University of Toronto
Department of Electrical and Computer Engineering
ECE557F Systems Control
Problem Set #3

1. $Q_c = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

   $\mathcal{R}(Q_c) = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ so that the system is not controllable. Now $\begin{bmatrix} 4 \\ 1 \\ 4 \end{bmatrix}$ is not in $\mathcal{R}(Q_c)$ so that the vector is not reachable from the origin.

2. $Q_c = \begin{bmatrix} 1 & 1 & 4 & 0 & \\ 0 & 0 & 1 & 0 & \ldots \\ 0 & 1 & 1 & 1 & \\ 1 & 0 & 1 & 1 & \end{bmatrix}$

   The 1st 4 columns of Q_c are linearly independent so that (A, B) is controllable.

3. $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$

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\begin{aligned}
\det(sI - A) &= \det \begin{bmatrix} s & -1 & -1 \\ -1 & s+1 & 0 \\ 0 & -2 & s \end{bmatrix} \\
&= s(s^2 + s - 1) - 2 = s^3 + s^2 - s - 2 \\
V &= [A^2B \;\; AB \;\; B] \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \quad \text{(Hence } (A, B) \text{ is controllable)} \\
&= \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \\
V^{-1} &= \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}
\end{aligned}
$$

The desired char. poly. $r(s) = (s+1)^3 = s^3 + 3s^2 + 3s + 1$. Thus

$$\begin{aligned}
K &= [-3 \quad -4 \quad -2]V^{-1} \\
&= [-3 \quad -4 \quad -2]\begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} = [-1 \quad -1 \quad -1] \\
A + BK &= \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 2 & 0 \end{bmatrix} + \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & -1 \end{bmatrix}
\end{aligned}$$