University of Toronto
Department of Electrical and Computer Engineering
ECE410F Control Systems
Problem Set #5

1. An unstable robot system is described by the state equation

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \,.$$

Assume that the initial condition is $x(0) = [1\ 1]^T$. Suppose the control is set to

$$u = Kx \,.$$

Design the gain K so that the cost function

$$J = \int_0^\infty (x^T x + \epsilon u^T u) dt$$

is minimized. Plot the magnitude of the control $\|u(0)\|$ at the initial time for $\epsilon \in (0, 100]$.

2. Consider the plant

$$Y(s) = \frac{1}{s - \lambda} U(s) \,,$$

where λ is an arbitrary parameter. Find a controller that stabilizes the system and minimizes the performance index

$$J = \int_0^\infty \left[y^2(\tau) + \epsilon u^2(\tau) \right] d\tau \,.$$

Next, examine the closed-loop eigenvalues of the system as $\epsilon \longrightarrow 0$ and $\epsilon \longrightarrow \infty$ for the case when (i) $\lambda > 0$, i.e. the plant is open-loop unstable, and (ii) $\lambda < 0$, i.e. the plant is open-loop stable.

3. Consider a radar tracking problem. Given the system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w \,,$$

$$y = [1\ 0] x + v \,,$$

where white noise w has intensity $W > 0$ and v has intensity $V > 0$. Assume that $\beta = \sqrt{\frac{W}{V}}$. Design, if possible, a Kalman filter for the system and find the optimal Kalman gain. What happens to the poles of the Kalman filter for the case when $W = 1$, $V \longrightarrow 0$?

4. (Matlab) An approximate model of a helicopter is given by:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \,, \end{aligned}$$

where

$$A = \begin{bmatrix} -3.66e^{-2} & 2.71e^{-2} & 1.88e^{-2} & -4.56e^{-1} \\ 4.92e^{-2} & -1.01 & 2.4e^{-3} & -4.02 \\ 1e^{-1} & 3.68e^{-1} & -7.07e^{-1} & 1.42 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 4.42e^{-1} & 1.76e^{-1} \\ 3.54 & -7.59 \\ -5.52 & 4.491 \\ 0 & 0 \end{bmatrix}.$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

This system has a highly unstable oscillatory mode, i.e. the system is open loop unstable. It is desired to design a controller to stabilize the system such that the resultant closed loop system has a "smooth, fast" dynamic response.

(a) Design a state feedback controller using pole placement (use the Matlab command `place`). Compare the eigenvalues of the open loop system with the closed loop system $\dot{x} = (A + BK)x$. Simulate the closed-loop system for initial condition $x(0) = (1, 1, 1, 1)$ using the Matlab command `impulse(A,B,C,D,1,T)`, where `T = [0:.1:10]'`.

(b) Repeat the design of a state-feedback controller using optimal control to obtain $u = Kx$ (using the Matlab command `lqr`) with the cost function

$$J = \int_0^\infty y^T y + \epsilon u^T u)dt\,,$$

where $\epsilon > 0$ is set to some reasonable value. Compare the eigenvalues of the open-loop system with the closed-loop system after applying the optimal feedback control. Simulate the closed loop system for the initial condition $x(0) = (1, 1, 1, 1)$.

(c) Implement the optimal controller obtained in the previous step using an observer and find the eigenvalues of the overall system. Simulate the resulting closed-loop system (controller+observer) for the initial condition $x(0) = (1, 1, 1, 1)$ and with the observer initial conditions set to zero.

(d) Find the minimal realization of the system above.

(e) Design an observer for the minimal realization obtained in the previous problem so that the observer is asymptotically stable with poles all equal to -100.