University of Toronto
Department of Electrical and Computer Engineering
ECE557F Systems Control
Problem Set #2

1. Consider the 2 matrices:

$$\begin{bmatrix} 4 & 1 & -1 \\ 3 & 2 & -3 \\ 1 & 3 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 1 & 2 \\ 3 & 4 & 5 & 0 & 0 \end{bmatrix}$$

   For each matrix, find (i) its rank, (ii) a basis for its image, (iii) a basis for its kernel.

2. Let $A, U \in \mathbb{R}^n$, $U \neq I$, and U nonsingular.
   (a) $\mathcal{N}(A) = \mathcal{N}(UA)$. T or F?
   (b) $\mathcal{N}(A) = \mathcal{N}(AU)$. T or F?
   (c) $\mathcal{N}(A^2) \subseteq \mathcal{N}(A)$. T or F?

3. Let x_i be real numbers. Is $\{ (x_1, x_2, x_3) : 2x_1 + 3x_2 + 6x_3 - 5 = 0\}$ a subspace of $\mathbb{R}^3$? Explain.

4. You are given $A : \mathbb{R}^2 \to \mathbb{R}^2$, $A \neq 0$ and $\mathcal{R}(A) \subseteq \mathcal{N}(A)$. Can you find A? Explain.

5. You are given the n eigenvalues of the matrix $A \in \mathbb{R}^{n \times n}$. Can you determine $rank(A)$? If yes, given an expression for $rank(A)$. If no, can you give bounds on $rank(A)$?

6. Suppose you have a ray starting at $x(0) = x_0$ and not including the origin. Can it be the trajectory of a linear system $\dot{x} = Ax$, $x \in \mathbb{R}^2$? What about in higher dimensions? Explain.

7. Two matrices $A, B \in \mathbb{R}^{n \times n}$ each have distinct eigenvalues. Also they share the same set of eigenvectors. Is $AB = BA$?

8. Suppose that $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times m}$ with $m \leq n$ and $rankA = rankB = m$. Find a necessary and sufficient condition that AB is full rank. Provide a proof.

9. This problem describes a method of determining functions $\phi_i(t)$, $i = 0, 1, \cdots, n-1$ such that

$$e^{At} = \sum_{i=0}^{n-1} \phi_i(t) A^i$$

   Let the characteristic polynomial of A be written as

$$det(sI - A) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0$$

   Define the matrix F by

$$F = \begin{bmatrix} 0 & \cdots & 0 & -\alpha_0 \\ 1 & 0 & \cdots & \vdots \\ \vdots & \ddots & \vdots & \vdots \\ \cdots & \cdots & 1 & -\alpha_{n-1} \end{bmatrix}$$

and let $\phi(t) = [\phi_0(t) \;\; \phi_1(t) \cdots \phi_{n-1}(t)]^T$ satisfy the differential equation

$$\dot{\phi}(t) = F\phi(t)$$

$$\phi(0) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

(a) Show that

$$G(t) = \sum_{i=0}^{n-1} \phi_i(t) A^i$$

is precisely the transition matrix e^{At}.

(Hint: Show that $G(t)$ satisfies the same differential equation and initial condition as e^{At}.)

(b) For

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$$

compute e^{At} using $\sum_{i=0}^{n-1} \phi_i(t) A^i$, as in part (a). Verify your calculation by directly computation of e^{At} using Laplace transforms.