CONTROL SYSTEMS LABORATORY
ECE410F
LAB2 : Controller Design Using Pole Placement and Full Order Observers

1 Purpose

The purpose of this experiment is to design a control system for the cart system (IP02), studied in Lab 1, such that a square wave is tracked using state feedback and observer theory. The state feeddback design is carried out using both pole placement and full order observers.

2 Introduction

In a third year control course, you would have designed P and PID control laws for a linear system such as a motor. Such control laws may have performance limitations. In this experiment, we would like to improve control performance using pole placement and observer theory.

Suppose we would like the cart to track a square wave input and suppose the performance specifications are given as

(a) The closed loop system is stable.

(b) The output of the closed loop system tracks step reference inputs.

(c) The overshoot ($\%OS$)is less than 15% and the peak time (t_p) is less than 0.2 second.

First we set up state equations to describe the system. Let x_1 be the position of the cart on the track and let x_2 be the cart velocity. We assume that the position of the cart can be measured as an output. The state equation of the cart is

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{T} \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{K_p}{T} u$$
$$y = [1\ 0]x\,.$$

The values for the parameters K_p and T are normally determined through experimentation (in a process called system identification). In this case the values will be provided to you. Tracking a square wave is equivalent to tracking a constant set point y_d provided the settling time is much faster than the switching time of the square wave. For this lab, the square wave generated by the computer has been chosen to be $\pm 15mm$ with a frequency of 0.5 Hz. So let $y_d(t) = y_d$ be the set point to be tracked. We want to use pole placement to achieve this tracking task, but pole placement in normally used for stabilization: that is, to send all the states to zero by selecting poles in the open left half complex plane. To transform the tracking problem to a stabilization we define new state variables

$$z_1 = x_1 - y_d$$
$$z_2 = x_2$$

This essentially shifts the origin of the new coordinate system. We obtain the new state equation

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{T} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{K_p}{T} u \tag{1}$$
$$\tilde{y} = [1\ 0]z\,. \tag{2}$$

Observe that if $z_1(t) \longrightarrow 0$ then $y(t) \longrightarrow y_d$, as desired. That is, asymptotic tracking correponds to regulating $z_1(t)$ to 0. We have, therefore, transformed the tracking specifications into stabilization requirements in terms of the new state $[z_1\ z_2]$. You will design a controller, using pole placement, to stabilize the system (1).

2.1 Choosing the closed loop poles

We can easily find the location of the closed loop poles from the constraints placed on the response of our system. First, recall that the characteristic equation of a second order system can be written in the form:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 \tag{3}$$

where ζ is the damping ratio and ω_n is the natural frequency. Second, both the damping ratio and the natural frequency can be found from the specifications for percent overshoot and peak time (Figure 1). That is,

$$\zeta = \frac{-ln(\%OS/100)}{\sqrt{\pi^2 + ln^2(\%OS/100)}} \tag{4}$$

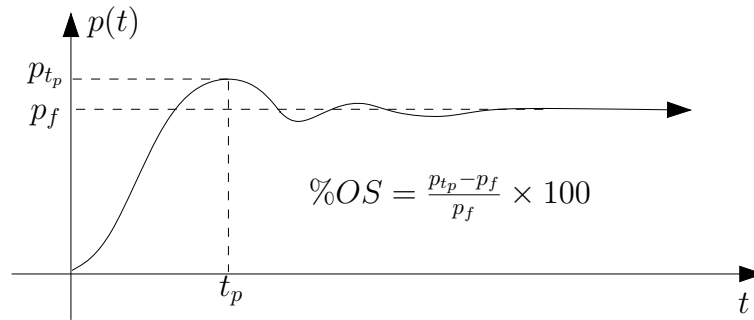$$\omega_n = \frac{\pi}{t_p\sqrt{1-\zeta^2}}. \tag{5}$$

Figure 1: Illustration of percent overshoot and peak time.

The last step, to obtain the location of the closed loop poles, is to substitute the values we have found in (4) and (5) into (3), and solve for the roots of the characteristic equation.

3 Preparation

(a) Check that the system (1) is controllable by computing $rank(Q_c)$. Do this calculation assuming K_p and T are parameters. Next, using the values $T = \frac{1}{17.13}$ and $K_p = 2.46T$, determine if the system is controllable both using your computed matrix Q_c and using the MATLAB command `ctrb`. Compare the two answers.

(b) We shall use (1) to design our observer-based output feedback controller. Using pole placement design (see the Appendix for a review of the procedure) and assuming T and K_p are unknown parameters, design a state feedback for the closed loop poles that you have found in the preceding section. Since y_d is available, we can consider $z_1(t)$ as the measured output. Using the Appendix as a guide, design a full order observer with observer poles both placed at $\{-40, -40\}$. Determine also the compensator transfer function in the form

$$U(s) = C(s)\tilde{Y}(s)$$

where $\tilde{Y}(s) = Y(s) - Y_d(s)$. Write a small MATLAB program to determine the compensator $C(s)$ using the parameters K_p and T as inputs (in case your servomotor parameter values have drifted). You may even want to include the closed loop poles as input parameters if you want to experiment with different choices.

(c) Design a simulink model for your system (plant, controller, and observer), and simulate the response to a step input. You can assume that T and K_p have the values given in (a). To ease your design process, a simulink file has been provided *lab2_prep*. Make sure that you understand the entire file, as your TA will ask you questions about it.

Each student must hand in their preparation in the beginning of the lab. The preparation report should include:

1. Determination of controllability of the system: Q_c matrix, rank of Q_c.

2. Controller gain (K), observer gain (L), and the transfer function $U(s)/\tilde{Y}(s) = C(s)$ as a function of T and K_p, including your derivations.

3. The damping ratio ζ, the natural frequency ω_n, the state feedback poles, the controller gain (K), observer gain (L), and the transfer function $U(s)/\tilde{Y}(s)(s) = C(s)$ with $T = \frac{1}{17.13}$ and $K_p = 2.46T$.

4. The response of the system to a step input (i.e. the plot of the position, $y = x_1$ vs. time). Identify the percent overshoot and peak time on your plot.

4   Experiment

CAUTION: Before running your experiment

- Please ensure that the cart is positioned at the middle of the track!

- Do not change any of the calibration gains in the simulink model, as they are placed there for safety!

- Do not change the frequency and gain of the square input, unless you have permission from your TA!

- When running the experiment ensure that one student from the group is firmly holding the track!

- If for any reason your model goes unstable, turn off the experiment right away!

(a) Using the values for parameters K_p and T given above, compute the gains of the state feedback controller and the observer using the MATLAB command `place`. Compare the results you obtain with MATLAB with the result you computed in the lab preparation. (The answers should be the same. Note, Matlab uses the convention that $u = -Kx$ while we use $u = Kx$)

(b) Copy all the files from the X drive (soft on 'ugsparc48') `/ ECE / ECE410 / Lab2` into your own directory.

(c) Open the Matlab file `lab2.m`. At the bottom of the file enter your controller gain (K) and your observer gain (L).

(d) Open the SIMULINK model `lab2_part1.mdl` and build the closed-loop system (including the observer) in SIMULINK (notice the similarities between this file and `lab2_prep`). Then plot the step response of the closed loop system and determine if the performance specs are satisfied. If not, adjust the desired closed loop poles until they are satisfied.

(e) Using your final design from step (c), build and run your state-feedback observer-based controller on the actual system. For this step you must open the file `lab2_part2.mdl`. Apply a square wave reference input for $y_d(t)$ (this has been set up for you). Save the actual position $y(t)$ and the reference $y_d(t)$ to a data file.

(f) Load the experimental data into MATLAB, and plot $y_d(t)$ and $y(t)$ versus time t. Does your controller satisfy the design specifications? If so, demonstrate the step response to the TA. Show the TA the MATLAB plot too.

(g) Compare the experimental response to the simulated response computed earlier. Do the experimental and simulated step responses agree? If not, re-tune your pole placement controller.

5 Report

Please follow the report format instructions provided by your TA for Lab 2.

A Pole Placement Controller Design

We consider the single input open-loop state space model

$$\dot{x} = Ax + bu\,.$$

We want to design a controller of the form $u = Kx$, where $K \in \mathbb{R}^{1 \times n}$ is a constant matrix, in order to stabilize the system. This is called a *state feedback* control law. With this control law the closed-loop system is

$$\dot{x} = (A + BK)x\,.$$

We know that the stability of the closed-loop system is determined by $eig(A + BK)$.

Suppose that we have computed the characteristic polynomial of A as:

$$p(s) = s^n + a_n^o s^{n-1} + \cdots a_2^o s + a_1^o\,,$$

where the superscript o stands for "open-loop". Suppose that the desired eigenvalues of $A + bK$ are $\{\lambda_1, \cdots, \lambda_n\}$. The associated desired characteristic polynomial for these eigenvalues is denoted

$$r(s) = s^n + a_n s^{n-1} + \cdots a_2 s + a_1\,.$$

Now define the following *Toeplitz* matrix:

$$\hat{Q} = \begin{bmatrix} a_2^o & a_3^o & \cdots & a_n^o & 1 \\ a_3^o & a_4^o & \cdots & 1 & 0 \\ \vdots & & \cdots & & \vdots \\ a_n^o & & \cdots & & \\ 1 & 0 & \cdots & & 0 \end{bmatrix}\,.$$

Also define the coefficient row vectors

$$\begin{aligned} a^o &= (a_1^o, a_2^o, \ldots, a_n^o) \\ a &= (a_1, a_2, \ldots, a_n)\,. \end{aligned}$$

Here is our main result about pole placement.

Theorem A.1. *The controller $u = Kx$ with*

$$K = [a^o - a](Q_c \hat{Q})^{-1}$$

has the property

$$eig(A + bK) = \{\lambda_1, \cdots, \lambda_n\}\,.$$

4

B Full State Observer Design

The purpose of an observer is to reconstruct the state vector when only the output y can be measured directly. The reconstructed state or *state estimate* is denoted $\hat{x}$. We want our observer to have the property that $x - \hat{x} \longrightarrow 0$ as $t \to \infty$. The full state observer equation is:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(y - \hat{y}), & \hat{x}(0) = \hat{x}_0 \\ \hat{y} &= C\hat{x} \end{aligned} \tag{6}$$

The initial condition $\hat{x}_0$ can be arbitrarily chosen (often taken to be 0). The error in the state estimate, defined as $e = x - \hat{x}$, is governed by the equation

$$\dot{e} = (A - LC)e, \qquad e(0) = x_0 - \hat{x}_0. \tag{7}$$

Observe that if the eigenvalues of $A - LC$ all lie in the left half plane, then regardless of $e(0)$, $e(t) \to 0$ exponentially and the goal of accurate state estimation is achieved.

So to design the observer you just have to design the coefficients of L so that $(A - LC)$ is stable. The eigenvalues of $(A - LC)$ are called the poles of the observer. You can use pole placement to design the poles because of the fact that $det(sI - A^T + C^T L^T) = det(sI - A + LC)$. That is, you will do pole placement design for the matrix $A^T - C^T L^T$ using the method you learned in class.

Your final observer-based pole placement controller will be $u = K\hat{x}$ rather than $u = Kx$. Once you finish the design of the observer, you can write a transfer function for the feedback controller $u = K\hat{x}$, called the "compensator", from the input of the compensator y to the output of the compensator u.

For example, let's take the case of a single input u and a single output y. Then the state feedback is $u = K\hat{x}$. Plug this control into the full state observer equation (6) to get

$$\dot{\hat{x}} = (A - LC + BK)\hat{x} + Ly.$$

Taking Laplace transforms:

$$\hat{X}(s) = (sI - A + LC - BK)^{-1}LY(s)$$

Plugging this into the state feedback $u = K\hat{x}$ we get the transfer function for the compensator $C(s)$ to be:

$$C(s) := \frac{U(s)}{Y(s)} = K(sI - A + LC - BK)^{-1}L.$$

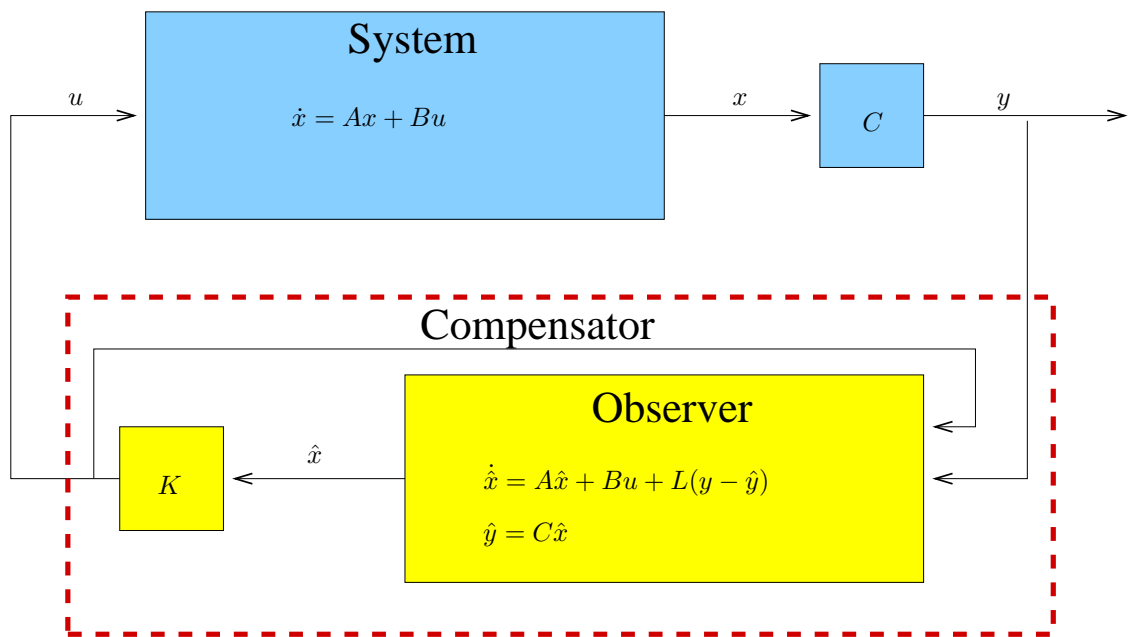A block diagram of the system together with the compensator is shown in Figure 2.

Figure 2: Block diagram of system with compensator.