Problem Set 10 Solutions

Problem 1

Let $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$. We need to show that the eigenvalues of $A+BK$ can be assigned to be the eigenvalues of any real 2x2 matrix by a suitable choice of K. To do this, we simply need to show that with a suitable K, the characteristic polynomial of $A+BK$ can made into the characteristic polynomial of any real 2x2 matrix. The characteristic polynomial of a real 2x2 matrix will have the form $x^2 + a_1 x + a_0$ for real numbers a_0 and a_1.

We have

$$A + BK = \begin{bmatrix} 3+3k_1 & 4+3k_2 \\ -2+k_1 & 6+k_2 \end{bmatrix}$$

The characteristic polynomial of $A+BK$ is $x^2 + (-9-3k_1-k_2)x + (26+14k_1+9k_2)$. Equating this polynomial to the general polynomial given above, we get a linear system of equations:

$$\begin{aligned} 3k_1 + k_2 &= -9 - a_1 \\ 14k_1 + 9k_2 &= -26 + a_0 \end{aligned}$$

which is solvable for any real a_0, a_1 since the matrix

$$\begin{bmatrix} 3 & 1 \\ 14 & 9 \end{bmatrix}$$

is invertible. Hence, the eigenvalues of $A+BK$ can be arbitrarily assigned.

Problem 2

Let $K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$. We have

$$A + BK = \begin{bmatrix} -2+k_1 & k_2 & k_3 \\ 3+2k_1 & -5+2k_2 & 2k_3 \\ -9+k_1 & k_2 & 7+k_3 \end{bmatrix}$$

The characteristic polynomial of $A+BK$ is $x^3 + (-k_1 - 2k_2 - k_3)x^2 + (-39 + 2k_1 + 7k_2 + 2k_3)x + (-70 + 35k_1 + 49k_2 + 35k_3)$.

If we evaluate this polynomial at $x = 7$ we get 0 for any k_1, k_2, k_3, which implies that 7 is an eigenvalue of $A+BK$ for any K. Hence, no linear state feedback can stabilize this system.

Another way to see why we cannot change the eigenvalue 7 is to change the state variable to z with $z = T^{-1}x$ where

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

We have the new system given by $\dot{z}(t) = T^{-1}ATz(t) + T^{-1}Bu(t)$ where

$$T^{-1}AT = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 7 \end{bmatrix} \quad T^{-1}B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

In this form, the dynamics on z_3 are $z_3(t) = 7z_3(t)$, which does not depend on the other states or the input. Hence, we cannot change the eigenvalue of the z_3 subsystem, which is 7. Hence, we cannot change the eigenvalue 7 of the original system.