Problem Set 4 Solutions

Problem 1

Compute e^{At} using the Laplace transform method and the eigenvalue/eigenvector method for the following matrix:

$$A = \begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix}.$$

Using the Laplace transform method, we must compute $(sI - A)^{-1}$. We have:

$$(sI - A) = \begin{bmatrix} s+2 & 2 & 0 \\ 0 & s & -1 \\ 0 & 3 & s+4 \end{bmatrix}.$$

We have that $\det(sI - A) = (s+2)(s+1)(s+3)$ and hence,

$$(sI - A)^{-1} = \frac{1}{(s+2)(s+1)(s+3)} \begin{bmatrix} (s+1)(s+3) & -2(s+4) & -2 \\ 0 & (s+2)(s+4) & (s+2) \\ 0 & -3(s+2) & s(s+2) \end{bmatrix}.$$

Next, we take the inverse Laplace transform of each entry. Firstly, we must simplify some of the entries using partial fraction expansions. We have

$$\begin{aligned}
\frac{-2(s+4)}{(s+2)(s+1)(s+3)} &= \frac{4}{s+2} - \frac{3}{s+1} - \frac{1}{s+3} \\
\frac{-2}{(s+2)(s+1)(s+3)} &= \frac{2}{s+2} - \frac{1}{s+1} - \frac{1}{s+3} \\
\frac{s+4}{(s+1)(s+3)} &= \frac{3/2}{s+1} - \frac{1/2}{s+3} \\
\frac{1}{(s+1)(s+3)} &= \frac{1/2}{s+1} - \frac{1/2}{s+3} \\
\frac{-3}{(s+1)(s+3)} &= -\frac{3/2}{s+1} + \frac{3/2}{s+3} \\
\frac{s}{(s+1)(s+3)} &= -\frac{1/2}{s+1} + \frac{3/2}{s+3}
\end{aligned}$$

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Lastly, we have e^{At} equals the inverse Laplace transform of the entries of $(sI - A)^{-1}$. Hence,

$$e^{At} = \begin{bmatrix} e^{-2t} & 4e^{-2t} - 3e^{-t} - e^{-3t} & 2e^{-2t} - e^{-t} - e^{-3t} \\ 0 & \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t} & \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} \\ 0 & -\frac{3}{2}e^{-t} + \frac{3}{2}e^{-3t} & -\frac{1}{2}e^{-t} + \frac{3}{2}e^{-3t} \end{bmatrix}.$$

Now, we solve the problem using the eigenvalue/eigenvector method. The MATLAB command $[V, D] =$ eig(A) produces

$$V = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & 3 \end{bmatrix}$$

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

So, we have $A = VDV^{-1}$. Hence, $e^{At} = e^{VDV^{-1}t} = Ve^{Dt}V^{-1}$. We have

$$e^{Dt} = \begin{bmatrix} e^{-2t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} 1 & 4 & 2 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

Hence

$$e^{At} = \begin{bmatrix} e^{-2t} & 4e^{-2t} - 3e^{-t} - e^{-3t} & 2e^{-2t} - e^{-t} - e^{-3t} \\ 0 & \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t} & \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} \\ 0 & -\frac{3}{2}e^{-t} + \frac{3}{2}e^{-3t} & -\frac{1}{2}e^{-t} + \frac{3}{2}e^{-3t} \end{bmatrix}.$$

Problem 2

Determine the best method to compute e^{At} for the following A matrix and then compute it:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

In this problem, the matrix A is not diagonalizable. It has only one eigenvalue, i.e. zero, with multiplicity 3, but only one linearly independent eigenvector. Hence, the eigenvalue/eigenvector method will not work in this case. Hence, we use the Laplace transform method. We have:

$$(sI - A)^{-1} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} & \frac{1}{s^3} \\ 0 & \frac{1}{s} & \frac{1}{s^2} \\ 0 & 0 & \frac{1}{s} \end{bmatrix}$$

So, taking the inverse Laplace transform of each entry, we get

$$e^{At} = \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 3

You are given the SISO system

$$\frac{Y(s)}{G(s)} = \frac{(s-1)}{(s^2+2s-3)}\,.$$

Show that it is possible for this system to generate an unbounded initial state reponse and a bounded input response.

We have the transfer function given by

$$G(s) = \frac{(s-1)}{(s^2+2s-3)}$$

The corresponding differential equation model is

$$\ddot{v} + 2\dot{v} - 3v = \dot{u} - u\,.$$

Taking $x_1 = v$ and $x_2 = \dot{v}$, we have

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = 3x_1 - 2x_2 + u, \quad y = -x_1 + x_2$$

and thus

$$A = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & 1 \end{bmatrix}, \quad D = 0$$

We have

$$e^{At} = \begin{bmatrix} \frac{3}{4}e^{t} + \frac{1}{4}e^{-3t} & \frac{1}{4}e^{t} - \frac{1}{4}e^{-3t} \\ \frac{3}{4}e^{t} - \frac{3}{4}e^{-3t} & \frac{1}{4}e^{t} + \frac{3}{4}e^{-3t} \end{bmatrix}$$

If we choose the initial condition $x_o = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ with zero input we get

$$x(t) = e^{At}x_o = \begin{bmatrix} e^{t} \\ e^{t} \end{bmatrix}$$

which is unbounded.
Now if we choose $x_o = 0$ and $u(t) = e^{-t} - 2te^{-t}$ we get

$$x(t) = \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = \int_0^t \begin{bmatrix} \frac{1}{4}e^{(t-\tau)} - \frac{1}{4}e^{-3(t-\tau)} \\ \frac{1}{4}e^{(t-\tau)} + \frac{3}{4}e^{-3(t-\tau)} \end{bmatrix} (e^{-\tau} - 2te^{-\tau})d\tau = \begin{bmatrix} \frac{1}{4}(e^{-3t} - e^{-t} + 2te^{-t}) \\ -\frac{1}{4}(3e^{-3t} - 3e^{-t} + 2te^{-t}) \end{bmatrix}$$

which is bounded.

Problem 4

Consider the closed loop system in Figure 1, where K and z are real numbers which you'll have to pick in part 2 of this problem.

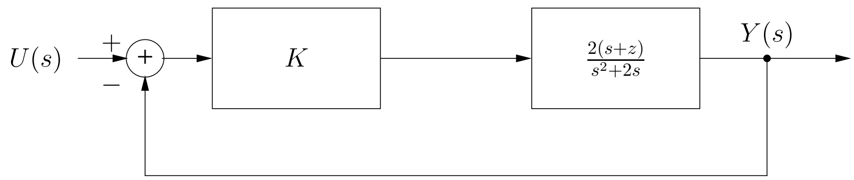

Figure 1: System block diagram

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1. The closed-loop transfer function is

$$T(s) = \frac{2K(s+z)}{s^2+2s+2K(s+z)}$$

Notice that this transfer function is not in the standard form

$$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}.$$

Thus, the formulas for settling time and percent overshoot do not actually apply. Nevertheless, we will carry out the instructions of the problem to find the region of the complex plane where the closed-loop poles should lie. We have

$$T_s = \tfrac{4}{\zeta\omega_n}, \quad \%OS = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

Solve the two equations imposing $T_s = 0.8$, $\%OS = 0.01$ and get ζ and ω_n

$$\zeta\omega_n = \frac{4}{0.8} = 5 = \sigma$$

$$\begin{aligned} e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} = 0.01 \quad &\Leftrightarrow \quad \frac{\zeta\pi}{\sqrt{1-\zeta^2}} = 4.6 \Leftrightarrow \frac{\zeta^2\pi^2}{1-\zeta^2} = 21.2 \\ &\Leftrightarrow \quad \zeta^2\pi^2 = 21.2 - 21.2\zeta^2 \\ &\Leftrightarrow \quad (21.2+\pi^2)\zeta^2 = 21.2 \Leftrightarrow \zeta = \sqrt{\frac{21.2}{31.07}} = 0.83 \end{aligned}$$

2. The poles of the CLS are the roots of $s^2+2s+2K(s+z) = 0 \Leftrightarrow s^2+2(K+1)s+2Kz = 0$. By identifying the coefficients of this polynomial with those of $s^2+2\zeta\omega_n+\omega_n^2 = 0$ we express ζ and ω_n as a function of K, z:

Let $\zeta\omega_n > 1$, then

$$\left.\begin{aligned} \zeta\omega_n &= K+1 \\ \omega_n^2 &= 2Kz \end{aligned}\right| \Rightarrow \quad \begin{aligned} K &= \zeta\omega_n - 1 \\ z &= \frac{\omega_n^2}{2\zeta\omega_n - 2} \end{aligned}$$

Conclusion: We can assign ANY ζ and ω_n by choosing K and z as above. Pick, for example, $\zeta = 1, \omega_n = 7$. With this choice of ζ and ω_n the poles of CLS are in the region found in part (1). The values of K and z yielding $\zeta = 1, \omega_n = 7$ are $K = 7-1 = 6$ and $z = \frac{7^2}{2\cdot 7-2} = \frac{49}{12}$.

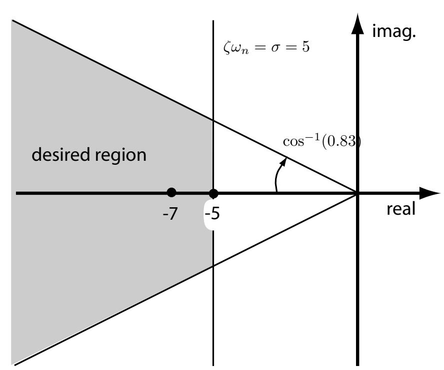

Figure 2: Region for the poles when $T_s \leq 0.8s$ and %OS≤ 0.01.