Problem Set 1 Solutions

Problem 1

The mathematical model is

$$u - v_C + L\frac{di_L}{dt} = 0$$
$$i_L + C\frac{dV_C}{dt} + h(v_C) = 0.$$

The state space model is

$$\frac{dx_1}{dt} = \frac{1}{L}x_2 - \frac{1}{L}u$$
$$\frac{dx_2}{dt} = -\frac{1}{C}x_1 - \frac{1}{C}h(x_2).$$

Problem 2

Free-body diagram: there are two masses, m_1 and m_2, hence we will draw two diagrams:

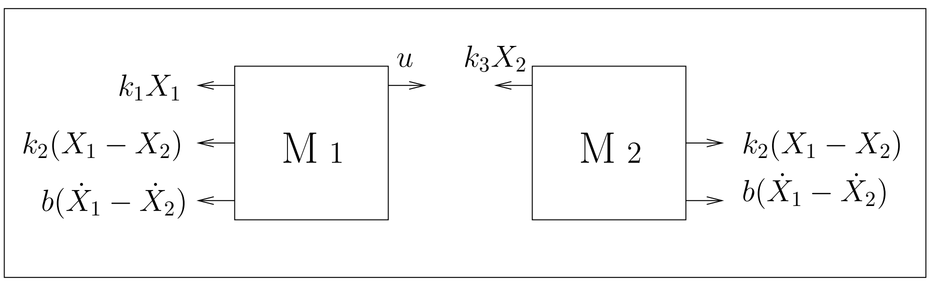


Figure 1: Free-body diagrams

Note that, when $x_1 > x_2$ and hence $x_1 - x_2 > 0$, the spring k_2 pushes m_1 to the left, and m_2 to the right. Hence the orientation of the forces in the free-body diagram. A similar reasoning holds for the damper b.

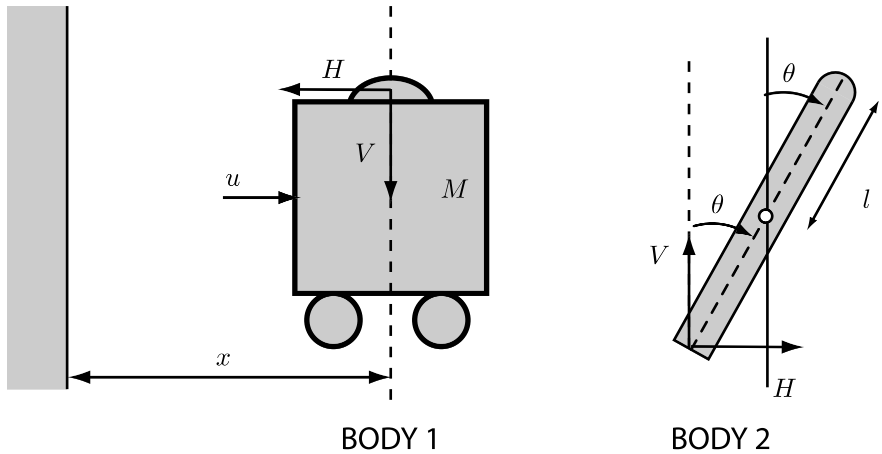
Applying Newton's law to the free-body diagram we get:

$$m_1\ddot{x}_1 = -k_1x_1 - k_2(x_1 - x_2) - b(\dot{x}_1 - \dot{x}_2) + u$$
$$m_2\ddot{x}_2 = -k_3x_2 + k_2(x_1 - x_2) + b(\dot{x}_1 - \dot{x}_2)$$

NOTE: Suppose that one wants to position m_2 at a desired location, i.e., to control x_2. In this case, the control input is the force u, the output is x_2.

Problem 3

Free-body diagram:



We need to characterize three things:

(i) The translational motion of body 1

(ii) The rotational motion of the body 2

(iii) The translational motion of body 2

Part (i). Fix an inertial reference frame and let x denote the corresponding displacement of the cart, as in the figure above. Apply Newton's law to body 1:

$$M\ddot{x} = -H + u. \tag{1}$$

Part (ii). Pass a vertical axis through the center of gravity of body 2, as in the figure above. Let I denote the moment of inertia of the rod measured at its center of gravity. Then Newton's law for rotational motion gives:

$$I\ddot{\theta} = Vl\sin\theta - Hl\cos\theta. \tag{2}$$

Part (iii). The displacement of the center of gravity of body 2 with respect to the inertial reference frame is $x + l\sin\theta$. Apply Newton's law to characterize the translational motion of the center of gravity of body 2. We write two equations for the horizontal and vertical motions, respectively.

$$\begin{aligned} H &= m\frac{d^2}{dt^2}(x + l\sin\theta) \\ &= m\ddot{x} + m\frac{d}{dt}(l\cos\theta\dot{\theta}) \\ &= m\ddot{x} - ml\sin\theta(\dot{\theta})^2 + ml\cos\theta\ddot{\theta} \end{aligned}$$

$$\begin{aligned} V - mg &= m\frac{d^2}{dt^2}(l\cos\theta) \\ \Longleftrightarrow V &= mg + ml(-\cos\theta(\dot{\theta})^2 - \sin\theta\ddot{\theta}) \end{aligned}$$

We now replace the expressions for H and V just found into (1) and (2). In conclusion:

$$\begin{aligned} M\ddot{x} &= -m\ddot{x} + ml\sin\theta(\dot{\theta})^2 - ml\cos\theta\ddot{\theta} + u \\ I\ddot{\theta} &= ml^2[-\sin\theta\cos\theta(\dot{\theta})^2 - \sin^2\theta(\ddot{\theta}) + \sin\theta\cos\theta(\dot{\theta})^2 - \cos^2\theta(\ddot{\theta})] - ml\ddot{x}\cos\theta + mgl\sin\theta \\ &= -ml^2\ddot{\theta} - ml\ddot{x}\cos\theta + mgl\sin\theta \end{aligned}$$

In order to get the equation of motion we need to solve the two equations above for $\ddot{x}$ and $\ddot{\theta}$. This part is left as an exercise.

Problem 4

(a)<u>KVL to loop</u>

$v_c + v_L + Ri_L - U = 0$

Noting that $v_L = L\frac{di_L}{dt}$, we get $\boxed{V_c + L\frac{di_L}{dt} + Ri_L - U = 0}$ (1)

KCL to node 1

$i_L = i_C + i_R$

Noting that $i_c = C\frac{dv_C}{dt}$ and $i_R = h(v_R) = h(v_C)$

we have that $\boxed{i_L = C\frac{dv_C}{dt} + h(v_C)}$ (2)

(b) Choose state variables $x_1 = v_C$, $x_2 = i_L$.

By massaging (1) and (2) we get:

$$\dot{x}_1 = \frac{1}{C}[-h(x_1) + x_2]$$
$$\dot{x}_2 = \frac{1}{L}[-x_1 - Rx_2 + u]$$

(c) Equilibria are found by setting $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$.
In other words,

$$\frac{1}{C}[-h(x_1) + x_2] = 0 \text{ or } x_2 = h(x_1)$$
$$\frac{1}{L}[-x_1 - Rx_2 + U] = 0 \text{ or } x_2 = -\frac{1}{R}x_1 + \frac{U}{R}$$
$$-\frac{1}{R}x_1 + \frac{U}{R} = h(x_1) \ \rightarrow \ \text{gives } x_1$$
$$x_2 = h(x_1) \ \rightarrow \ \text{gives } x_2$$

Graphically the equilibria are given by:
Equilibria: $(\overline{x_1^1}, \overline{x_2^1})$, $(\overline{x_1^2}, \overline{x_2^2})$, $(\overline{x_1^3}, \overline{x_2^3})$

Depending on U and R there may be one, two, or three equilibria. In other words, the circuit may have more than one operating point. This type of circuit is said to be MULTISTABLE.

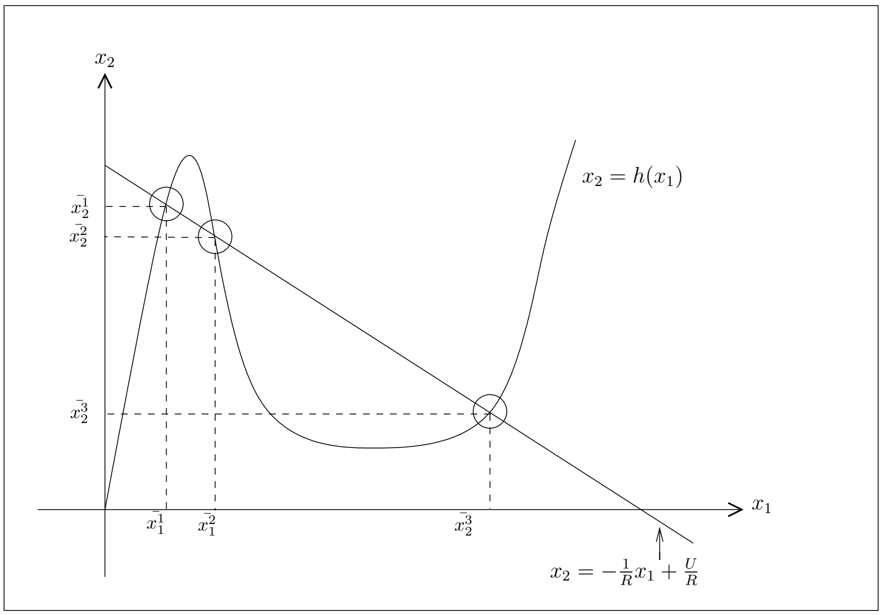


Figure 2: Equilibria of the Tunnel-Diode