University of Toronto
Department of Electrical and Computer Engineering
ECE557F Systems Control
Problem Set #4

1. $A = \begin{bmatrix} 5 & 2 & 1 \\ -2 & 1 & -1 \\ -4 & 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} -2 & 2 \\ 2 & -1 \\ 2 & -2 \end{bmatrix}$

$$
\begin{aligned}
[b_1 \ \ Ab_1 \ \ A^2b_1] &= \begin{bmatrix} -2 & -4 & -6 \\ 2 & 4 & 6 \\ 2 & 6 & 10 \end{bmatrix} \quad \text{which has rank 2} \\
[b_1 \ \ Ab_2 \ \ A^2b_2] &= \begin{bmatrix} 2 & 6 & \star \\ -1 & -3 & \star \\ -2 & -6 & \star \end{bmatrix} \quad \text{which has rank 1}
\end{aligned}
$$

$$
\begin{aligned}
Q &= \begin{bmatrix} -2 & -4 & 2 \\ 2 & 4 & -1 \\ 2 & 6 & -2 \end{bmatrix} \qquad Q^{-1} = \begin{bmatrix} -\frac{1}{2} & 1 & -1 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 1 & 0 \end{bmatrix} \\
S &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad K_1 = SQ^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \\
A + BK_1 &= \begin{bmatrix} 5 & 2 & 1 \\ -2 & 1 & -1 \\ -4 & 0 & -1 \end{bmatrix} + \begin{bmatrix} -2 & 2 \\ 2 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} 5 & 2 & 1 \\ -2 & 1 & -1 \\ -4 & 0 & -1 \end{bmatrix} - \begin{bmatrix} -1 & 0 & -1 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 6 & 2 & 2 \\ -\frac{5}{2} & 1 & -\frac{3}{2} \\ -5 & 0 & -2 \end{bmatrix}
\end{aligned}
$$

$\det(sI - A - BK_1) = s^3 - 5s^2 + 7s - 3$

Desired $r(s) = (s+10)((s+2)^2+1) = s^2 + 14s^2 + 45s + 50$

$$
\begin{aligned}
V &= \begin{bmatrix} -4 & -4 & -2 \\ 5 & 4 & 2 \\ 8 & 6 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 7 & -5 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 6 & -2 \\ -1 & -6 & 2 \\ -8 & -4 & 2 \end{bmatrix} \\
V^{-1} &= \begin{bmatrix} 1 & 1 & 0 \\ 3.5 & 3 & 0.5 \\ 11 & 10 & 1.5 \end{bmatrix} \\
k_1 &= [-53 \quad -38 \quad -19]V^{-1} = [-395 \quad -357 \quad -47.5] \\
K &= K_1 + e_1 k_1 = \begin{bmatrix} -395 & -357 & -47.5 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}
\end{aligned}
$$

2. $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$

$$Q_c = \begin{bmatrix} 1 & 1 & 1 & 0 & \\ 1 & 0 & 1 & 0 & \cdots \\ 1 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 0 & \end{bmatrix}$$

which is of rank 2

$$\mathcal{R}(Q_c) = \text{span}\,\{ \underset{V_1}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}}, \underset{V_2}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}} \}$$

Take $V_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad V_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$$V = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now

$$\begin{aligned} Av_1 &= v_1 \\ Av_2 &= 0 \\ Av_3 &= \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} = v_1 - v_2 - 2v_3 + v_4 \\ Av_4 &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = v_2 + v_4 \\ \tilde{A} &= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = V^{-1}AV \qquad \tilde{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = V^{-1}B \end{aligned}$$

Since the lower right 2×2 block is uncontrolled and unstable, the system is not stabilizable.

3. (i) To show necessity, suppose there exists an eigenvalue λ with $Re\,\lambda \geq 0$ such that

$$Rank \begin{bmatrix} A - \lambda I & B \end{bmatrix} < n$$

Then there exists a complex vector v such that

$$v^* \begin{bmatrix} A - \lambda I & B \end{bmatrix} = 0$$

where v^* denotes the conjugate transpose of v. But then for any K,

$$v^*(A - BK - \lambda I) = 0$$

so that λ is an eigenvalue of $A - BK$ for any K. Hence (A, B) is not stabilizable. To show sufficiency, recall that if (A, B) is not controllable, there exists a nonsingular V such that

$$\begin{aligned} V^{-1}AV &= \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \\ V^{-1}B &= \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \end{aligned}$$

with (A_{11}, B_1) controllable. Assume that for all $Re\,\lambda \geq 0$,

$$Rank \begin{bmatrix} A - \lambda I & B \end{bmatrix} = n$$

Then we must also have

$$\begin{aligned} Rank\, V^{-1} \begin{bmatrix} A - \lambda I & B \end{bmatrix} \begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix} &= \begin{bmatrix} V^{-1}AV - \lambda I & V^{-1}B \end{bmatrix} \\ &= \begin{bmatrix} A_{11} - \lambda I & A_{12} & B_1 \\ 0 & A_{22} - \lambda I & 0 \end{bmatrix} \\ &= n \end{aligned}$$

This implies that λ cannot be an eigenvalue of A_{22} so that (A, B) is stabilizable.

(ii) We appeal to the fact that (C, A) is detectable if and only if (A^T, C^T) is stabilizable. The previous result then show that (C, A) is detectable if and only if for all $Re\,\lambda \geq 0$,

$$Rank \begin{bmatrix} A^T - \lambda I & C^T \end{bmatrix} = n$$

Transposing, we get (C, A) is detectable if and only if for all $Re\,\lambda \geq 0$,

$$Rank \begin{bmatrix} C \\ A - \lambda I \end{bmatrix} = n$$