2.5x_2 \in \mathcal{U}^{\text{flow}}(x) \) for all \( x \in \mathcal{F}_3 \). We conclude \( V(x) \) is a \( \mathcal{C}^\infty \) control flow function, and from Theorem 7.4 RCP is solvable by continuous state feedback. Notice that the above controls also achieve (46) with \( \alpha = 0.05 \).

Then to find the continuous feedback, we follow the following procedure. First, we construct a continuous feedback, \( u^{mv}(x) \), satisfying the invariance conditions of \( \mathcal{P} \). By solving the invariance conditions at vertices of \( \mathcal{P} \), we obtain the control inputs \( u_1 = 0 \), \( u_2 = -3 \), \( u_3 = -5 \), and \( u_4 = -2 \). Then we triangulate \( \mathcal{P} \) into two simplices: \( \mathcal{S}^1 := \text{co} \{ v_1, v_2, v_3 \} \) and \( \mathcal{S}^2 := \text{co} \{ v_1, v_3, v_4 \} \), and we obtain on each simplex the unique affine feedback based on the control values selected at the vertices [24]. We obtain

![Figure 8. \( \mathcal{P} \) for Example 8.3.](image-url)
has direct analogies to RCP via control flow functions, and this has allowed to extend a
where
The findings of the paper are put in context with Lyapunov stability theory in Table 1.
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\[ u^{inv}(x) = \begin{cases} 
-3 & x \in S^1 \\
-3 & x \in S^2 
\end{cases} \]

Second, we construct a continuous feedback, \( u^{flow}(x) \), to guarantee that closed-loop trajectories leave \( P \) in finite time. We have \( f(x) := Ax + a + Bu^{inv}(x) = (-0.5x_1 + 0.5x_2, -2x_1 - x_2 + 1) \), and \( L_f V(x) = 0.3x_1 - 0.6x_2 + 0.1 \). Then from (50) we have

\[ u^{flow}(x) = \begin{cases} 
0, & 0.3x_1 - 0.6x_2 \leq -0.125, \\
\frac{1}{3}x_1 - \frac{2}{3}x_2 + 0.13889, & 0.3x_1 - 0.6x_2 > -0.125.
\end{cases} \]

Notice that \( u^{flow}(x) \) is locally Lipschitz on a neighborhood of \( P \). The final continuous control law is

\[ u(x) = u^{inv}(x) + u^{flow}(x) = \begin{cases} 
-3x_1 - 2x_2, & 0.3x_1 - 0.6x_2 \leq -0.125, \\
\frac{8}{3}x_1 - \frac{8}{3}x_2 + 0.13889, & 0.3x_1 - 0.6x_2 > -0.125.
\end{cases} \]

The system behavior under \( u(x) \) is shown in Figure 9. The dashed line represents \( 0.3x_1 - 0.6x_2 = -0.125 \). The figure shows that closed-loop trajectories starting in \( P \) leave it through \( F_0 \) as expected. \( \triangle \)

9. Conclusion

The findings of the paper are put in context with Lyapunov stability theory in Table 1. We have introduced the notion of a flow function, which provides a necessary and sufficient condition for leaving a polytope in finite time. We have provided a set of results, including a Lasalle Principle for RCP, that can be used to analyze solvability of RCP by a given continuous feedback without the need for calculating the state trajectories. We have shown for continuous PWA feedback that a suitable flow function is of the form \( V(x) = \max\{V_i(x)\} \), where \( V_i(x) \) are affine functions. We have introduced the notion of control flow functions. Finally, we have shown that the Artstein-Sontag theorem of control Lyapunov functions has direct analogies to RCP via control flow functions, and this has allowed to extend a verification tool based on flow functions to a synthesis tool based on control flow functions.

References

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Table 1. Summary of results of the paper.

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