Problem Set 6 Solutions

Problem 1

$$T(s) = \frac{Ks^2 + 2Ks}{s^3 + (K-1)s^2 + (2K-4)s + 24}$$

s^3	1	$2K-4$
s^2	$K-1$	24
s^1	$\frac{2K^2-6K-20}{K-1}$	0
s^0	24	0

For stability, $K > 5$; row of zeros if $K = 5$. Therefore, $4s^2 + 24 = 0$ is the auxiliary polynomial. Its root are $s = \pm j\sqrt{6}$; hence, $\omega = \sqrt{6}$ for oscillation.

Problem 2

$$T(s) = \frac{K(s+2)}{s^4 + 3s^3 - 3s^2 + (K+3)s + (2K-4)}$$

s^4	1	-3	$2K-4$
s^3	3	$K+3$	0
s^2	$-\frac{K+12}{3}$	$2K-4$	0
s^1	$\frac{K(K+33)}{K+12}$	0	0
s^0	$2K-4$	0	0

For $K < -33$: one sign change; for $-33 < K < -12$: one sign change; for $-12 < K < 0$: one sign change; for $0 < K < 2$: three sign changes; for $K > 2$: two sign changes. Therefore, $K > 2$ yields two right-half-plane poles.

Problem 3

$$T(s) = \frac{K}{s^4 + 8s^3 + 17s^2 + 10s + K}$$

s^4	1	17	K
s^3	8	10	0
s^2	$\frac{126}{8}$	K	0
s^1	$-\frac{32K}{63} + 10$	0	0
s^0	K	0	0

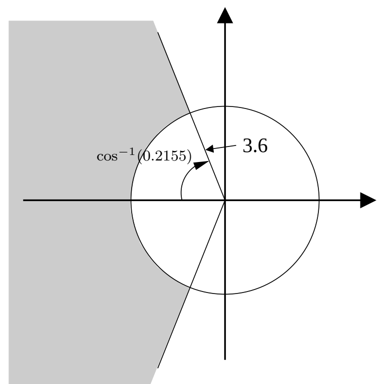
a. For stability $0 < K < 19.69$.
b,c. Row of zeros when $K = 19.69$. Therefore, $\frac{126}{8}s^2 + 19.69 = 0$ is the auxiliary polynomial. Thus, $s = \pm j\sqrt{1.25}$, or 1.18 rad/s. Two other poles are at -3.5 and -4.5.

Problem 4

(a) In order to meet the performance specifications, the TWO poles of the closed-loop system must have the following properties:

$$w_n > 3.6 \text{ and } \zeta > 0.2155.$$

Geometrically, the poles of the closed-loop system should lie in the shaded region below.

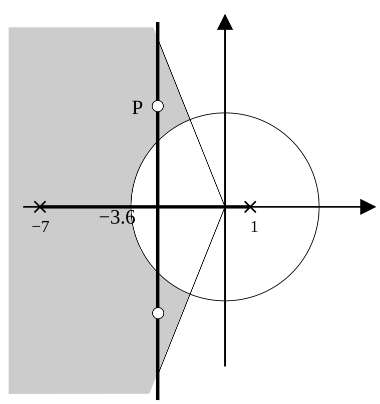


(b) The poles of the closed-loop system are the roots of

$$s^2 + 6s + K - 7,$$

$$s_{1,2} = -3 \pm \sqrt{16 - K}. \quad (1)$$

One way to solve the problem is to draw the root locus



and choose a location for the two closed-loop system poles which is compatible with the region found in part (a) of the problem. For instance, we can choose closed-loop system poles at $P = 3 + 4j$ (and its complex conjugate) as in the figure above. So, referring to (1), the problem is solved by setting $\sqrt{16 - K} = 4j$, or $K = 32$.

Problem 5

(a) The closed-loop transfer function is

$$T(s) = \frac{s(s + a + 1)}{(s + b)(s^2 + as + 1) + s(s + a + 1)} = \frac{s(s + a + 1)}{s^3 + (a + b + 1)s^2 + (ab + a + 2)s + b}$$

(b) $b = 1 \Rightarrow$ closed-loop poles are the roots of $s^3 + (a+2)s^2 + (2a+2)s + 1 = 0$, then

$$\begin{array}{c|ccc} s^3 & 1 & 2a+2 & 0 \\ s^2 & a+2 & 1 & 0 \\ s & * & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}$$

where

$$* = \frac{(2a+2)(a+2)-1}{a+2} = \frac{2a^2+4a+2a+4-1}{a+2} = \frac{2a^2+6a+3}{a+2}$$

The closed-loop system is unstable if $a+2 < 0$ or $2a^2+6a+3 < 0 \Leftrightarrow -\frac{3}{2} - \frac{\sqrt{3}}{2} < a < -\frac{3}{2} + \frac{\sqrt{3}}{2}$ because in either case we have at least one sign variation.

Conclusion: The CLS is unstable for $a < -\frac{3}{2} + \frac{\sqrt{3}}{2}$.

The only situation left to study is the case when $a = -\frac{3}{2} + \frac{\sqrt{3}}{2}$. In this case the Routh array is given by:

$$\begin{array}{c|ccc} s^3 & 1 & -1+\sqrt{3} & 0 \\ s^2 & \frac{1}{2}+\frac{\sqrt{3}}{2} & 1 & 0 \\ s & 0 & 0 & 0 \end{array}$$

One row is zero. The auxiliary polynomial is: $a(s) = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) s^2 + 1 = 0$. Its roots are $s = \pm \frac{j}{\sqrt{\frac{1}{2}+\frac{\sqrt{3}}{2}}} \Rightarrow$ the CLS in this case is marginally stable (not BIBO stable).

(c) Set $a = 0 \Rightarrow$ closed-loop poles are the roots of $s^3 + (b+1)s^2 + 2s + b = 0$

$$\begin{array}{c|ccc} s^3 & 1 & 2 & 0 \\ s^2 & b+1 & b & 0 \\ s & \frac{2b+2-b}{b+1} & 0 & 0 \\ 1 & b & & \end{array}$$

For BIBO stability:

$$\left.\begin{array}{ccc} b+1>0 & \Leftrightarrow & b>-1 \\ \& & & \& \\ b+2>0 & \Leftrightarrow & b>-2 \\ \& & & \\ b>0 & & \end{array}\right| \Leftrightarrow \quad b>0$$

Problem 6

The poles of the system are the roots of

$$a(s) = s^6 + s^5 + 5s^4 + s^3 + 2s^2 - 2s - 8 = 0.$$

We form the Routh table.

$$\begin{array}{c|cccc} s^6 & 1 & 5 & 2 & -8 \\ s^5 & 1 & 1 & -2 & 0 \\ s^4 & 4 & 4 & -8 & 0 \\ s^3 & 0 & 0 & 0 & 0 \end{array}$$

We get a row of zeros. This indicates that some roots of $a(s)$ are symmetric with respect to $s = 0$. The auxiliary polynomial is

$$\bar{a}(s) = 4s^4 + 4s^2 - 8,$$

and so we can write $a(s) = \bar{a}(s)b(s)$, where $b(s)$ is a polynomial of degree 2 with NO symmetric roots with respect to $s = 0$.

We replace the row of zeros by the coefficients of the derivative of $\bar{a}(s)$,

$$\frac{d\bar{a}}{ds} = 16s^3 + 8s,$$

and complete the Routh array. We thus obtain,

s^6	1	5	2	−8
s^5	1	1	−2	0
s^4	4	4	−8	0
s^3	16	8	0	0
s^2	2	−8	0	0
s^1	72	0	0	0
s^0	−8	0	0	0

We divide the table in two parts. There are 0 sign variations in the first part, so the two roots of $b(s)$ are in the open LHP. There is one sign variation in $\bar{a}(s)$ and so $\bar{a}(s)$ has one root in the open RHP. Since all roots of $\bar{a}(s)$ must be symmetric with respect to $s = 0$, it follows that $\bar{a}(s)$ must also have one root in the open LHP. The remaining 2 roots of $\bar{a}(s)$ must necessarily lie on the imaginary axis.

Conclusion: $a(s)$ has: 3 poles in open LHP
1 pole in open RHP
2 poles on the imaginary axis

Since not all roots of $a(s)$ are in the open LHP, it follows that $G(s)$ is not BIBO stable.