Patterned Linear Systems

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Patterned Linear Systems

- Special category of distributed control

- **Broad definition**: Collections of identical subsystems with distinct patterns of interaction.

- **Precise definition**: LTI control systems with state, input, and output transformations that are functions of a common base transformation.
Applications: Multi-Agent Systems

City Car, MIT Media Lab

Multi-satellite Darwin Mission, ESA
Applications: Cross-Directional Control

- Papermaking, Der Grüne Punkt
- Steel Rolling, Ray Jacobs Machinery
- Paper Gloss Control, VIB Systems
- Plastic Extrusion, Honeywell
Discretized PDE Models

- Controlled diffusion process

\[ \frac{\partial x(t, d)}{\partial t} = k \frac{\partial^2 x(t, d)}{\partial d^2} + u(t, d) \]

- Lumped approximation

\[ \frac{dx_i(t)}{dt} = \frac{k}{h^2} (x_{i+1}(t) - 2x_i(t) + x_{i-1}(t)) + u_i(t), \quad i = 1, \ldots, n - 1. \]

- A.M. Turing, The Chemical Basis of Morphogenesis (1952)
Circulant Systems

• Linear ring systems are represented by circulant models

\[ \dot{x} = Ax = \begin{bmatrix} a_0 & a_1 & a_2 \\ a_2 & a_0 & a_1 \\ a_1 & a_2 & a_0 \end{bmatrix} x \]

• Every circulant matrix is a polynomial of the shift operator \( \Pi \).

\[ A = a_0 I + a_1 \Pi + a_2 \Pi^2 \]

\[ \Pi = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \]

• Eigenvectors of \( \Pi \) are eigenvectors of every circulant matrix.
Patterned Linear Systems

A broader class of systems:

Any set of matrices that are polynomials of a common base matrix will share the eigenvectors of the base.
Main Control Question

Problem. Given a patterned linear system, does there exist a control theory for synthesis of feedbacks to solve various classical control synthesis problems, with the requirement that the system pattern is preserved by the feedback?
Previous Control Research

- **Decentralized Control**
  - Controllers use only local state information. Global objective achieved by exploiting dynamic coupling of subsystems.

- **Structured Systems**
  - Studies effect of zero/non-zero entries of system matrices. Insufficient for solving stabilization problems.
Previous Control Research

- **R. Brockett and J.L. Willems (1974)**
  
  - Used block diagonalization property of block circulant systems.
  - Studied properties of $n$ modal subsystems in an eigenvector basis rather than studying full system.

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx \\
\dot{x}_i &= \alpha_i \tilde{x}_i + \beta_i \tilde{u}_i \\
\tilde{y}_i &= \kappa_i \tilde{x}_i, \quad i = 1, \ldots, n.
\end{align*}
\]
Geometric Control Approach

Shared eigenvectors $\implies$ Shared invariant subspaces

*Patterned linear systems can be studied using linear geometric control theory.*

This entails:

1. Define patterned controllable and unobservable subspaces.
2. Characterize patterned decomposition and patterned pole placement.
3. Control synthesis with patterned feedback.
**M-Patterned Systems**

Given a linear map $M : \mathcal{X} \to \mathcal{X}$, the set of polynomial functions of $M$ is

$$\mathfrak{F}(M) := \left\{ T \mid (\exists t_i \in \mathbb{R}) \; T = t_0 I + t_1 M + t_2 M^2 + \ldots + t_{n-1} M^{n-1} \right\}.$$  

Called the set of **M-patterned maps**. Members have **M-patterned spectra**.

Consider the linear system

$$\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*}$$

If $A$, $B$, and $C$ are $M$-patterned, we call it an **$M$-patterned system**.
Patterned Maps and Invariant Subspaces

Given $T \in \mathfrak{F}(M)$. Then

- If $\mathcal{V} \subset \mathcal{X}$ is $M$-invariant, then it is $T$-invariant, but not vice versa.
- $\text{Im} \, T$ and $\text{Ker} \, T$ are $M$-invariant.
- Spectral subspaces of $T$ are $M$-invariant and $M$-decoupling.
- $T \mathcal{V}$, the restriction of $T$ to an $M$-invariant subspace $\mathcal{V}$, belongs to $\mathfrak{F}(M \mathcal{V})$.

Given $\mathcal{V} \subset \mathcal{X}$, $T \mathcal{V} \in \mathfrak{F}(M \mathcal{V})$. Then

- Under certain conditions, there is a lifting procedure to $T$, an $M$-patterned map.
Example: Invariant Subspaces

\[ M = \begin{bmatrix} 4 & 2 & -5 \\ 1 & 2 & -2 \\ 1 & 2 & -2 \end{bmatrix} \]

\[ T := 2I - 0.5M + 0.5M^2 = \begin{bmatrix} 6.5 & 0 & -4.5 \\ 1.5 & 2 & -1.5 \\ 1.5 & 0 & 0.5 \end{bmatrix} \]

Let \( v = (1, 0, 1) \), \( \mathcal{V} = \text{span} \{ v \} \). Then \( Tv = (2, 0, 2) = 2v \), but \( Mv = (-1, -1 - 1) \). Thus \( \mathcal{V} \) is \( T \)-invariant, but not \( M \)-invariant.
First Decomposition Theorem

**Theorem.** Let \( \mathcal{V}, \mathcal{W} \subset \mathcal{X} \) be \( \mathbb{M} \)-decoupling subspaces such that \( \mathcal{X} = \mathcal{V} \oplus \mathcal{W} \). Let \( A \in \mathfrak{F}(\mathbb{M}) \). There exists a coordinate transformation \( T : \mathcal{X} \to \mathcal{X} \) such that the representation of \( A \) in the new coordinates is given by

\[
T^{-1}AT = \begin{bmatrix}
A_{\mathcal{V}} & 0 \\
0 & A_{\mathcal{W}}
\end{bmatrix}, \quad A_{\mathcal{V}} \in \mathfrak{F}(\mathbb{M}_{\mathcal{V}}), \ A_{\mathcal{W}} \in \mathfrak{F}(\mathbb{M}_{\mathcal{W}}).
\]

The spectrum splits into \( \sigma(A) = \sigma(A_{\mathcal{V}}) \cup \sigma(A_{\mathcal{W}}) \).
System Properties

- Controllable subspace $\mathcal{C} = \text{Im } B$
- Patterned controllable subspace:
  \[ C_M := \sup D^\diamond (M; C) = \sum_{\lambda \in \sigma(B), \lambda \neq 0} S_{\lambda}(B). \]
  In general $C_M \subset C$.

- Unobservable subspace $\mathcal{N} = \text{Ker } C$
- Patterned unobservable subspace:
  \[ \mathcal{N}_M := \inf D^\diamond (M; \mathcal{N}) = S_0(C). \]
  In general $\mathcal{N} \subset \mathcal{N}_M$. 

Patterned Linear Systems
Example: Patterned Controllable Subspace

\[
M = \begin{bmatrix}
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
6 & 1 & 1 & 0 & -4 & 0 & -4 \\
0 & -1 & -1 & 0 & 2 & 0 & 0 \\
-3 & 0 & 0 & 3 & -2 & 1 & -1 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
6 & 0 & 0 & -2 & -4 & 1 & -4 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
A \doteq -4I + M + 3.5M^2 - 2.7M^3 - 1.2M^4 + 1.5M^5 - 0.44M^6
\]

\[
B \doteq 2M + 3.7M^2 - 3.0M^3 - 1.5M^4 + 1.7M^5 - 0.42M^6.
\]
There exists $\Omega$ such that $\Omega^{-1}M\Omega = J$, and

$$\mathcal{X} = \mathcal{J}_1(M) \oplus \mathcal{J}_2(M) \oplus \mathcal{J}_3(M) \oplus \mathcal{J}_4(M) \oplus \mathcal{J}_5(M).$$

$$\Omega^{-1}B\Omega = 
\begin{bmatrix}
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3+j & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3-j & 0 \\
\end{bmatrix}$$
\[ \mathcal{C}_M = S_1(B) + S_{3+j}(B) + S_{3-j}(B) \]
\[ = \mathcal{J}_2(M) \oplus \mathcal{J}_3(M) \oplus \mathcal{J}_4(M) \oplus \mathcal{J}_5(M) \]
\[ \mathcal{C} = \mathrm{Im} \, B = \mathcal{C}_M \oplus \mathrm{span} \, \{v_1\} . \]
Second Decomposition Theorem

**Theorem.** Let \((A, B)\) be an \(M\)-patterned pair. There exists a coordinate transformation \(T : \mathcal{X} \rightarrow \mathcal{X}\) for the state and input spaces \((\mathcal{U} \simeq \mathcal{X})\), which decouples the system into two subsystems, \((A_1, B_1)\) and \((A_2, B_2)\), such that

1. pair \((A_1, B_1)\) is \(M_{CM}\)-patterned and controllable,
2. pair \((A_2, B_2)\) is \(M_R\)-patterned,
3. \(\sigma(A) = \sigma(A_1) \uplus \sigma(A_2)\),
4. \(\sigma(A_2)\) is unaffected by patterned state feedback in the class \(F(M_R)\),
5. \(B_2 = 0\) if \(C_M = C\).

Patterned Linear Systems
Patterned Pole Placement

**Theorem.** The $\mathcal{M}$-patterned pair $(A, B)$ is controllable if and only if, for every $\mathcal{M}$-patterned spectrum $\mathcal{L}$, there exists a map $F : \mathcal{X} \rightarrow \mathcal{U}$ with $F \in \mathcal{F}(\mathcal{M})$ such that $\sigma(A + BF) = \mathcal{L}$. 
Patterned Control Synthesis

Given a patterned linear system

\[ \dot{x} = Ax + Bu + Ew \]
\[ y = Cx \]
\[ z = Dx. \]

- **Stabilization:**
  Find a patterned feedback \( u = Kx \) such that \( x(t) \to 0 \).

- **Stabilization by Measurement Feedback:**
  Find a patterned measurement feedback \( u = Ky \) such that \( x(t) \to 0 \).

- **Output Stabilization:**
  Find a patterned feedback \( u = Kx \) such that \( z(t) \to 0 \).
• **Output Stabilization by Measurement Feedback:**
  Find a patterned measurement feedback $u = Ky$ such that $z(t) \to 0$.

• **Restricted Regulator Problem:**
  Find a patterned feedback $u = Kx$ such that $\mathcal{N}_M \subset \ker K$ and $z(t) \to 0$.

• **Disturbance Decoupling:**
  Find a patterned feedback $u = Kx$ such that
  \[ D \int_0^t e^{(A+BK)(t-\tau)} Ew(\tau) d\tau = 0. \]
Patterned Control Synthesis

For all synthesis problems studied, if there exists a general feedback, then there exists a patterned feedback.
Stabilization Problem

**Problem.** Given a linear system

\[
\dot{x} = Ax + Bu.
\]

*Find a state feedback* \( u = Kx \) *such that* \( x(t) \to 0 \) *as* \( t \to \infty \).

**Theorem.** The SP is solvable if and only if

\[
\mathcal{X}^+(A) \subset \mathcal{C}.
\]
By S.D.T. there exists \((x_1, x_2) = T^{-1}x\) such that

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
A_1 & \ast \\
0 & A_2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
B_1 \\
0
\end{bmatrix}u.
\]

where \(A_1 = A_C\), \(A_2 = A_{\mathcal{X}/\mathcal{C}}\) and \((A_1, B_1)\) is c.c.

By P.P.T. \(\exists K_1\) such that \(\sigma(A_1 + B_1K_1) \subset \mathbb{C}^-\).

Define \(K = \begin{bmatrix}
K_1 & 0 \\
0 & 0
\end{bmatrix} T^{-1}\)

\[
\dot{x} = T \begin{bmatrix}
A_1 + B_1K_1 & \ast \\
0 & A_2
\end{bmatrix} T^{-1}x.
\]

\(\mathcal{X}^+(A) \subset \mathcal{C} \implies \sigma(A_2) \subset \mathbb{C}^-\).
Patterned Stabilization Problem

**Problem.** Given a patterned linear system

\[ \dot{x} = Ax + Bu. \]

*Find a patterned state feedback* \( u = Kx \) *such that* \( x(t) \to 0 \) *as* \( t \to \infty \).

**Theorem.** The PSP is solvable if and only if

\[ \chi^+(A) \subset \mathcal{C}. \]
Let $\mathcal{X} = \mathcal{C}_M \oplus \mathcal{R}$. By S.D.T. there exists $T$ such that

$$
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
A_1 & 0 \\
0 & A_2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} +
\begin{bmatrix}
B_1 \\
B_2
\end{bmatrix} u
$$

where $A_1, B_1 \in \mathcal{F}(M_{\mathcal{C}_M})$, $A_2, B_2 \in \mathcal{F}(M_{\mathcal{R}})$, and $(A_1, B_1)$ is c.c.

- By P.P.P.T. $\exists K_1 \in \mathcal{F}(M_{\mathcal{C}_M})$ such that $\sigma(A_1 + B_1 K_1) \subset \mathbb{C}^-$.
- Define $K = S_{\mathcal{C}_M} K_1 N_{\mathcal{C}_M} \in \mathcal{F}(M)$.
- $(A + BK)_{\mathcal{C}_M} = A_1 + B_1 K_1$.
- $(A + BK)_\mathcal{R} = A_2$.
- $\mathcal{X}^+(A) \subset \mathcal{C} \implies \mathcal{X}^+(A) \subset \mathcal{C}_M \implies \sigma(A_2) \subset \mathbb{C}^-$. 

Patterned Linear Systems
Rings

General Rings

Symmetric Rings

Hierarchy of Rings

\[ \Pi_5 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ \Sigma_6 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \]

\[ H_r = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]
Chains and Trees

\[
N = \begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

\[
H = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}
\]
Example: Multiagent Consensus

Robots model: \( \dot{x}_i = u_i, i = 1, \ldots, n. \)

\[
\begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \\
\vdots & \ddots & \ddots & \\
0 & 0 & 0 & \\
\end{bmatrix}
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & 0 & \\
\vdots & \ddots & \ddots & \\
0 & 0 & 1 & \\
\end{bmatrix}
\begin{bmatrix}
x \\
u
\end{bmatrix}
\]

Measurement model: \( y = Cx, C \in \mathcal{F}(\Pi) \)

Global objective is rendezvous:

\[
\begin{bmatrix}
-1 & 1 & \cdots & 0 & 0 \\
0 & -1 & 0 & 0 & \\
\vdots & \ddots & \ddots & \ddots & \\
1 & 0 & \cdots & 0 & -1
\end{bmatrix}
\begin{bmatrix}
x
\end{bmatrix}
\]

Find \( u = Ky, K \in \mathcal{F}(\Pi) \) such that \( z(t) \to 0 \) as \( t \to \infty. \)
• This is the Patterned Restricted Regulator Problem.

• Solution exists iff

\[ \mathcal{X}^+(A) \cap N_M \subset \ker D \]

\[ \mathcal{X}^+(A) \subset \mathcal{C} + \mathcal{V}^* \]

where \( \mathcal{V}^* := \sup I(A, B; \ker D) \).

• We have \( \mathcal{X}^+(A) = \mathbb{R}^n \), \( \mathcal{C} = \mathbb{R}^n \), \( N_M = \ker \mathcal{C} \), and

\( \mathcal{V}^* = \ker D = \text{span} \{(1, 1, \ldots, 1)\} \).

• A controller exists iff

\[ N_M \subset \text{span} \{(1, 1, \ldots, 1)\} \]
Future Research Directions

- Patterned Robust Regulator Problem.
- Block patterned systems.
- Infinite dimensional patterned systems.
- Patterned identification problem.