# ECE311S: Dynamic Systems and Control

# Problem Set 2

## Problem 1

You are given a nonlinear system

$$\begin{aligned} \dot{x}_1 &= -x_1 + u \\ \dot{x}_2 &= -2x_2 + x_3 \\ \dot{x}_3 &= e^{x_1} x_2 + u \\ y &= x_1 + x_2. \end{aligned}$$

(i) Setting $u = 1$, find the equilibrium point of the system.

(ii) Find a linearized model of the system about the equilibrium point you found in part (i).

## Main Properties of the Laplace Transform

Memorize this table of Laplace transform pairs

| $f(t)$ | $F(s)$ |
|---|---|
| $\delta(t)$ | $1$ |
| $\mathbf{1}(t)$ | $1/s$ |
| $t\mathbf{1}(t)$ | $1/s^2$ |
| $t^k\mathbf{1}(t)$($k$ integer) | $k!/s^{k+1}$ |
| $e^{at}\mathbf{1}(t)$ | $\frac{1}{s-a}$ |
| $t^k e^{at}\mathbf{1}(t)$ | $\frac{k!}{(s-a)^{k+1}}$ |
| $\sin(at)\mathbf{1}(t)$ | $\frac{a}{s^2+a^2}$ |
| $\cos(at)\mathbf{1}(t)$ | $\frac{s}{s^2+a^2}$ |

where $\mathbf{1}(t)$ denotes the unit step, i.e.

$$\mathbf{1}(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

The following is a list of the main properties of Laplace transforms. Using these properties you'll be able to solve Problems 1 and 2 without computing any integral. Throughout the following we denote $F(s) := \mathcal{L}\{f(t)\}$, $G(s) := \mathcal{L}\{g(t)\}$.

| | |
|---|---|
| P1: Linearity | $(\forall c_1, c_2 \in \Re)\ \ \mathcal{L}\{c_1 f(t) + c_2 g(t)\} = c_1 F(s) + c_2 G(s)$. |
| P2: Differentiation | $\mathcal{L}\left\{\frac{d}{dt} f(t)\right\} = sF(s) - f(0^-)$. |
| P3: Integration | $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}F(s)$. |
| P4: Convolution | $\mathcal{L}\{f(t) \star g(t)\} = F(s)G(s)$. |
| P5: Time shift | Let $T > 0$. Then, $\mathcal{L}\{f(t-T)\mathbf{1}(t-T)\} = e^{-Ts}F(s)$. |
| P6: Shift in $s$ | $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$. |
| P7: Multiplication by $t$ | $\mathcal{L}\{tf(t)\} = -\dfrac{dF(s)}{ds}$. |

## Problem 2

Find the Laplace transform of each of the following time-signals.

1. $f(t) = 3t^2 e^{-t} \cdot \mathbf{1}(t)$
2. $f(t) = \sin t \cos t \cdot \mathbf{1}(t)$
3. $f(t) = \sin(t-3) \cdot \mathbf{1}(t)$
4. $f(t) = \sin(t-3) \cdot \mathbf{1}(t-3)$
5. $f(t) = t\sin(t-3)\mathbf{1}(t)$
6. $f(t) = (t-3)e^{t-3}\mathbf{1}(t)$
7. $f(t) = (t-3)e^{t-3}\mathbf{1}(t-3)$
8. $f(t) = te^{-at} + 2t\cos t$
9. $f(t) = t^2 + e^{-at}\sin bt$

## Problem 3

Find the inverse Laplace transform of each of the following functions.

1. $$F(s) = \frac{s^2+s+1}{(s+1)(s+2)(s+3)} \tag{1}$$
2. $$F(s) = \left(\frac{s^2+s+1}{(s+1)(s+2)(s+3)}\right)e^{-s} \tag{2}$$
3. $$F(s) = \frac{e^{-2s}}{s-3} \tag{3}$$
4. $$F(s) = \frac{e^{1-s}}{s} \tag{4}$$
5. $$F(s) = \frac{1}{s^6} \tag{5}$$
6. $$F(s) = \frac{10}{s(s+1)(s+10)} \tag{6}$$
7. $$F(s) = \frac{1}{s(s+2)^2} \tag{7}$$
8. $$F(s) = \frac{2(s+2)}{(s+1)(s^2+4)} \tag{8}$$

## Problem 4

Given $F(s) = \frac{2-s}{(s-1)(s+2)}$, find $\lim_{t\to+\infty} f(t)$.

## Problem 5

Find the solution $y(t)$ to each of the following differential equations

1. $\ddot{y} + 2\dot{y} + 5y = \exp(-t)\,\mathbf{1}(t)$ $\qquad y(0) = \dot{y}(0) = 0$
2. $\ddot{y} - y = \mathbf{1}(t)$ $\qquad y(0) = 1, \dot{y}(0) = 1$
3. $y^{(4)} + 4y^{(3)} + 8y^{(2)} + 8y^{(1)} + 4y = 3\dot{u} + 4u$ $\qquad y(0) = \dot{y}(0) = \cdots = y^{(3)}(0) = 0$, $u = \mathbf{1}(t)$
4. $\ddot{y} + \dot{y} + 3y(t) = 0;$ $\qquad y(0) = \alpha, \dot{y}(0) = \beta$