University of Toronto
Department of Electrical and Computer Engineering
ECE557F Systems Control
Problem Set #1
Selected Solutions

2. Solve $\det(\lambda I - A) = 0$, we get two eigenvalues

$$\lambda_1 = \sigma - i\omega, \lambda_2 = \sigma + i\omega.$$

The eigenvectors are given by (up to scalar multiplication)

$$v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

by solving the equations $(\lambda_i I - A)v_i = 0, \ \ i = 1, 2$.
Let $T = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$. We have $T^{-1} = \begin{bmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix}$.
Since

$$e^{T^{-1}ATt} = T^{-1}e^{At}T = e^{\Lambda t} = \begin{bmatrix} e^{(\sigma - i\omega)t} & 0 \\ 0 & e^{(\sigma + i\omega)t} \end{bmatrix},$$

we have

$$e^{At} = Te^{\Lambda t}T^{-1} = e^{\sigma t} \begin{bmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{bmatrix}.$$

which is the same result as that given in the course notes.

3. (a) Let

$$A = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}$$

The eigenvalues are given by $-1 \pm 2i$. The eigenvector corresponding to $-1 + 2i$ satisfies the equation

$$\begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -1 + 2i \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

From the 2nd equation, we see that $v_1 = 2iv_2$ so that the eigenvector is given by

$$v = \begin{bmatrix} 2i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

(b) Let

$$P = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

Then

$$P^{-1} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$$

with

$$D = P^{-1}AP = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$$

From problem 1, we see that

$$e^{Dt} = e^{-t} \begin{bmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{bmatrix}$$

Hence

$$e^{At} = Pe^{Dt}P^{-1} = e^{-t} \begin{bmatrix} \cos 2t & -2\sin 2t \\ \frac{1}{2}\sin 2t & \cos 2t \end{bmatrix}$$

(c)

$$\begin{aligned} (sI - A)^{-1} &= \begin{bmatrix} s+1 & 4 \\ -1 & s+1 \end{bmatrix}^{-1} \\ &= \frac{\begin{bmatrix} s+1 & -4 \\ 1 & s+1 \end{bmatrix}}{s^2 + 2s + 5} \\ &= \begin{bmatrix} \frac{s+1}{s^2+2s+5} & \frac{-4}{s^2+2s+5} \\ \frac{1}{s^2+2s+5} & \frac{s+1}{s^2+2s+5} \end{bmatrix} \end{aligned}$$

From Laplace transform table, we have the following transform pairs:

$$\frac{s+a}{(s+a)^2 + b^2} \longleftrightarrow e^{-at}\cos bt$$

$$\frac{1}{(s+a)^2 + b^2} \longleftrightarrow \frac{1}{b}e^{-at}\sin bt$$

Noting that $s^2 + 2s + 5 = (s+1)^2 + 2^2$, we obtain, by inverting the various entries

$$e^{At} = e^{-t} \begin{bmatrix} \cos 2t & -2\sin 2t \\ \frac{1}{2}\sin 2t & \cos 2t \end{bmatrix}$$

4. The eigenvalues are

$$\lambda_1 = 4, \quad \lambda_2 = 1.$$

Their corresponding eigenvectors are

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Then solving the equation

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix},$$

we have

$$c_1 = \frac{8}{3}, \quad c_2 = \frac{1}{3}.$$

Thus the modal decomposition is given by

$$x(t) = \frac{8}{3}e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{3}e^{t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

5. We must determine the transfer function $H(s) = c(sI - A)^{-1}b$.

a)

$$
\begin{aligned}
H(s) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s & b \\ -1 & s+a \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ c \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{1}{s^2 + as + b} \begin{bmatrix} s+a & -b \\ 1 & s \end{bmatrix} \begin{bmatrix} 1 \\ c \end{bmatrix} \\
&= \frac{1}{s^2 + as + b} \begin{bmatrix} 1 & s \end{bmatrix} \begin{bmatrix} 1 \\ c \end{bmatrix} \\
&= \frac{cs + 1}{s^2 + as + b}
\end{aligned}
$$

b)

$$
\begin{aligned}
H(s) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ b & s+a \end{bmatrix}^{-1} \begin{bmatrix} c \\ 1 - ac \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{s^2 + as + b} \begin{bmatrix} s+a & 1 \\ -b & s \end{bmatrix} \begin{bmatrix} c \\ 1 - ac \end{bmatrix} \\
&= \frac{1}{s^2 + as + b} \begin{bmatrix} s+a & 1 \end{bmatrix} \begin{bmatrix} c \\ 1 - ac \end{bmatrix} \\
&= \frac{cs + 1}{s^2 + as + b}
\end{aligned}
$$

So (a) and (b) describe the same input/output behavior.

c)

Part (a): By the output equation, $y = x_2$. Take the derivatives to get:

$$\dot{y} = \dot{x_2} = x_1 - ax_2 + cu = x_1 - ay + cu$$

Thus,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \dot{y} + ay - cu \\ y \end{bmatrix}$$

Part (b): By the output equation, $y = x_1$. Take the derivatives to get:

$$\dot{y} = \dot{x_1} = x_2 + cu$$

Thus,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} - cu \end{bmatrix}$$

6. **a)**

$$\begin{aligned}
H(s) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s & -1 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{1}{s^2-1} \begin{bmatrix} s & 1 \\ 1 & s \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
&= \frac{1}{s^2-1} \begin{bmatrix} 1 & s \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
&= \frac{s-1}{s^2-1} = \frac{1}{s+1}
\end{aligned}$$

Observe that $s = -1$ is the only pole of $H(s)$ and hence it is stable.

b) Take $u = 0$, which is obviously bounded. We have $x(t) = e^{At}x_0$.

$$\begin{aligned}
e^{At} &= \mathcal{L}^{-1} \begin{bmatrix} \frac{s}{s^2-1} & \frac{1}{s^2-1} \\ \frac{1}{s^2-1} & \frac{s}{s^2-1} \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} \mathcal{L}^{-1}(\frac{1}{s-1} + \frac{1}{s+1}) & \mathcal{L}^{-1}(\frac{1}{s-1} - \frac{1}{s+1}) \\ \mathcal{L}^{-1}(\frac{1}{s-1} - \frac{1}{s+1}) & \mathcal{L}^{-1}(\frac{1}{s-1} + \frac{1}{s+1}) \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} e^t + e^{-t} & e^t - e^{-t} \\ e^t - e^{-t} & e^t + e^{-t} \end{bmatrix}
\end{aligned}$$

Choosing $x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ will give us

$$y(t) = ce^{At}x_0 = \frac{1}{2}(e^t + e^{-t})$$

which grows exponentially without bound as $t \to \infty$.

7. Let $\dot{x} = f(x, u)$. Suppose this system can be linearized to

$$\begin{aligned}
\triangle\dot{x} &= A\triangle x + B\triangle u, \\
\triangle y &= C\triangle x.
\end{aligned}$$

An equilibrium point $x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$, u_0 satisfies $f(x_0, u_0) = 0$, i.e.,

$$\begin{aligned}
0 &= x_{10}(-\alpha_1 + \sin x_{20}) + x_{20}\sin x_{20} + u_0, \\
0 &= x_{10}\sin x_{10} + x_{20}(-\alpha_2 + \sin x_{20}) + u_0.
\end{aligned}$$

Taking one equation minus the other, we get

$$-\alpha_1 x_{10} + \alpha_2 x_{20} + x_{10}\sin x_{20} - x_{10}\sin x_{10} = 0$$

Now let's take $x_{10} = x_{20} = 0$, which means $u_0 = 0$. Then we compute the Jacobian matrices:

$$\begin{aligned}
A &= \left.\frac{\partial f}{\partial x}\right|_{(x_0,u_0)} \\
&= \begin{bmatrix} -\alpha_1 + \sin x_{20} & x_{10}\cos x_{20} + \sin x_{20} + x_{20}\cos x_{20} \\ \sin x_{10} + x_{10}\cos x_{10} & -\alpha_2 + \sin x_{20} + x_{20}\cos x_{20} \end{bmatrix}_{(x_0,u_0)} \\
&= \begin{bmatrix} -\alpha_1 & 0 \\ 0 & -\alpha_2 \end{bmatrix}, \\
B &= \left.\frac{\partial f}{\partial u}\right|_{(x_0,u_0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\
C &= \begin{bmatrix} 1 & 0 \end{bmatrix}.
\end{aligned}$$

The next part is to determine the response to a step input $\mathrm{e}^{-3}\overline{u}(t)$. First, we calculate the state transition matrix:

$$\begin{aligned}
(sI - A)^{-1} &= \begin{bmatrix} \frac{1}{s+\alpha_1} & 0 \\ 0 & \frac{1}{s+\alpha_2} \end{bmatrix}, \\
e^{At} &= \begin{bmatrix} e^{-\alpha_1 t} & 0 \\ 0 & e^{-\alpha_2 t} \end{bmatrix}, t \geq 0
\end{aligned}$$

Since the initial state is zero and $D = 0$, an approximate solution corresponding to the linearized model can be expressed as

$$\begin{aligned}
y(t) &= \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau \\
&= \int_0^t \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} e^{-\alpha_1(t-\tau)} & 0 \\ 0 & e^{-\alpha_2(t-\tau)} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3} d\tau \\
&= \frac{1}{\alpha_1} e^{-3}(1 - e^{-\alpha_1 t}).
\end{aligned}$$

8. Consider the autonomous system $\dot{x} = Ax$. In these problems we are asked to find the Jordan form of A. Generally this would require computing the eigenvectors and generalized eigenvectors of A. You have not been taught about generalized eigenvectors, but you are provided with extra data about A that enables you to solve these problems.

   (a) Suppose that $eigs(A) = \{-1, -3, -3, -1 + j2, -1 - j2\}$. Also, suppose the rank of $(A - \lambda I)_{\lambda=-3}$ is 4. Now we know that the Jordan form must look like one of the following two cases. Either

$$\Lambda = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -1+2j & 0 \\ 0 & 0 & 0 & 0 & -1-2j \end{bmatrix}$$

or

$$\Lambda = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -1+2j & 0 \\ 0 & 0 & 0 & 0 & -1-2j \end{bmatrix}.$$

Basically you need to determine how many 1's are on the upper diagonal corresponding to the Jordan block for eigenvalue -3, which is the only eigenvalue that is repeated. This can be determined directly from the rank information provided. We are told that the rank of $(A - \lambda I)_{\lambda=-3}$ is 4. Now the relationship between A and Λ is by way of a similarity transformation $A = P\Lambda P^{-1}$ and this does not change the rank. In other words $\text{rank}(A - \lambda I)_{\lambda=-3} = \text{rank}(\Lambda - \lambda I)_{\lambda=-3}$. Knowing that $\text{rank}(\Lambda - \lambda I)_{\lambda=-3} =$ rules out the first choice for Λ above since we would get two columns zeroed out, resulting in a rank of 3. Therefore, we obtain the solution

$$\Lambda = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -1+2j & 0 \\ 0 & 0 & 0 & 0 & -1-2j \end{bmatrix}.$$

(b) Suppose that $eigs(A) = \{-1, -2, -2, -2\}$. Also, suppose the rank of $(A - \lambda I)_{\lambda=-2}$ is 3. Following the same procedure as described above,

$$\Lambda = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

(c) Suppose that $eigs(A) = \{-1, -2, -2, -2, -3\}$. Also, suppose the rank of $(A - \lambda I)_{\lambda=-2}$ is 3. Then,

$$\Lambda = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}.$$

1. We have:

$$\begin{aligned} (sI - A)(sI - A)^{-1} &= I \\ (sI - A)\frac{adj(sI - A)}{det(sI - A)} &= I \\ (sI - A)adj(sI - A) &= det(sI - A)I \end{aligned}$$

Write $adj(sI-A)=\sum_{k=1}^{n} B_k s^{n-k}$ and $det(sI-A)=s^n+\sum_{k=1}^{n} a_k s^{n-k}$ to get:

$$\begin{aligned}
(sI-A)\sum_{k=1}^{n} B_k s^{n-k} &= Is^n + I\sum_{k=1}^{n} a_k s^{n-k} \\
\sum_{k=1}^{n} B_k s^{n-k+1} - \sum_{k=1}^{n} AB_k s^{n-k} &= Is^n + \sum_{k=1}^{n}(a_k I)s^{n-k} \\
\sum_{k=0}^{n-1} B_{k+1} s^{n-k} - \sum_{k=1}^{n} AB_k s^{n-k} &= Is^n + \sum_{k=1}^{n}(a_k I)s^{n-k} \\
B_1 s^n + \sum_{k=1}^{n-1}(B_{k+1}-AB_k)s^{n-k} - AB_n &= Is^n + \sum_{k=1}^{n-1}(a_k I)s^{n-k} + a_n I
\end{aligned}$$

Equating powers of s, we will obtain:

$$\left|\begin{array}{l}
B_1 = I \\
B_{k+1} = AB_k + a_k I \quad ; \quad 1 \le k \le n-1 \\
AB_n + a_n I = 0
\end{array}\right.$$

The first two equations can be solved recursively to determine $B_k,\ k=1,\ldots,n$. The matrix B_n thus obtained will satisfy the third equation by Cayley-Hamilton theorem:

$$A^n + \sum_{k=1}^{n} a_k A^{n-k} = 0$$

For the given matrix $A=\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}$ we have:

$$det(sI-A) = det\begin{bmatrix} s-4 & -2 \\ -3 & s-3 \end{bmatrix} = s^2-7s+6$$

Therefore $a_1=-7$ and $a_2=6$.

$$\left|\begin{array}{l}
B_1 = I \\
B_2 = AB_1 + a_1 I = \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix} - 7I = \begin{bmatrix} -3 & 2 \\ 3 & -4 \end{bmatrix}
\end{array}\right.$$

Thus,

$$\begin{aligned}
adj(sI-A) = B_1 s + B_2 = Is + \begin{bmatrix} -3 & 2 \\ 3 & -4 \end{bmatrix} &= \begin{bmatrix} s-3 & 2 \\ 3 & s-4 \end{bmatrix} \\
(sI-A)^{-1} &= \frac{1}{s^2-7s+6}\begin{bmatrix} s-3 & 2 \\ 3 & s-4 \end{bmatrix}
\end{aligned}$$