University of Toronto
Department of Electrical and Computer Engineering
ECE557F Systems Control
Problem Set #1

1. A system has three inputs u_1, u_2, and u_3 and three outputs y_1, y_2, and y_3. The input-output equations are

$$\begin{aligned} \dddot{y}_1 + a_1\ddot{y}_1 + a_2(\dot{y}_1 + \dot{y}_2) + a_3(y_1 - y_3) &= u_1(t) \\ \ddot{y}_2 + a_4(\dot{y}_2 - \dot{y}_1 + 2\dot{y}_3) + a_5(y_2 - y_1) &= u_2(t) \\ \dot{y}_3 + a_6(y_3 - y_1) &= u_3(t)\,. \end{aligned}$$

Find a state space model for this system.

2. Let the matrix A be given by

$$A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

In the course notes, the transition matrix e^{At} is determined using the Laplace transform method. Use the diagonalization method (extending the scalar field to the complex numbers) to determine e^{At} and verify that it is the same as that shown in the notes.

3. In the previous problem, we saw that when

$$A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

its matrix exponential is easy to write down. This problem demonstrates that a matrix with distinct complex eigenvalues can be transformed into the above form using a nonsingular transformation. Let the matrix A be given by

$$A = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}$$

   (a) Determine the eigenvalues and eigenvectors of A, noting that they form complex conjugate pairs. Let the first eigenvalue be written as $a+ib$ with the corresponding eigenvector $v_1 + iv_2$. v_1 and v_2 are 2-dimensional vectors.

   (b) Set

$$P = [v_1 \quad v_2]$$

   Determine $D = P^{-1}AP$. Hence determine e^{At}.

   (c) Also determine e^{At} using the Laplace transform method, and verify that you get the same answer.

4. Consider the homogeneous state equation $\dot{x} = Ax$. Let

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$$

and $x_0 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Determine the modal decomposition of $x(t)$.

5. (a) Determine the transfer function from u to y for the linear system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & -b \\ 1 & -a \end{bmatrix} x + \begin{bmatrix} 1 \\ c \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} x \end{aligned}$$

(b) Repeat for the linear system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} x + \begin{bmatrix} c \\ 1 - ac \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{aligned}$$

and verify that the 2 transfer functions are the same.

(c) The above results show that the choice of state variables is not unique. Indeed, both state representations describe the second order scalar differential equation

$$\ddot{y} + a\dot{y} + by = u + c\dot{u}$$

Determine how the state variables from parts 1 and 2 are defined in terms of y and u.

6. Consider the linear system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} x \end{aligned}$$

(a) Determine its transfer function from u to y, and verify that it is stable.

(b) Show that in general, the total response, which includes the initial condition response, results in the output y increasing exponentially without bound even when the input u is bounded. This shows one must be careful in interpreting results based on input-output transfer functions.

7. Find a linearized model about the equilibrium $x = 0$ of the following system:

$$\dot{x} = \begin{bmatrix} -\alpha_1 + \sin x_2 & \sin x_2 \\ \sin x_1 & -\alpha_2 + \sin x_2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u , \qquad y = [1\ 0]x\,.$$

where $\alpha_i > 0$ are constant parameters. Next, assume $x(0) = 0$ and let $u(t) = \mathrm{e}^{-3}\overline{\mathrm{u}}(\mathrm{t})$ be applied at $t = 0$, where $\overline{u}$ is a unit step function. Find an approximate solution to the system above.

8. Consider the autonomous system $\dot{x} = Ax$.

(a) Suppose that $eigs(A) = \{-1, -3, -3, -1 + j2, -1 - j2\}$. Also, suppose the rank of $(A - \lambda I)_{\lambda=-3}$ is 4. Determine Λ, the (complex) Jordan form of A.

(b) Suppose that $eigs(A) = \{-1, -2, -2, -2\}$. Also, suppose the rank of $(A - \lambda I)_{\lambda=-2}$ is 3. Determine Λ, the Jordan form of A.

(c) Suppose that $eigs(A) = \{-1, -2, -2, -2, -3\}$. Also, suppose the rank of $(A - \lambda I)_{\lambda=-2}$ is 3. Determine Λ, the Jordan form of A.

9. The Laplace transform method requires the computation of $(sI - A)^{-1}$. For matrices of dimension ≥ 3, sometimes the following recursive method is useful. First note that

$$(sI - A)^{-1} = \frac{adj(sI - A)}{det(sI - A)}$$

where adj(M) is the adjoint matrix of the matrix M. Now we can express

$$adj(sI - A) = B_1 s^{n-1} + B_2 s^{n-2} + \cdots + B_n$$

and

$$det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_n$$

Assume that the coefficients a_i have already been determined, derive a set of recursive equations for the matrices B_i, $i = 1, \ldots, n$ by writing

$$det(sI - A)I = (sI - A)adj(sI - A)$$

and equating coefficients of the powers of s. Note that B_1 is a matrix independent of the a_is (what is it?). Using this method, determine $(sI - A)^{-1}$ for

$$A = \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}$$

(In fact, it is also true that the coefficients a_i can be determined recursively using the equation

$$a_i = -\frac{1}{i} tr(AB_i)$$

which you can verify for the above example.)