ECE356 - LINEAR SYSTEMS AND CONTROL
MIDTERM SOLUTIONS

1. (i) Write the logic definition of "the equilibrium $x = 0$ is stable i.s.L.".

   (ii) Write the definition of asymptotic stability of an equilibrium point.

   (iii) State necessary and sufficient conditions for an equilibrium point $x = 0$ of $\dot{x} = Ax$ to be asymptotically stable.

   (iv) Consider the system
   $$\dot{x} = Ax\,,$$
   where $x \in \mathbb{R}^n$. Suppose that $x = 0$ and $x = v \neq 0$ are both equilibria. Using (iii), prove that $x = 0$ is not asymptotically stable.

(i) $(\forall \epsilon > 0)(\exists \delta > 0)(\forall t \geq 0)\|x_0\| < \delta \implies \|x(t)\| < \epsilon$.

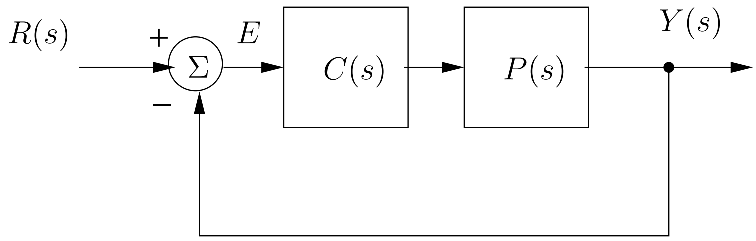
(ii) We are given a system $\dot{x} = f(x)$ and a point $x_0 \in \mathbb{R}^n$ such that $f(x_0) = 0$. We say x_0 is *asymptotically stable* if it is stable i.s.L. and it is attractive.

(iii)

Theorem 1. *The equilibrium $x = 0$ of $\dot{x} = Ax$ is asymptotically stable if and only if the eigenvalues of A lie in the open left-half complex plane.*

(iv) Since $v \in \mathbb{R}^n$ is an equilibrium of $\dot{x} = Ax$, we have
$$Av = 0 = 0v\,.$$
Since $v \neq 0$, it must be an eigenvector of A, and 0 is an eigenvalue. By Theorem 1, $x = 0$ is not asymptotically stable.

2. Consider the system



Prove that if the closed-loop transfer function $\frac{Y(s)}{R(s)}$ has a pole-zero cancellation, then $P(s)C(s)$ has a pole-zero cancellation.

Proof. Let

$$P(s) = \frac{N(s)}{D(s)}, \qquad C(s) = \frac{N_c(s)}{D_c(s)}$$

where we assume without loss of generality that $\{N(s), D(s)\}$ are coprime (i.e. they have no common factors) and $\{N_c(s), D_c(s)\}$ are coprime. The closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{P(s)C(s)}{1 + P(s)C(s)} = \frac{N_c(s)N(s)}{N_c(s)N(s) + D_c(s)D(s)} =: H(s)\,.$$

Suppose there is a pole-zero cancellation in $H(s)$. That is $H(s)$ has the form

$$H(s) = \frac{\widehat{N}(s)(s+a)}{\widehat{D}(s)(s+a)}$$

for some polynomials $\widehat{N}(s)$ and $\widehat{D}(s)$ and for some $a \in \mathbb{C}$. Comparing with the previous formula for $H(s)$ we get

$$H(s) = \frac{\widehat{N}(s)(s+a)}{\widehat{N}(s)(s+a) + D_c(s)D(s)}\,.$$

Comparing the denominators we get

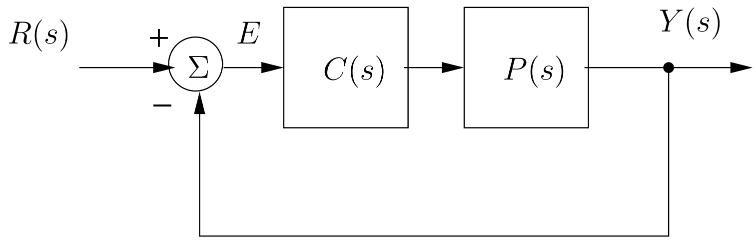
$$D_c(s)D(s) = (\widehat{D}(s) - \widehat{N}(s))(s+a)\,.$$

Since

$$N_c(s)N(s) = \widehat{N}(s)(s+a)$$

it follows that $P(s)C(s)$ has a pole-zero cancellation of the factor $(s+a)$. □

3. Consider the system



where

$$P(s) = \frac{1}{(s+7)(s+11)}\,, \quad C(s) = \frac{K}{s}\,.$$

(i) Find the range of K such that the closed-loop system is stable.

(ii) Suppose you are given a reference signal $r(t)$, $t \geq 0$. What are the conditions on K and on $PC(s)$ in order that perfect asymptotic tracking, i.e. $e_{ss} = 0$, is achieved?

(iii) Suppose $r(t) = 1, t \geq 0$. Can the given system achieve $e_{ss} = 0$? Justify briefly using part (ii).

(i) We use the Routh Criterion. The Routh table is

s^3	1	77	0
s^2	18	K	0
s^1	$\frac{1386-K}{18}$	0	
s^0	K		

In order to have no sign changes in the first column, we must have

$$0 < K < 1386\,.$$

(ii) For $e_{ss} = 0$ is is necessary that

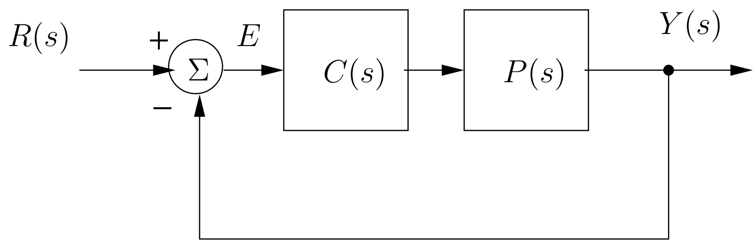
(a) No unstable pole of the exosystem $R(s)$ is a zero of $PC(s)$.

(b) $PC(s)$ contains a copy of the unstable poles of $R(s)$. This is the *internal model principle.*

(c) The poles of $sE(s)$ are in the OLHP. We achieve this requirement by designing $C(s)$ so that the closed-loop system is asymptotically stable.

(iii) Yes, the system can achieve perfect asymptotic tracking. Conditions (a) and (b) and are met by inspection. Also we can use the data from part (i) to set the loop gain so that the closed-loop system is asymptotically stable.

4. Consider the system



where

$$P(s) = \frac{1}{s^2+4}\,, \quad C(s) = K\,.$$

Use the Nyquist criterion to find the range of K such that the closed-loop system is stable.

The Nyquist contour is shown in Figure 1.

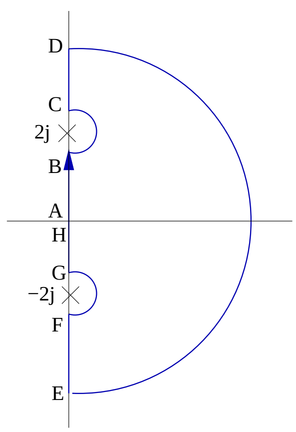


Figure 1: Nyquist contour.

- $\overline{AB}$, $\overline{CD}$: Set $s = j\omega$, where $\omega : 0 \to \infty$. We have

$$PC(j\omega) = \frac{1}{4-\omega^2}\,.$$

  Thus, we have

$$\begin{aligned} \Re(PC(j\omega)) &= \frac{1}{4-\omega^2} \\ \Im(PC(j\omega)) &= 0\,. \end{aligned}$$

  The plot of $\Re((PC(j\omega))$ for positive values of ω is shown in Figure 2.

- $\overline{DE}$: Set $s = Re^{j\theta}$ where $R \to \infty$ and $\theta : \frac{\pi}{2} \to -\frac{\pi}{2}$. Since PC is strictly proper, the semi-circle at ∞ will collapse to zero.
- $\overline{EF}$, $\overline{GH}$: Use rules about complex conjugate.

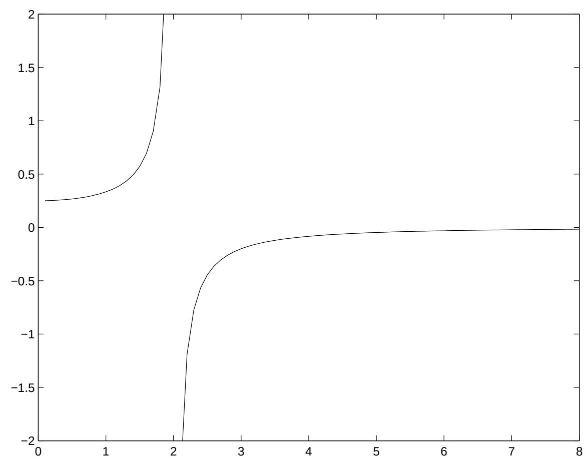


Figure 2: $\Re(PC)$ v.s. ω.

- $\overline{BC}$: Set $s = 2j + \epsilon e^{j\theta}$ where $\epsilon \to 0$ and $\theta : -\frac{\pi}{2} \to \frac{\pi}{2}$. This gives

$$PC(s)|_{s=2j+\epsilon e^{j\theta}} \sim \frac{1}{2j\epsilon} e^{-j\theta}.$$

Now we can try a test point $s = 2j + \epsilon$ and we get

$$PC(\epsilon) \sim \frac{-j}{\epsilon}.$$

This is a negative complex number which tells us which way the infinite radius semicircle turns after departing from point B.

Putting all this together, the Nyquist plot is shown in Figure 3.

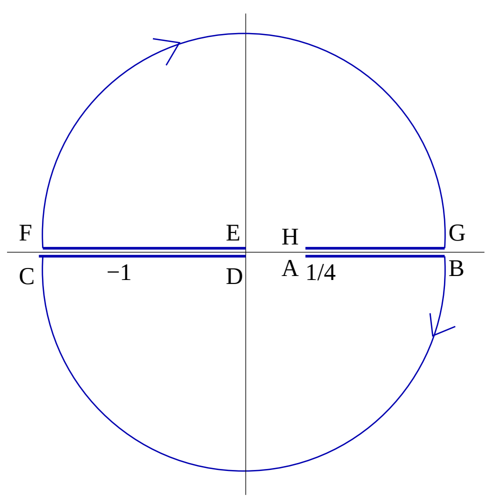


Figure 3: Nyquist plot for Problem 4

Now we have the following cases.

- $-\infty < -\frac{1}{K} < 0$
  The nyquist plot passes through $-\frac{1}{K}$, so the c.l.s. is not stable.
- $0 < -\frac{1}{K} < \frac{1}{4}$
  $n = -1$, $p = 0$, $z = p - n = 1$. The c.l.s. is not stable.
- $\frac{1}{4} < -\frac{1}{K} < \infty$
  The nyquist plot passes through $-\frac{1}{K}$, so the c.l.s. is not stable.

In sum, the closed-loop system is not stable for any value of K. Notice that you can also write the closed-loop transfer function

$$\frac{Y(s)}{R(s)} = \frac{K}{s^2 + 4 + K} \, .$$

The poles are always on the $j\omega$-axis, and we conclude that this system cannot be stabilized by a P controller.

5. Sketch the Bode plot (magnitude and phase) for

$$G(s) = \frac{1000}{s(s+100)} .$$

We must put the transfer function in Bode form

$$G(s) = \frac{10}{s(0.01s+1)} .$$

The straightline approximations of the gain and phase plots are: