

**University of Toronto**  
**Department of Electrical and Computer Engineering**  
**ECE410F – Control Systems**

**Experiment 4**

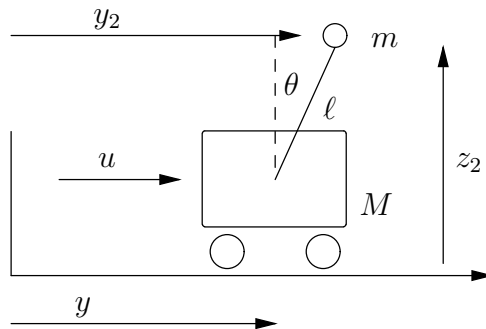
**CONTROL OF AN INVERTED PENDULUM ON A CART**

## 1 Purpose

The purpose of this experiment is to apply controller and observer design techniques to a physical “difficult-to-control” system, namely, the inverted pendulum on a cart system. The controller will mainly be designed via optimal control techniques. The control objective is for the cart to track a square wave while the pendulum remains upright. Also, the controller must be improve the system’s robustness to external disturbances.

## 2 Introduction

An inverted pendulum on a cart experiment is very often used as an illustrative model for control system design. The following figure provides an illustration of the physical system.



Ignoring inertia and friction effects, the simplified dynamics of the inverted pendulum is given by the following differential equations:

$$\ddot{y} = \frac{u + ml\dot{\theta}^2 \sin \theta - mg \cos \theta \sin \theta}{M + m \sin^2 \theta} \quad (1)$$

$$\ddot{\theta} = \frac{(M + m)g \sin \theta + u \cos \theta - ml\dot{\theta}^2 \sin \theta \cos \theta}{l(M + m \sin^2 \theta)}. \quad (2)$$

The above system assumes a force input that horizontally pushes the cart. The pendulum system in our lab, however, takes a voltage input that drives the motor, which in turn drives the cart. So, to convert the former to the latter, we use the following relationship between

the driving force generated by the DC motor and the force acting on the cart through the motor pinion:

$$u = \alpha_1 \left( \frac{dy}{dt} \right) + \alpha_2 V, \quad (3)$$

where  $\alpha_1$  and  $\alpha_2$  are characteristics related to the motor, and  $V$  is the voltage input into the motor. Hence, substituting (3) into (1) and (2), we get the following nonlinear dynamic equations of the inverted pendulum system with a voltage input:

$$\ddot{y} = \frac{(\alpha_1 \dot{y} + \alpha_2 V) + ml\dot{\theta}^2 \sin \theta - mg \cos \theta \sin \theta}{M + m \sin^2 \theta} \quad (4)$$

$$\ddot{\theta} = \frac{(M + m)g \sin \theta + (\alpha_1 \dot{y} + \alpha_2 V) \cos \theta - ml\dot{\theta}^2 \sin \theta \cos \theta}{l(M + m \sin^2 \theta)}, \quad (5)$$

where  $V$  is a voltage and is now our input to the system.

For the actual inverted pendulum in the lab we have the following physical parameters:

- $M = 1.0731$  Kg
- $m = 0.2300$  Kg
- $l = 0.3302$  m
- $g = 9.8$  m/s<sup>2</sup>
- $\alpha_1 = -7.7443 \frac{Ns}{m}$
- $\alpha_2 = 1.7265 \frac{N}{V}$

The system has two sensors which can measure the position of the cart and the angle of the pendulum.

## 2.1 Objectives:

It is well-known from one's common experience with trying to balance a broom that the inverted pendulum on a cart is a difficult system to control. The control design objectives of this experiment are:

- (a) Balance the inverted pendulum about its vertical position;
- (b) Control the cart to track a square wave of amplitude  $\pm 20$  mm with a frequency of 0.05 Hz;
- (c) Achieve good transient response (Note: this is deliberately vague, and better transient response is often obtained by tuning control parameters); and
- (d) Achieve a certain level of robustness to physical disturbances.

### 3 Preparation

In the preparation you will linearize the nonlinear model of the pendulum and cart and use this linear model to design a pole placement controller, full order observer, and an optimal controller. You will simulate both the pole placement controller and optimal controller on the nonlinear model.

- (a) Assume that the state vector is  $x = [y \ \dot{y} \ \theta \ \dot{\theta}]^T$  and derive a nonlinear state equation for  $x$  with input  $V$ . Then, determine the equilibrium point(s) of the system,  $(y_o, \theta_o, \dot{y}_o, \dot{\theta}_o)$  and  $V_o$ . Next, compute the linearized model

$$\begin{aligned}\tilde{x} &= A\tilde{x} + B\tilde{V} \\ \tilde{y} &= C\tilde{x}\end{aligned}$$

where  $\tilde{x} = x - x_o$ ,  $\tilde{y} = y - y_o$ ,  $\tilde{V} = V - V_o$ . In this step of the prep, leave the model parameters  $M$ ,  $m$ ,  $l$ , and  $g$  as variables, i.e. do not plug in the numerical values.

- (b) Write a Matlab script that sets the proper values of the model parameters and the equilibrium point, and defines the  $(A, B, C, D)$  matrices in terms of these parameters. Use this script to carry out the following steps, which should be done using MATLAB and SIMULINK.
- (c) Assume that the state of the pendulum system is fully observed. Design a state feedback law,  $u = Kx$ , so that the closed loop poles are located at  $\{-2, -4, -10, -20\}$ .
- (d) Now assume that only the cart position,  $y$ , and the pendulum angle,  $\theta$ , are measured. Design a full-order observer to estimate the states by computing the full-order observer gain,  $L$ , so that observer poles are located at  $\{-40, -40, -50, -50\}$ .
- (e) Now open the SIMULINK model `Lab4_prelab.mdl` and modify the compensator block to include your controller and observer to give the output feedback control law,  $u = K\hat{x}$ . First run the `init4.m` file in MATLAB to define the system parameters, and then simulate the response of the nonlinear pendulum system in SIMULINK. The initial conditions of the plant are,  $(y_o, \theta_o, \dot{y}_o, \dot{\theta}_o) = (0, 10^\circ \times \frac{\pi}{180^\circ}, 0, 0)$  (i.e. the inverted pendulum starts with an initial angle of  $10^\circ$ ), and for the observer,  $(\hat{y}_o, \hat{\theta}_o, \hat{\dot{y}}_o, \hat{\dot{\theta}}_o) = (0, 0, 0, 0)$ . Plot your results on 4 plots, each plot containing one state component and its observer estimate. For the plot with  $y$  and  $\hat{y}$ , also include the square wave input,  $y_d$ .

Note that the nonlinear model used in this file actually includes inertia and friction terms excluded from our modeling in (4) and (5). This nonlinear model better approximates the real physical system. Try to make sense of the results you obtain, i.e. does the cart track a square wave, why or why not?

**Feel free to tune the poles of the controller and see how it affects the response.**

- (f) Next you will explore the effects of the cost function parameters  $Q$  and  $R$  on the response when using an optimal controller. Let

$$Q = \begin{bmatrix} q_1 & 0 & 0 & 0 \\ 0 & q_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad R = r$$

The parameters  $q_1$  and  $q_2$  will influence the speed of response of the cart position and the pendulum angle, respectively, while  $r$  will affect the control signal,  $V$ .

Design an optimal controller with  $q_1 = 1$ ,  $q_2 = 4$  and  $r = 0.0005$ . This time, plot on 2 plots only the cart position response and the angle response. Likewise, for the plot with the cart position response,  $y$ , also include the square wave input,  $y_d$ .

**Again, feel free to play with  $q_1$ ,  $q_2$  and  $r$  to see how it affects the response.**

**Each** student must hand in their preparation in the beginning of the lab. The preparation should include:

1. The nonlinear state equations, the equilibrium point(s), and the linearized model as a function of the model parameters.
2. A MATLAB script for the pre-lab.
3. The controller gain ( $K$ ) and observer gain ( $L$ ) obtained in parts (c),(d), and (e).
4. Four plots for the response with the pole placement controller and observer.
5. Two plots for the response with the optimal controller and observer.

## 4 Experiment

Assuming that in your pre-lab you have tested out your optimal controller in simulation and obtained satisfactory response, it is now time to try the real thing.

**CAUTION:** Before running your experiment

- Please ensure that the cart is positioned at the middle of the track!
- Do not change any of the calibration gains in the SIMULINK model, as they are placed there for safety!
- Do not change the frequency and gain of the square input, unless you have permission from your TA!
- When running the experiment ensure that one student from the group is firmly holding the track!

- If for any reason your model goes unstable, turn off the experiment right away!
- (a) First, copy all the files from the X drive (soft on 'ugsparc48') / ECE / ECE410 / Lab4 into your own directory.
  - (b) Open the SIMULINK file `Lab4_Part1.mdl`. Study the diagram and try to make sense of the blocks. Modify the compensator subsystem to implement the controller and observer you designed in the pre-lab (you may use  $q_1$ ,  $q_2$ , and  $r$  given in the pre-lab, or you may use your own as long as you have verified that they work and achieve satisfactory response in simulation). Get a TA to verify your diagram and controller/observer design.
  - (c) Now it's time to build.
    1. Run the initialization files (i.e. `init4.m` and your pre-lab script) to define the system parameters, the controller and observer gain.
    2. In SIMULINK, Wincon→Build
    3. Wait for the Wincon server to start (i.e. see the green 'Start' button).
  - (d) To start the system
    1. First let the pendulum stick hang down (i.e. in the 'gantry' position).
    2. Then press the green **Start** button.
    3. Then slowly bring the pendulum stick to the upright inverted position. Once the stick reaches  $180^\circ$  from the hanging down position, the controller will 'kick' in, at that point gently release the stick. The system should now be stable.
 

**Caution:** the transition of when the controller 'kicks' in, may be violent if you don't let go of the stick in time or continue to grip it when the controller is working, so beware.

## 4.1 Tuning Optimal Controller for better Tracking

The system should now be stable, and the inverted pendulum is balanced in the upright position while the cart tracks a square wave. However, note that the cart is not tracking the square wave very well. We now need to tune the values of  $q_1$ ,  $q_2$ , and  $r$  to try to get a better response.

- (a) Firstly, using the  $q_1$ ,  $q_2$ , and  $r$  obtained in the pre-lab, save the real-time data to a file and plot on two graphs the actual and the simulated position and angle responses. For the plot with position response, also include the reference square wave. How well does the cart track the square wave?

Also, plot on two graphs the cart velocity,  $\dot{y}$ , and the pendulum velocity,  $\dot{\theta}$ , along with their estimated states from the observer,  $\hat{y}$  and  $\hat{\theta}$ . How well is the observer estimating the states?

- (b) Change (increase and decrease) the values of  $q_1$ ,  $q_2$ , and  $r$  individually and observe how the resulting response varies. Give a brief description (i.e. tracking, oscillations, overshoot, settling time, etc.).

Note: to change the values in real-time (i.e. while the system is still running), update the values in the MATLAB command prompt, and then type ‘**wc\_update**’ to update the corresponding variables in SIMULINK.

- (c) Try to find a set of  $q_1$ ,  $q_2$ , and  $r$  that achieves better tracking than the controller obtained in the prelab. Provide a plot of the position and angle response, along with the reference signal.

## 4.2 Testing the Robustness of the system to Physical Perturbations

In this part of the lab, we will balance the inverted pendulum without tracking, and tune our controller to improve the system’s robustness to external disturbances.

- (a) Open the SIMULINK file `Lab4_Part2.mdl`. Modify the compensator block to implement your controller and observer from 4.1(c).
- (b) Build and start the system as before.
- (c) When the system is stable, gently tap the pendulum stick and observe the response. Save the data to a file and plot the position and angle response of the ‘tap’. Briefly explain how that system reacts to the tap to recover stability.
- (d) Change the values of  $q_1$ ,  $q_2$ , and  $r$  and tap again. Does the system have a smoother recovery, or is the system less robust? Try a few more different values. (no plots required).
- (e) Finally, try to find a set of  $q_1$ ,  $q_2$ , and  $r$  (if possible) that makes the system more robust to disturbances. Plot the position and angle response.

## 5 Report

Please follow the report format instructions for Lab 4 provided by your TA.