

University of Toronto
Department of Electrical and Computer Engineering
ECE410F Control Systems
Problem Set #4
Solutions

1. Notice that if you only use the second input of the system, i.e. set $u_1 = 0$, then the system becomes

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2.$$

This system is in controllable canonical form. In particular, it is controllable and it is very easy to compute a feedback control $u = Kx$. Let $K = [k_1 \ k_2 \ k_3]$. The characteristic polynomial of A is $s^3 = 0$ and the characteristic polynomial of $A + e_3K$ is $s^3 - k_3s^2 - k_2s - k_1$. The desired characteristic polynomial is $s^3 + 30s^2 + 300s + 1000$. Therefore the feedback control is

$$u = [-1000 \ -300 \ -30]x.$$

2. Find a feedback gain K such that the closed-loop poles are assigned to $\{-10, -2 \pm i\}$. We have the system matrices:

$$A = \begin{bmatrix} 5 & 2 & 1 \\ -2 & 1 & -1 \\ -4 & 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} -2 & 2 \\ 2 & -1 \\ 2 & -2 \end{bmatrix}$$

First, one can verify using MATLAB that this system is controllable but it is *not* controllable using any one column of B . Therefore, we must use the multi-input pole-placement procedure to design the feedback $u = Kx$.

$$[\ b_1 \quad Ab_1 \quad A^2b_1 \] = \begin{bmatrix} -2 & -4 & -6 \\ 2 & 4 & 6 \\ 2 & 6 & 10 \end{bmatrix} \text{ which has rank 2}$$

We are, in particular, interested in the linearly independent columns of this matrix. Notice that $2(\text{column}2) - \text{column}1 = \text{column}3$, as one would expect from the rank test.

$$[\ b_2 \quad Ab_2 \quad A^2b_2 \] = \begin{bmatrix} 2 & 6 & * \\ -1 & -3 & * \\ -2 & -6 & * \end{bmatrix} \text{ which has rank 1}$$

$$Q = \begin{bmatrix} -2 & -4 & 2 \\ 2 & 4 & -1 \\ 2 & 6 & -2 \end{bmatrix}, \quad Q^{-1} = \begin{bmatrix} -\frac{1}{2} & 1 & -1 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 1 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad K_1 = SQ^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$\begin{aligned}
A + BK_1 &= \begin{bmatrix} 5 & 2 & 1 \\ -2 & 1 & -1 \\ -4 & 0 & -1 \end{bmatrix} + \begin{bmatrix} -2 & 2 \\ 2 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} 6 & 2 & 2 \\ -\frac{5}{2} & 1 & -\frac{3}{2} \\ -5 & 0 & -2 \end{bmatrix}.
\end{aligned}$$

Now we have a new single input system

$$\dot{x} = (A + BK_1)x + b_1v,$$

so we can do single-input pole placement to design $v = Fx$.

Open loop: $\det(sI - A - BK_1) = s^3 - 5s^2 + 7s - 3$

Desired: $r(s) = (s + 10)((s + 2)^2 + 1) = s^3 + 14s^2 + 45s + 50$.

$$Q_c \hat{Q} = \begin{bmatrix} -2 & -4 & -4 \\ 2 & 4 & 5 \\ 2 & 6 & 8 \end{bmatrix} \begin{bmatrix} 7 & -5 & 1 \\ -5 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 6 & -2 \\ -1 & -6 & 2 \\ -8 & -4 & 2 \end{bmatrix}.$$

$$(Q_c \hat{Q})^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 3.5 & 3 & 0.5 \\ 11 & 10 & 1.5 \end{bmatrix}.$$

$$F = [-53 \quad -38 \quad -19](Q_c \hat{Q})^{-1} = [-395 \quad -357 \quad -47.5].$$

$$K = K_1 + e_1 F = \begin{bmatrix} -395 & -357 & -47.5 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

3. We can assume without loss of generality that A_1 and A_2 are both in Jordan form. For controllability, we consider only the state equation, not the output equation. The overall system is

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u.$$

First suppose the subsystems are controllable. Then use the PBH test. We must test whether

$$\text{rank} \begin{bmatrix} A_1 - \lambda I & 0 & B_1 \\ 0 & A_2 - \lambda I & B_2 \end{bmatrix} = 2n \quad (1)$$

for all $\lambda \in \{eig(A_1)\} \cup \{eig(A_2)\}$. But we know that $\text{rank}[A_1 - \lambda I \ B_1] = n$ for all $\lambda \in eig(A_1)$ and $\text{rank}[A_2 - \lambda I \ B_2] = n$ for all $\lambda \in eig(A_2)$. So, if λ is an eigenvalue of A_1 in (1) then the first n rows are linearly independent. Also, the second n rows have no zero rows because the eigenvalues of A_1 and A_2 are disjoint. This means the second set of n rows are linearly independent and also linearly independent of the first n rows. This shows the matrix has rank $2n$. The same argument can be used when substituting an eigenvalue of A_2 in (1). We conclude the overall system is controllable.

Next, suppose the overall system is controllable. Construct the controllability matrix:

$$Q_c = \begin{bmatrix} B_1 & A_1 B_1 & \dots & A_1^{n-1} B_1 \\ B_2 & A_2 B_2 & \dots & A_2^{n-1} B_2 \end{bmatrix}.$$

This matrix is by assumption of rank $2n$. Therefore the first n rows must be linearly independent as well as the second n rows. Immediately we obtain that the subsystems are controllable.

For observability, we can let $u = 0$ for simplicity. Again we can assume without loss of generality that A_1 and A_2 are each in Jordan form. The overall system is

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix}, \quad y = [C_1 \ C_2] \begin{bmatrix} x \\ \xi \end{bmatrix},$$

First suppose the subsystems are observable. Then use the PBH test. We must test whether

$$\text{rank} \begin{bmatrix} A_1 - \lambda I & 0 \\ 0 & A_2 - \lambda I \\ C_1 & C_2 \end{bmatrix} = 2n \quad (2)$$

for all $\lambda \in \{eig(A_1)\} \cup \{eig(A_2)\}$. We know that $\text{rank} \begin{bmatrix} A_1 - \lambda I \\ C_1 \end{bmatrix} = n$ for all $\lambda \in eig(A_1)$

and $\text{rank} \begin{bmatrix} A_2 - \lambda I \\ C_2 \end{bmatrix} = n$ for all $\lambda \in eig(A_2)$. So, if λ is an eigenvalue of A_1 in (2) then the first n columns are linearly independent. Also, the second n columns have no zero columns because the eigenvalues of A_1 and A_2 are disjoint. This means the second set of n columns are linearly independent and also linearly independent of the first n columns. This shows the matrix has rank $2n$. The same argument can be used when substituting an eigenvalue of A_2 in (2). We conclude the overall system is observable.

Next, suppose the overall system is observable. Construct the observability matrix:

$$Q_o = \begin{bmatrix} C_1 & C_2 \\ C_1 A_1 & C_2 A_2 \\ \vdots & \vdots \\ C_1 A_1^{n-1} & C_2 A_2^{n-1} \end{bmatrix}.$$

This matrix is by assumption of rank $2n$. Therefore the first n columns must be linearly independent as well as the second n columns. Immediately we obtain that the subsystems are observable.

Now suppose the spectra of the A_i 's are not disjoint. Then the results above will not hold in some cases. For example, controllability does not hold for the overall system if S_1 is given by

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u,$$

and S_2 is given by

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u.$$

4. The overall system (A, B, C, D) after a series connection is given by

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} &= \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 D_1 \end{bmatrix} u \\ y &= \begin{bmatrix} D_2 C_1 & C_2 \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + D_2 D_1 u. \end{aligned}$$

The corresponding controllability matrix and observability matrices are

$$\begin{aligned} Q_c &= \begin{bmatrix} B & AB & A^2 B & \cdots \end{bmatrix} \\ &= \begin{bmatrix} B_1 & A_1 B_1 & A_1^2 B_1 & \cdots \\ B_2 D_1 & B_2 C_1 B_1 + A_2 B_2 D_1 & A_2^2 B_2 D_1 + B_2 C_1 A_1 B_1 + A_2 B_2 C_1 B_1 & \cdots \end{bmatrix}, \\ Q_o &= \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix} \\ &= \begin{bmatrix} D_2 C_1 & C_2 \\ D_2 C_1 A_1 + C_2 B_2 C_1 & C_2 A_2 \\ D_2 C_1 A_1^2 + C_2 B_2 C_1 A_1 + C_2 A_2 B_2 C_1 & C_2 A_2^2 \\ \vdots & \vdots \end{bmatrix}. \end{aligned}$$

From Q_c , it is obvious that one necessary condition for controllability is that S_1 is controllable. From Q_o , one necessary condition for observability is that S_2 is observable.

Unlike the previous problem, the controllability and observability of each subsystem is not sufficient to preserve controllability and observability of the systems connected in series. For example, if $A_1 = 2$, $A_2 = 1$, $B_1 = B_2 = C_1 = C_2 = D_1 = D_2 = 1$, it is easy to check that $\text{rank} Q_c = 1$, so that the overall system is not controllable.

5. First, we use the given feedback controller $u(t) = kx$ to get an expression for $x(t)$:

$$x(t) = \sqrt{2}e^{(1+k)t}.$$

To render the origin stable clearly we require that $k < -1$. The cost function can be written as

$$J = \int_0^{\infty} 2e^{2(1+k)t} dt.$$

Under the condition $k < -1$, J is well-defined so we calculate

$$J = -\frac{1}{k+1}.$$

Now to minimize this J it is clear that we should set $k = -\infty$. In other words, to minimize J we require infinite control effort. Such a controller is not physically realizable.

The problem with the previous example is that we did not put a cost on the control effort. Now let us revise the performance index (or cost function) to be:

$$J = \int_0^{\infty} (x^2(t) + u^2(t)) dt.$$

This is the standard linear quadratic regulator problem (LQR), with $A = 1$, $B = 1$, $Q = 1$ and $R = 1$. It is obvious that (A, B) is controllable, hence stabilizable, which guarantees this problem is solvable. The optimal control is $u^*(t) = kx(t)$ where $k = -R^{-1}B^T P$, and P is the positive semi-definite solution of the algebraic Riccati equation (ARE):

$$A^T P + PA - PBR^{-1}B^T P + Q = 0.$$

In this case (ARE) yields

$$P^2 - 2P - 1 = 0$$

and the positive solution is $P = 1 + \sqrt{2}$. Therefore, the optimal control is

$$u^*(t) = -(1 + \sqrt{2})x.$$

We can also use the MATLAB command `lqr` to compute k .

6. In this question, our objective is to design using MATLAB a controller that minimizes the performance index

$$\begin{aligned} J &= \int_0^\infty [y^2(\tau) + \epsilon u^2(\tau)] d\tau \\ &= \int_0^\infty [x^T(\tau)C^T Cx(\tau) + \epsilon u^2(\tau)] d\tau \end{aligned}$$

The optimal controller that minimizes J is $u = -R^{-1}B^T P$ where P is the solution of (ARE). Below is a MATLAB script to solve the problem.

```
>> A=[0 1 0 -1; 0 0 0 0; 0 0 0 1; 0 0 -4.4 0];
B=[0 -4.2e-3 0 -2.8e-2]';
C=[1 0 0 0];
R=1e-5;
Q=C'*C;
P=care(A,B,Q,R)
```

P =

```
    2.7398    3.7533    2.8341   -0.4501
    3.7533    5.9892    4.4617   -0.5890
    2.8341    4.4617    3.4564   -0.4456
   -0.4501   -0.5890   -0.4456    0.0990
```

```
>> K=-inv(R)*B'*P
```

K =

```
    316.2278   866.4054   626.2078   29.8190
```

```
>> eig(A+B*K)
```

```
ans =
```

```
-1.4743 + 2.6659i  
-1.4743 - 2.6659i  
-0.7626 + 0.2195i  
-0.7626 - 0.2195i
```

For the observer design, we assume the desired poles are $\{-20, -20, -20, -20\}$. Then, we use the pole-placement technique to find the observer gain. Below is a MATLAB script. At the end we show a simulink diagram and the response of the closed-loop system for the initial condition $x_0 = (1, 1, 1, 1)$.

```
>> L=acker(A',C',[-20 -20 -20 -20])'
```

```
L =
```

```
1.0e+04 *  
  
0.0080  
3.6364  
0.7193  
3.3968
```

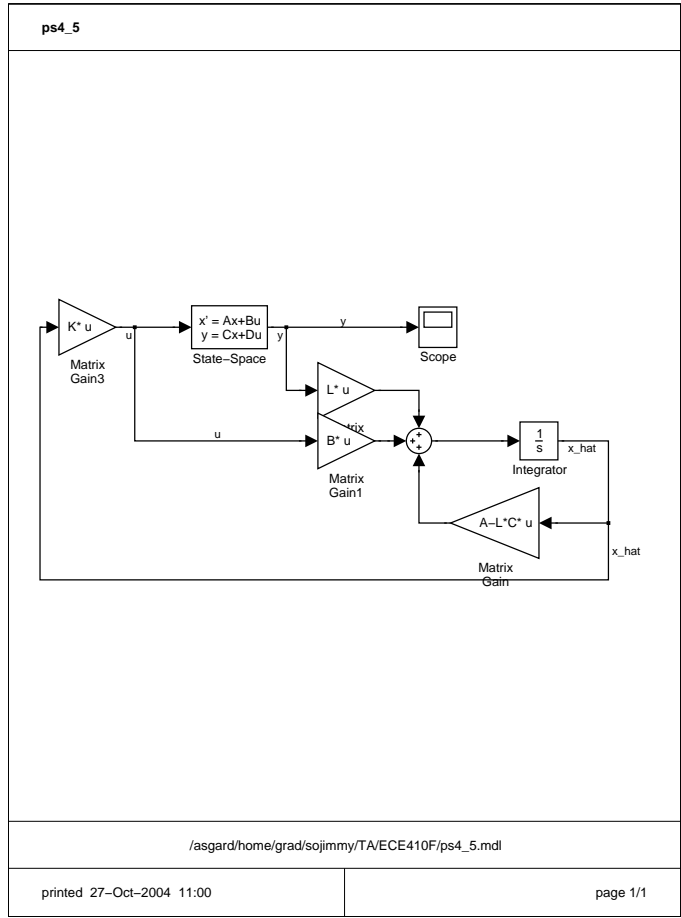


Figure 1: Simulink Diagram

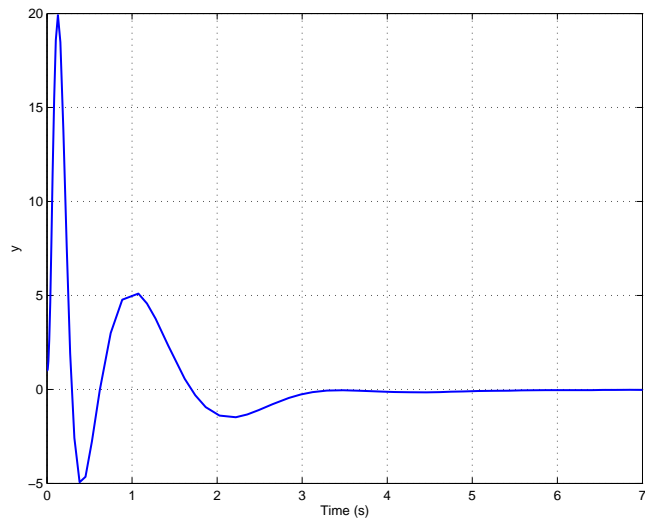


Figure 2: Initial Response