ECE410 - CONTROL SYSTEMS
Midterm Solutions
November, 2004

1. Suppose you are given the linear system

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u\,.$$

   (a) Write the modal decomposition of $x(t)$ when $u = 0$. [10 marks]

   (b) Suppose $u = \overline{u}$, where $\overline{u}$ is a constant. Show that for any initial condition the solution $(x_1(t), x_2(t))$ moves along a circle described by the equation

$$(x_1(t) + \overline{u})^2 + x_2^2(t) = \textit{constant}.$$

   [15 marks]

(a) First compute the eigenvalues and eigenvectors of A:

$$eig(A) = \{j, -j\}\,, \quad v_1 = \begin{bmatrix} 1 \\ -j \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ j \end{bmatrix}.$$

Let $x_0 = \xi_1 v_1 + \xi_2 v_2$ where $\xi_i \in \mathbb{C}$. Then the modal decomposition is:

$$x(t) = \mathrm{e}^{jt}\xi_1 \begin{bmatrix} 1 \\ -j \end{bmatrix} + \mathrm{e}^{-jt}\xi_2 \begin{bmatrix} 1 \\ j \end{bmatrix}.$$

(b) First compute the state transition matrix:

$$P = \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2j} \\ \frac{1}{2} & \frac{1}{2j} \end{bmatrix},$$

$$\Lambda = P^{-1}AP = \begin{bmatrix} j & 0 \\ 0 & -j \end{bmatrix}.$$

$$\mathrm{e}^{At} = P\mathrm{e}^{\Lambda t}P^{-1} = \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix}\begin{bmatrix} j & 0 \\ 0 & -j \end{bmatrix}\begin{bmatrix} \frac{1}{2} & -\frac{1}{2j} \\ \frac{1}{2} & \frac{1}{2j} \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}.$$

Now use the variation of constants formula to compute the solution:

$$\begin{aligned}
x(t) &= \mathrm{e}^{At}x_0 + \int_0^\infty \mathrm{e}^{A(t-\tau)}Bu(\tau)d\tau \\
&= \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} x_0 + \int_0^t \begin{bmatrix} \cos(t-\tau) & -\sin(t-\tau) \\ \sin(t-\tau) & \cos(t-\tau) \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\overline{u}d\tau \\
&= \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} x_0 + \int_0^t \begin{bmatrix} -\sin(t-\tau) \\ \cos(t-\tau) \end{bmatrix}\overline{u}d\tau \\
&= \begin{bmatrix} \cos t x_{10} - \sin t x_{20} \\ \sin t x_{10} + \cos t x_{20} \end{bmatrix} + \begin{bmatrix} -\overline{u} + \cos t\overline{u} \\ \sin t\overline{u} \end{bmatrix} \\
&= \begin{bmatrix} \cos t\ (x_{10} + \overline{u}) - \sin t\ x_{20} - \overline{u} \\ \sin t\ (x_{10} + \overline{u}) + \cos t\ x_{20} \end{bmatrix}.
\end{aligned}$$

Next we verify that

$$(x_1(t) + \overline{u})^2 + x_2^2(t) = \textit{const}.$$

Substituting the solutions in the expression above

$\cos^2 t(x_{10} + \overline{u})^2 + \sin^2 t x_{20}^2 - 2\sin t\cos t x_{20}(x_{10} + \overline{u}) + \sin^2 t(x_{10} + \overline{u})^2 + \cos^2 t x_{20}^2 + 2\sin t\cos t x_{20}(x_{10} + \overline{u})$

$= (\cos^2 t + \sin^2 t)(x_{20}^2 + (x_{10}^2 + \overline{u})^2)$

$= x_{20}^2 + (x_{10}^2 + \overline{u})^2 = \textit{const}$

2. Suppose the eigenvalues of A are $\{-1, -1, -1, 2\}$. Suppose

$$rank[A - \lambda I]_{\lambda=-1} = 2.$$

Find the Jordan form of A. [10 marks]

The Jordan form is

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

3. Consider the control system in Jordan form:

$$\dot{x} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} x + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} u \,.$$

$$y = \begin{bmatrix} 1 & 1 & 2 & 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 2 & 0 & 1 & 1 \\ 1 & 0 & 2 & 3 & 0 & 2 & 2 \end{bmatrix} x \,.$$

(a) Give a test for determining by inspection (without a computer) if the system is completely controllable? completely observable? [10 marks]

(b) Using the test you wrote above, determine if the system is completely observable. [10 marks]

(a) To test controllability by inspection when A is in Jordan form one uses PBH test:

$$rank\,[A - \lambda I \quad B] = n \qquad\qquad \forall \lambda \in eig(A)\,.$$

To test observability by inspection when A is in Jordan form one uses the dual version of PBH test:

$$rank \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n \qquad\qquad \forall \lambda \in eig(A)\,.$$

(b) Since

$$rank \begin{bmatrix} A - \lambda_2 I \\ C \end{bmatrix} = 6$$

the system is not completely observable.

4. Consider the system

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u \,.$$

Can this system be stabilized to the origin? Explain. If yes, design a stabilizing controller to achieve exponential convergence to the origin with rate e^{-5t}. Note: use the multi-input procedure, i.e. design an inner feedback loop to make the system single input first. [20 marks]

To stabilize the control system to the origin, it must either be controllable or stabilizable. First let's check controllability. We get

$$rank(Q_c) = rank \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & -2 & 0 \end{bmatrix} = 2 \,,$$

so the system is completely controllable. This means we can do pole placement design.

Following the multi-input procedure, we first compute $u = K_1 x + e_1 v$.

$$Q = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix} \qquad S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} .$$

so $K_1 = SQ^{-1} = 0$. The new single input system is

$$\dot{x} = Ax + b_1 v \,.$$

Now design $v = Fx = [\ f_1 \ f_2\ ]x$.

$$A + b_1 F = \begin{bmatrix} f_1 + 1 & f_2 + 1 \\ f_1 & f_2 - 2 \end{bmatrix}$$

The characteristic polynomial is:

$$det(sI - A - b_1 F) = s^2 + s(1 - f_1 - f_2) + (-2 - 3f_1 + f_2) \,.$$

In order to achieve exponential convergence to the origin with rate e^{-5t} we must set both eigenvalues of $(A + b_1 F)$ to -5. Thus, the desired characteristic polynomial is:

$$s^2 + 10s + 25 \,.$$

Equating coefficients,

$$\begin{aligned} 1 - f_1 - f_2 &= 10 \\ -2 - 3f_1 + f_2 &= 25 \,. \end{aligned}$$

We get $F = [\ -9 \ \ 0\ ]$. The overall controller is

$$u = Kx = e_1 F x = \begin{bmatrix} 0 & 0 \\ -9 & 0 \end{bmatrix} .$$