Problem Set 3 Solutions

Problem 1

$$
\begin{aligned}
\dddot{y} + 3\ddot{y} - 2\dot{y} + y &= \ddot{u} - 3\dot{u} + 2u \\
&\Downarrow \\
G(s) &= \frac{Y(s)}{U(s)} = \frac{s^2 - 3s + 2}{s^3 + 3s^2 - 2s + 1}
\end{aligned}
$$

A state space representation is

$$
\begin{aligned}
\dot{\mathbf{x}} &= A\mathbf{x} + Bu \\
y &= C\mathbf{x} + Du
\end{aligned}
$$

where

$$
A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 2 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & -3 & 1 \end{bmatrix} \quad D = 0
$$

Another possible state space representation is

$$
A = \begin{bmatrix} -3 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -3 & 2 \end{bmatrix} \quad D = 0
$$

Problem 2

$$
\begin{aligned}
&\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u \\
&y = [1 \;\; 0 \;\; 0]\mathbf{x},
\end{aligned}
$$

Rewrite the above in the equivalent scalar form,

$$
\begin{aligned}
&\dot{x}_1 = x_2 + u \\
&\dot{x}_2 = x_3 + u \\
&\dot{x}_3 = -x_1 - 3x_2 - 2x_3 + u \\
&y = x_1.
\end{aligned}
$$

Take the Laplace transform of each equation assuming zero initial conditions

$$
\begin{aligned}
&sX_1(s) = X_2(s) + U(s) \\
&sX_2(s) = X_3(s) + U(s) \\
&sX_3(s) = -X_1(s) - 3X_2(s) - 2X_3(s) + U(s) \\
&Y(s) = X_1(s).
\end{aligned}
$$

Since we want the transfer function from U to Y, and since $Y(s) = X_1(s)$, we seek to express $X_1(s)$ as a function of $U(s)$, eliminating $X_2(s)$ and $X_3(s)$.

$$\begin{aligned} X_2(s) &= sX_1(s) - U(s) \\ X_3(s) &= sX_2(s) - U(s) \\ X_1(s) &= -3X_2(s) - 2X_3(s) - sX_3(s) + U(s) \end{aligned}$$

$$\begin{aligned} & X_1(s) = -3(sX_1(s) - U(s)) - (s+2)(sX_2(s) - U(s)) + U(s) \\ \iff & X_1(s) = -3(sX_1(s) - U(s)) - (s+2)\left[s(sX_1(s) - U(s)) - U(s)\right] + U(s) \\ \iff & X_1(s)(1 + 3s + s^2(s+2)) = (3 + s(s+2) + (s+2) + 1)U(s) \\ \iff & X_1(s)/U(s) = G(s) = \frac{s^2 + 3s + 6}{s^3 + 2s^2 + 3s + 1}. \end{aligned}$$

Alternatively, one can get the same expression by the formula:

$$G(s) = C(sI - A)^{-1}B,$$

but this involves inverting a 3×3 matrix.

Problem 3

Part 1. Recall that the transfer function $G(s)$ of a linear system with input u and output y is found by taking all initial conditions to be at zero and computing

$$G(s) = \frac{Y(s)}{U(s)}.$$

Here we have

$$U(s) = \frac{1}{s^2+1}, \quad Y(s) = \frac{1}{2}\left(\frac{1}{s+1} + \frac{1}{s^2+1} - \frac{s}{s^2+1}\right),$$

and so

$$G(s) = \frac{1}{2}\frac{\frac{1}{s+1} + \frac{1}{s^2+1} - \frac{s}{s^2+1}}{\frac{1}{s^2+1}} = \frac{1}{s+1}.$$

Part 2. Let $G(s)$ denote the tranfer function that is sought. There are two ways to compute $G(s)$. Either perform matrix computations

$$G(s) = C(sI - A)^{-1}B + D,$$

or take Laplace transform of the ODEs above. Let us follow the latter route. We have

$$\begin{aligned} sX_1(s) &= X_2(s) + U(s) \\ sX_2(s) &= U(s) \\ Y(s) &= X_1(s) + X_2(s) + U(s). \end{aligned}$$

Express $X_1(s)$ and $X_2(s)$ in terms of $U(s)$:

$$X_1(s) = \frac{1}{s^2}U(s) + \frac{1}{s}U(s), \quad X_2(s) = \frac{1}{s}U(s).$$

Thus,

$$Y(s) = \frac{1}{s^2}U(s) + \frac{1}{s}U(s) + \frac{1}{s}U(s) + U(s),$$

or,

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2} + \frac{2}{s} + 1 = \frac{s^2 + 2s + 1}{s^2}.$$

Problem 4

Cross Multiplying, $(s^3 + 8s^2 + 9s + 15)X(s) = (s + 2)F(s)$.
Taking the inverse Laplace transform, $\frac{d^3x}{dt^3} + 8\frac{d^2x}{dt^2} + 9\frac{dx}{dt} + 15x = \frac{df(t)}{dt} + 2f(t)$.

Problem 5

(a) Using the standard form

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & -5 & -1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r(t)$$

$$c = [10\ 5\ 0\ 0]x$$

(b) Using the standard form

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -8 & -10 & -9 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} r(t)$$

$$c = [3\ 7\ 12\ 2\ 1]x$$

Problem 6

$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$

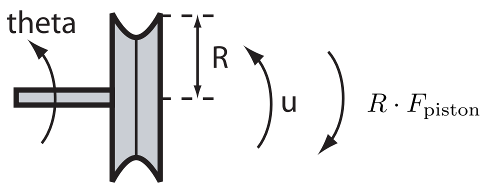
$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -2 & -5 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

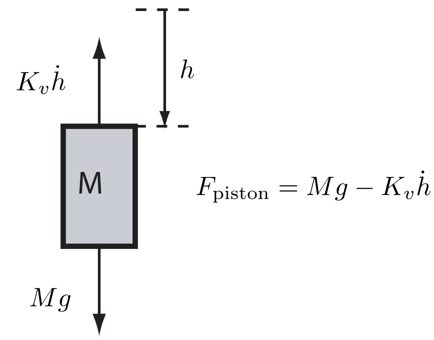
$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^3 + 5s^2 + 2s + 3} \begin{bmatrix} s^2 + 5s + 2 & s + 5 & 1 \\ -3 & s(s + 5) & s \\ -3s & -2s - 3 & s^2 \end{bmatrix}$$

Therefore, $G(s) = \frac{10}{s^3+5s^2+2s+3}$.

Problem 7

We draw the free-body diagrams of the system.





The pulley is subject to two moments, the torque u exerted by the DC motor and the moment $R \cdot F_{\text{piston}}$ resulting from the force exerted by the piston on the cable. The piston is subject to two forces, the force of gravity Mg and viscous friction which is proportional to the mass' velocity and opposes motion. Putting all together we obtain

$$\begin{aligned} I\ddot{\theta} &= u - RF_{\text{piston}} = u - R(Mg - K_v\dot{h}) \\ M\ddot{h} &= F_{\text{piston}} = Mg - K_v\dot{h}. \end{aligned} \tag{1}$$

Assuming that the cable is rigid and its mass is negligible and choosing the origin for the θ coordinates so that, when $\theta = 0$, $h = 0$ as well, we have

$$h = R\theta. \tag{2}$$

Since θ and h are related by the algebraic constraint (2), the second differential equation in (1) is redundant and thus the mathematical model of the system is

$$\begin{aligned} I\ddot{\theta} &= u - R(Mg - K_vR\dot{\theta}) \\ h &= R\theta. \end{aligned} \tag{3}$$

The transfer function from $\tilde{u} = u - RMg$ to h is found by taking the Laplace transform of (3):

$$s^2 I\Theta(s) = \tilde{U}(s) + sK_vR^2\Theta(s).$$

Solving for $\Theta(s)/\tilde{U}(s)$ we get

$$\frac{\Theta(s)}{\tilde{U}(s)} = \frac{1}{s^2 I - sK_vR^2}.$$

The desired transfer function is:

$$\frac{H(s)}{\tilde{U}(s)} = R\frac{\Theta(s)}{\tilde{U}(s)} = \frac{R}{s^2 I - sK_vR^2}.$$

On the other hand, if instead of measuring h we measure $\dot{\theta}$, then the transfer function from $\tilde{u}$ to $\dot{\theta}$ is:

$$\frac{s\Theta(s)}{\tilde{U}(s)} = \frac{s}{s^2 I - sK_vR} = \frac{1}{sI - K_vR^2}.$$