# Patterned Linear Systems

**Sarah C. Hamilton and Mireille E. Broucke**

Systems Control Group

Department of Electrical and Computer Engineering

University of Toronto

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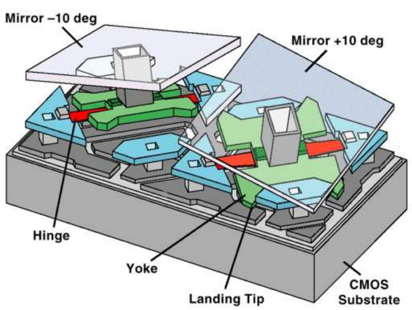
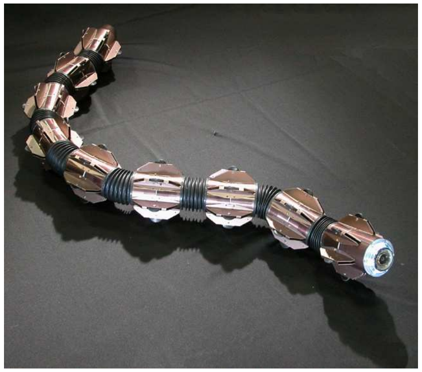
# Patterned Linear Systems

- Special category of distributed control
- **Broad definition**: Collections of identical subsystems with distinct patterns of interaction.




*DLP Chip*, Texas Instruments

*Solar Two*, US Dept. Energy

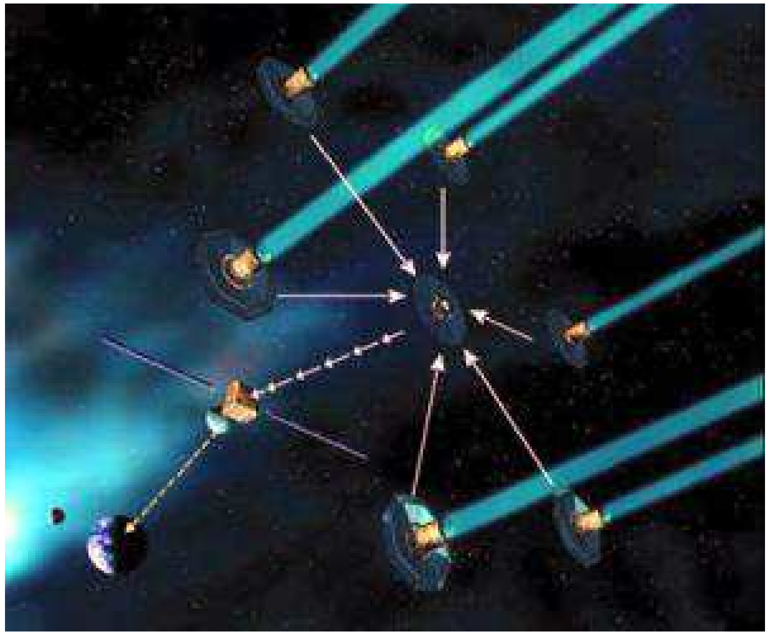


*ACM-R5*, Hirose Fukushima Lab

- **Precise definition**: LTI control systems with state, input, and output transformations that are functions of a common base transformation.

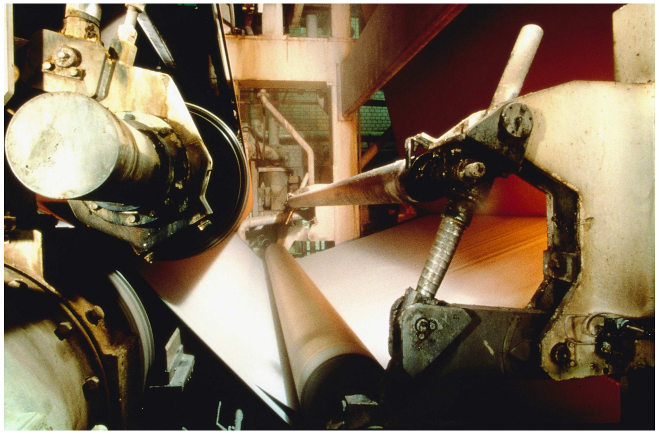
# Applications: Multi-Agent Systems

*City Car*, MIT Media Lab



*Multi-satellite Darwin Mission*, ESA

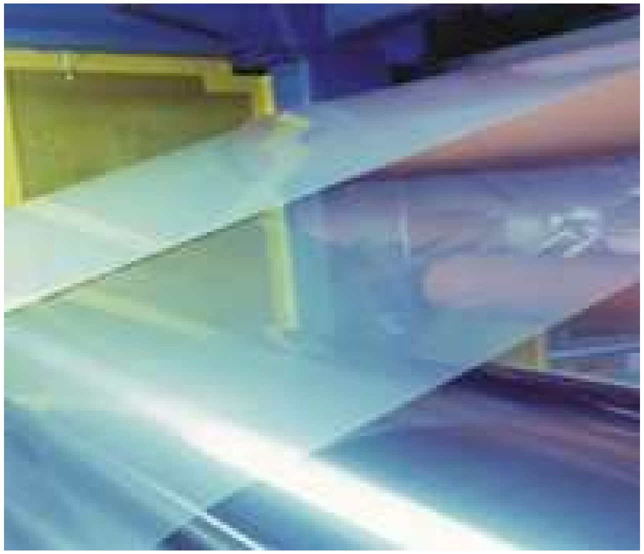
# Applications: Cross-Directional Control



*Papermaking*, Der Grüne Punkt

*Paper Gloss Control*, VIB Systems

*Steel Rolling*, Ray Jacobs Machinery



*Plastic Extrusion*, Honeywell

# Discretized PDE Models

- Controlled diffusion process

$$\frac{\partial x(t,d)}{\partial t} = k\frac{\partial^2 x(t,d)}{\partial d^2} + u(t,d)$$

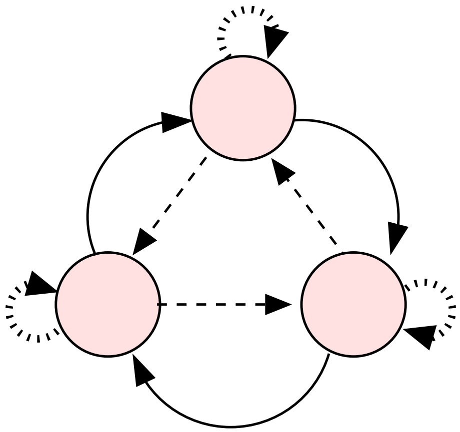
- Lumped approximation

$$\frac{dx_i(t)}{dt} = \frac{k}{h^2}\left(x_{i+1}(t) - 2x_i(t) + x_{i-1}(t)\right) + u_i(t)\,, \quad i = 1, \ldots, n-1\,.$$

- **A.M. Turing, The Chemical Basis of Morphogenesis (1952)**

# Circulant Systems

- Linear ring systems are represented by circulant models



$$\dot{x} = Ax = \begin{bmatrix} a_0 & a_1 & a_2 \\ a_2 & a_0 & a_1 \\ a_1 & a_2 & a_0 \end{bmatrix} x$$

- Every circulant matrix is a polynomial of the shift operator $\Pi$.

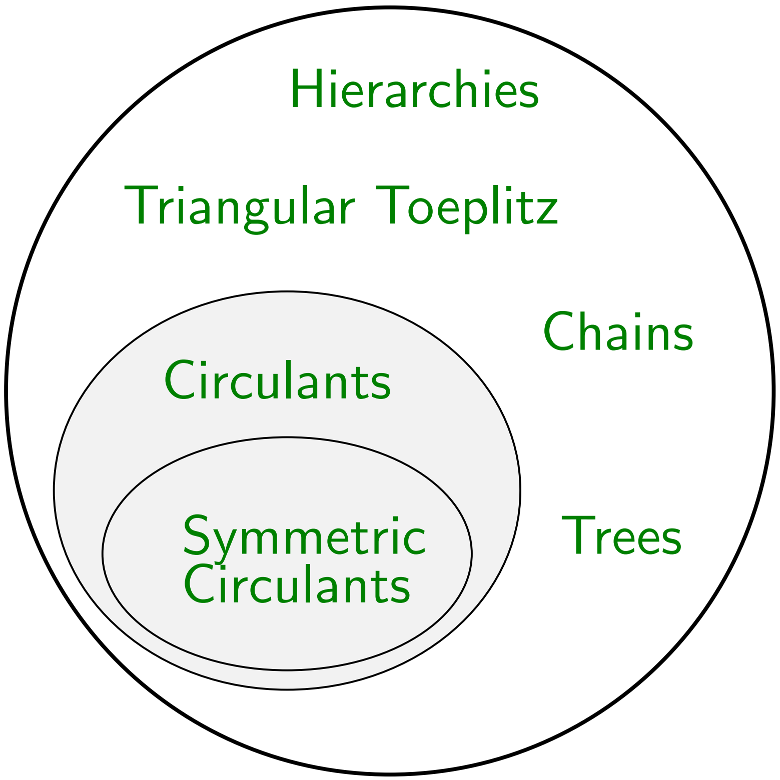
$$A = a_0 I + a_1 \Pi + a_2 \Pi^2 \qquad \Pi = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

- Eigenvectors of $\Pi$ are eigenvectors of *every* circulant matrix.

# Patterned Linear Systems

*A broader class of systems:*

Any set of matrices that are polynomials of a common base matrix will share the eigenvectors of the base.



# Main Control Question

**Problem.** *Given a patterned linear system, does there exist a control theory for synthesis of feedbacks to solve various classical control synthesis problems, with the requirement that the system pattern is preserved by the feedback?*

# Previous Control Research

- **Decentralized Control**
  - Controllers use only local state information. Global objective achieved by exploiting dynamic coupling of subsystems.
- **Structured Systems**
  - Studies effect of zero/non-zero entries of system matrices. Insufficient for solving stabilization problems.

# Previous Control Research

- **R. Brockett and J.L. Willems (1974)**
  - Used block diagonalization property of block circulant systems.
  - Studied properties of $n$ modal subsystems in an eigenvector basis rather than studying full system.

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \quad \longrightarrow \quad \begin{aligned} \dot{\tilde{x}}_i &= \alpha_i \tilde{x}_i + \beta_i \tilde{u}_i \\ \tilde{y}_i &= \kappa_i \tilde{x}_i\,, \qquad\qquad i = 1, \ldots, n\,. \end{aligned}$$

# Geometric Control Approach

Shared eigenvectors $\Longrightarrow$ Shared invariant subspaces

*Patterned linear systems can be studied using linear geometric control theory.*

This entails:

1. Define patterned controllable and unobservable subspaces.
2. Characterize patterned decomposition and patterned pole placement.
3. Control synthesis with patterned feedback.

# $\mathbf{M}$-Patterned Systems

Given a linear map $\mathbf{M} : \mathcal{X} \rightarrow \mathcal{X}$, the set of polynomial functions of $\mathbf{M}$ is

$$\mathfrak{F}(\mathbf{M}) := \left\{ \mathbf{T} \mid (\exists\, t_i \in \mathbb{R})\; \mathbf{T} = t_0\mathbf{I} + t_1\mathbf{M} + t_2\mathbf{M}^2 + \ldots + t_{n-1}\mathbf{M}^{n-1} \right\} .$$

Called the set of $\mathbf{M}$-patterned maps. Members have $\mathbf{M}$-patterned spectra.

Consider the linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

If $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are $\mathbf{M}$-patterned, we call it an $\mathbf{M}$-patterned system.

# Patterned Maps and Invariant Subspaces

Given $\mathbf{T} \in \mathfrak{F}(\mathbf{M})$. Then

- If $\mathcal{V} \subset \mathcal{X}$ is $\mathbf{M}$-invariant, then it is $\mathbf{T}$-invariant, but not vice versa.
- $\mathrm{Im}\,\mathbf{T}$ and $\mathrm{Ker}\,\mathbf{T}$ are $\mathbf{M}$-invariant.
- Spectral subspaces of $\mathbf{T}$ are $\mathbf{M}$-invariant and $\mathbf{M}$-decoupling.
- $\mathbf{T}_{\mathcal{V}}$, the restriction of $\mathbf{T}$ to an $\mathbf{M}$-invariant subspace $\mathcal{V}$, belongs to $\mathfrak{F}(\mathbf{M}_{\mathcal{V}})$.

Given $\mathcal{V} \subset \mathcal{X}$, $\mathbf{T}_{\mathcal{V}} \in \mathfrak{F}(\mathbf{M}_{\mathcal{V}})$. Then

- Under certain conditions, there is a lifting procedure to $\mathbf{T}$, an $\mathbf{M}$-patterned map.

# Example: Invariant Subspaces

$$M = \begin{bmatrix} 4 & 2 & -5 \\ 1 & 2 & -2 \\ 1 & 2 & -2 \end{bmatrix}$$

$$T := 2I - 0.5M + 0.5M^2 = \begin{bmatrix} 6.5 & 0 & -4.5 \\ 1.5 & 2 & -1.5 \\ 1.5 & 0 & 0.5 \end{bmatrix}$$

Let $v = (1, 0, 1)$, $\mathcal{V} = \text{span}\ \{v\}$. Then $Tv = (2, 0, 2) = 2v$ , but $Mv = (-1, -1 - 1)$. Thus $\mathcal{V}$ is T-invariant, but not M-invariant.

# First Decomposition Theorem

**Theorem.** *Let $\mathcal{V}, \mathcal{W} \subset \mathcal{X}$ be $\mathbf{M}$-decoupling subspaces such that $\mathcal{X} = \mathcal{V} \oplus \mathcal{W}$. Let $\mathbf{A} \in \mathfrak{F}(\mathbf{M})$. There exists a coordinate transformation $\mathbf{T} : \mathcal{X} \to \mathcal{X}$ such that the representation of $\mathbf{A}$ in the new coordinates is given by*

$$\mathrm{T}^{-1}\mathrm{A}\mathrm{T} = \begin{bmatrix} \mathrm{A}_{\mathcal{V}} & 0 \\ 0 & \mathrm{A}_{\mathcal{W}} \end{bmatrix}, \qquad \mathrm{A}_{\mathcal{V}} \in \mathfrak{F}(\mathrm{M}_{\mathcal{V}}),\ \mathrm{A}_{\mathcal{W}} \in \mathfrak{F}(\mathrm{M}_{\mathcal{W}}).$$

*The spectrum splits into $\sigma(\mathrm{A}) = \sigma(\mathrm{A}_{\mathcal{V}}) \uplus \sigma(\mathrm{A}_{\mathcal{W}})$.*

# System Properties

- Controllable subspace $\mathcal{C} = \operatorname{Im} \mathbf{B}$
- Patterned controllable subspace:

$$\mathcal{C}_M := \sup \mathfrak{D}^{\diamond}(\mathbf{M}; \mathcal{C}) = \sum_{\substack{\lambda \in \sigma(\mathbf{B}), \\ \lambda \neq 0}} \mathcal{S}_\lambda(\mathbf{B}) \,.$$

  In general $\mathcal{C}_M \subset \mathcal{C}$.

- Unobservable subspace $\mathcal{N} = \operatorname{Ker} \mathbf{C}$
- Patterned unobservable subspace:

$$\mathcal{N}_M := \inf \mathfrak{D}_{\diamond}(\mathbf{M}; \mathcal{N}) = \mathcal{S}_0(\mathbf{C}) \,.$$

  In general $\mathcal{N} \subset \mathcal{N}_M$.

# Example: Patterned Controllable Subspace

$$M = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 6 & 1 & 1 & 0 & -4 & 0 & -4 \\ 0 & -1 & -1 & 0 & 2 & 0 & 0 \\ -3 & 0 & 0 & 3 & -2 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 6 & 0 & 0 & -2 & -4 & 1 & -4 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A \doteq -4I + M + 3.5M^2 - 2.7M^3 - 1.2M^4 + 1.5M^5 - 0.44M^6$$

$$B \doteq 2M + 3.7M^2 - 3.0M^3 - 1.5M^4 + 1.7M^5 - 0.42M^6.$$

There exists $\Omega$ such that $\Omega^{-1}\mathrm{M}\Omega = \mathrm{J}$, and

$$\mathcal{X} = \mathcal{J}_1(\mathrm{M}) \oplus \mathcal{J}_2(\mathrm{M}) \oplus \mathcal{J}_3(\mathrm{M}) \oplus \mathcal{J}_4(\mathrm{M}) \oplus \mathcal{J}_5(\mathrm{M}).$$

$$\Omega^{-1}\mathrm{B}\Omega = \left[\begin{array}{cc|cc|c|c|c} 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 3+\mathrm{j} & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 3-\mathrm{j} \end{array}\right]$$

$$
\begin{aligned}
\mathcal{C}_M &= \mathcal{S}_1(\mathrm{B}) + \mathcal{S}_{3+\mathrm{j}}(\mathrm{B}) + \mathcal{S}_{3-\mathrm{j}}(\mathrm{B}) \\
&= \mathcal{J}_2(\mathrm{M}) \oplus \mathcal{J}_3(\mathrm{M}) \oplus \mathcal{J}_4(\mathrm{M}) \oplus \mathcal{J}_5(\mathrm{M}) \\
\mathcal{C} &= \operatorname{Im}\mathrm{B} = \mathcal{C}_M \oplus \operatorname{span}\{v_1\}\ .
\end{aligned}
$$

# Second Decomposition Theorem

**Theorem.** *Let* $(\mathbf{A}, \mathbf{B})$ *be an* $\mathbf{M}$*-patterned pair. There exists a coordinate transformation* $\mathbf{T} : \mathcal{X} \to \mathcal{X}$ *for the state and input spaces* $(\mathcal{U} \simeq \mathcal{X})$*, which decouples the system into two subsystems,* $(\mathbf{A}_1, \mathbf{B}_1)$ *and* $(\mathbf{A}_2, \mathbf{B}_2)$*, such that*

(1) *pair* $(\mathbf{A}_1, \mathbf{B}_1)$ *is* $\mathbf{M}_{\mathcal{C}_M}$*-patterned and controllable,*

(2) *pair* $(\mathbf{A}_2, \mathbf{B}_2)$ *is* $\mathbf{M}_{\mathcal{R}}$*-patterned,*

(3) $\sigma(\mathbf{A}) = \sigma(\mathbf{A}_1) \uplus \sigma(\mathbf{A}_2)$,

(4) $\sigma(\mathbf{A}_2)$ *is unaffected by patterned state feedback in the class* $\mathfrak{F}(\mathbf{M}_{\mathcal{R}})$,

(5) $\mathbf{B}_2 = 0$ *if* $\mathcal{C}_M = \mathcal{C}$.

# Patterned Pole Placement

**Theorem.** *The* $\mathbf{M}$*-patterned pair* $(\mathbf{A}, \mathbf{B})$ *is controllable if and only if, for every* $\mathbf{M}$*-patterned spectrum* $\mathfrak{L}$*, there exists a map* $\mathbf{F} : \mathcal{X} \to \mathcal{U}$ *with* $\mathbf{F} \in \mathfrak{F}(\mathbf{M})$ *such that* $\sigma(\mathbf{A} + \mathbf{BF}) = \mathfrak{L}$*.*

# Patterned Control Synthesis

Given a patterned linear system

$$\begin{aligned} \dot{x} &= Ax + Bu + Ew \\ y &= Cx \\ z &= Dx\,. \end{aligned}$$

- **Stabilization**:
  Find a patterned feedback $u = Kx$ such that $x(t) \longrightarrow 0$.
- **Stabilization by Measurement Feedback**:
  Find a patterned measurement feedback $u = Ky$ such that $x(t) \longrightarrow 0$.
- **Output Stabilization**:
  Find a patterned feedback $u = Kx$ such that $z(t) \longrightarrow 0$.

- **Output Stabilization by Measurement Feedback**:
  Find a patterned measurement feedback $u = Ky$ such that $z(t) \longrightarrow 0$.

- **Restricted Regulator Problem**:
  Find a patterned feedback $u = Kx$ such that $\mathcal{N}_M \subset \operatorname{Ker} K$ and $z(t) \longrightarrow 0$.

- **Disturbance Decoupling**:
  Find a patterned feedback $u = Kx$ such that $D \int_0^t \mathrm{e}^{(A+BK)(t-\tau)} E w(\tau) d\tau = 0$.

# Patterned Control Synthesis

For all synthesis problems studied, if there exists a general feedback, then there exists a patterned feedback.

# Stabilization Problem

**Problem.** *Given a linear system*

$$\dot{x} = \mathrm{A}x + \mathrm{B}u\,.$$

*Find a state feedback* $u = \mathrm{K}x$ *such that* $x(t) \to 0$ *as* $t \to \infty$*.*

**Theorem.** *The SP is solvable if and only if*

$$\mathcal{X}^{+}(\mathbf{A}) \subset \mathcal{C}\,.$$

- By S.D.T. there exists $(x_1, x_2) = \mathrm{T}^{-1}x$ such that

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \mathrm{A}_1 & \star \\ 0 & \mathrm{A}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \mathrm{B}_1 \\ 0 \end{bmatrix} u \,.$$

  where $\mathbf{A}_1 = \mathbf{A}_{\mathcal{C}}$, $\mathbf{A}_2 = \mathbf{A}_{\mathcal{X}/\mathcal{C}}$ and $(\mathbf{A}_1, \mathbf{B}_1)$ is c.c.

- By P.P.T. $\exists \mathbf{K}_1$ such that $\sigma(\mathbf{A}_1 + \mathbf{B}_1\mathbf{K}_1) \subset \mathbb{C}^-$.

- Define $\mathrm{K} = \begin{bmatrix} \mathrm{K}_1 & 0 \\ 0 & 0 \end{bmatrix} \mathrm{T}^{-1}$

$$\dot{x} = \mathrm{T} \begin{bmatrix} \mathrm{A}_1 + \mathrm{B}_1\mathrm{K}_1 & \star \\ 0 & \mathrm{A}_2 \end{bmatrix} \mathrm{T}^{-1} x \,.$$

- $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C} \quad \Longrightarrow \quad \sigma(\mathbf{A}_2) \subset \mathbb{C}^-$.

# Patterned Stabilization Problem

**Problem.** *Given a patterned linear system*

$$\dot{x} = \mathrm{A}x + \mathrm{B}u\,.$$

*Find a patterned state feedback* $u = \mathrm{K}x$ *such that* $x(t) \to 0$ *as* $t \to \infty$*.*

**Theorem.** *The PSP is solvable if and only if*

$$\mathcal{X}^{+}(\mathbf{A}) \subset \mathcal{C}\,.$$

- Let $\mathcal{X} = \mathcal{C}_M \oplus \mathcal{R}$. By S.D.T. there exists $\mathbf{T}$ such that

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \mathrm{A}_1 & 0 \\ 0 & \mathrm{A}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \mathrm{B}_1 \\ \mathrm{B}_2 \end{bmatrix} u$$

  where $\mathbf{A}_1, \mathbf{B}_1 \in \mathfrak{F}(\mathbf{M}_{\mathcal{C}_M})$, $\mathbf{A}_2, \mathbf{B}_2 \in \mathfrak{F}(\mathbf{M}_{\mathcal{R}})$, and $(\mathbf{A}_1, \mathbf{B}_1)$ is c.c.

- By P.P.P.T. $\exists \mathbf{K}_1 \in \mathfrak{F}(\mathbf{M}_{\mathcal{C}_M})$ such that $\sigma(\mathbf{A}_1 + \mathbf{B}_1\mathbf{K}_1) \subset \mathbb{C}^-$.

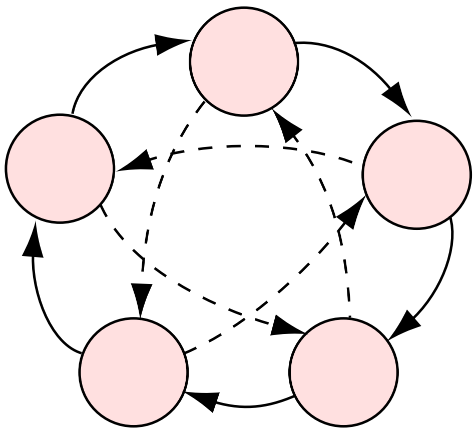
- Define $\mathbf{K} = \mathbf{S}_{\mathcal{C}_M}\mathbf{K}_1\mathbf{N}_{\mathcal{C}_M} \in \mathfrak{F}(\mathbf{M})$.

- $(\mathbf{A} + \mathbf{BK})_{\mathcal{C}_M} = \mathbf{A}_1 + \mathbf{B}_1\mathbf{K}_1$,

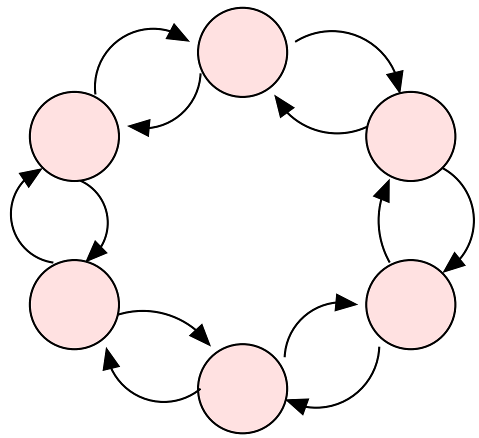
- $(\mathbf{A} + \mathbf{BK})_{\mathcal{R}} = \mathbf{A}_2$.

- $\mathcal{X}^+(\mathbf{A}) \subset \mathcal{C} \quad \Longrightarrow \quad \mathcal{X}^+(\mathbf{A}) \subset \mathcal{C}_M \quad \Longrightarrow \quad \sigma(\mathbf{A}_2) \subset \mathbb{C}^-$.

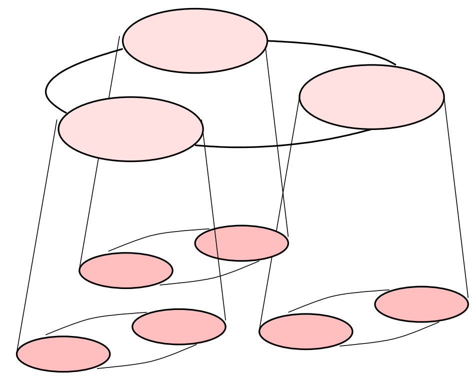
# Rings



General Rings



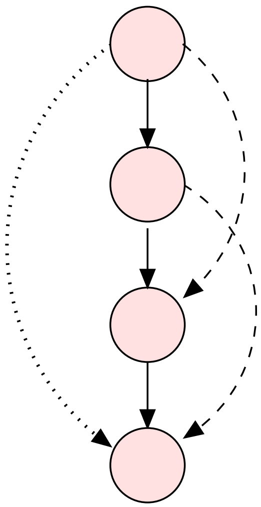
Symmetric Rings

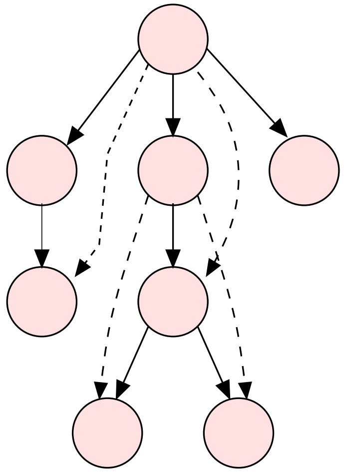


Hierarchy of Rings

$$\Pi_5 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \Sigma_6 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad H_r = \left[\begin{array}{cc|cc|cc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

# Chains and Trees





$$N = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

# Example: Multiagent Consensus

Robots model: $\dot{x}_i = u_i$, $i = 1, \ldots, n$.

$$\dot{x} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{bmatrix} u$$

Measurement model: $y = \mathrm{C}x$, $\mathrm{C} \in \mathfrak{F}(\Pi)$

Global objective is rendezvous:

$$z = \mathrm{D}x = \begin{bmatrix} -1 & 1 & \cdots & 0 & 0 \\ 0 & -1 & & 0 & 0 \\ & \vdots & & \vdots & \\ 1 & 0 & \cdots & 0 & -1 \end{bmatrix} x.$$

Find $u = \mathrm{K}y$, $\mathrm{K} \in \mathfrak{F}(\Pi)$ such that $z(t) \to 0$ as $t \to \infty$.

- This is the Patterned Restricted Regulator Problem.
- Solution exists iff

$$\mathcal{X}^+(\mathrm{A}) \cap \mathcal{N}_M \subset \operatorname{Ker} \mathrm{D}$$
$$\mathcal{X}^+(\mathrm{A}) \subset \mathcal{C} + \mathcal{V}^\star$$

  where $\mathcal{V}^\star := \sup \mathfrak{I}(\mathbf{A},\, \mathbf{B};\, \operatorname{Ker} \mathbf{D})$.

- We have $\mathcal{X}^+(\mathrm{A}) = \mathbb{R}^n$, $\mathcal{C} = \mathbb{R}^n$, $\mathcal{N}_M = \operatorname{Ker} \mathrm{C}$, and $\mathcal{V}^* = \operatorname{Ker} \mathrm{D} = \operatorname{span}\, \{(1, 1, \ldots, 1)\}$.
- A controller exists iff

$$\mathcal{N}_M \subset \operatorname{span}\, \{(1, 1, \ldots, 1)\}\ .$$

# Future Research Directions

- Patterned Robust Regulator Problem.
- Block patterned systems.
- Infinite dimensional patterned systems.
- Patterned identification problem.